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Numerical Mathematics II

Exercise Problems 07

The solutions have to be presented in the tutorial by participants of the course. In order to fulfill the tutorial requirements, each student has to present two correct solutions (depending on the number of subproblems, a 'solution' might cover only a part of the subproblems) and obtain a total of 4 points from their presentations. A fully correct solution is awarded 2 points, a partially correct solution is awarded 1 point, and an incorrect solution is awarded 0 points.
Prepare these presentations! All statements have to be proved, auxiliary calculations have to be presented. Statements given in the lectures can be used without proof.

1. *A-stability of θ-methods.* Consider the θ-method

$$y_{k+1} = y_k + h\left((1-\theta)f_k + \theta f_{k+1}\right)$$

for $\theta \in [0,1]$. Show that this method is A-stable if and only if $\theta \geq \frac{1}{2}$.

2. *Stability function of classical Runge–Kutta scheme.* Compute the stability function of the classical Runge–Kutta method and draw a sketch of the domain of stability.
Hint: For drawing the domain of stability, one can compute sufficiently many values of the stability function, which can be performed quickly with a short program. The domain of stability is simply connected. **4 points**

3. *A-stability of an implicit 2-stage Runge–Kutta method.*

   i) Show that when the general implicit 2-stage Runge–Kutta scheme is applied to the model problem for the studies of linear stability, then

$$K_1 = \frac{(1+\lambda h(a_{12}-a_{22}))\,\lambda y_k}{\Delta}, \quad K_2 = \frac{(1+\lambda h(a_{21}-a_{11}))\,\lambda y_k}{\Delta},$$

   where Δ is the determinant of the matrix $I - \lambda h A$ with

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

   ii) For the method defined by the Butcher tableau

$$\begin{array}{c|cc} \frac{1}{6}(3-\sqrt{3}) & \frac{1}{4} & \frac{1}{12}(3-2\sqrt{3}) \\ \frac{1}{6}(3+\sqrt{3}) & \frac{1}{12}(3+2\sqrt{3}) & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

   deduce that $y_{k+1} = R(\lambda h) y_k$, where

$$R(\lambda h) = \frac{1 + \frac{1}{2}\lambda h + \frac{1}{12}\lambda^2 h^2}{1 - \frac{1}{2}\lambda h + \frac{1}{12}\lambda^2 h^2}.$$

   By writing $R(z)$ in the factorized form $\frac{(z+p)(z+q)}{(z-p)(z-q)}$ deduce that this Runge–Kutta method is A-stable.

4. *Second order boundary value problem with first order term and upwind discretization; GMRES.* Continue Problem 3 from Exercise Sheet 06.

   (a) Consider a decomposition of $[0,1]$ by a grid as, e.g., in Problem 3, Exercise sheet 01. Show that the approximation (backward finite difference or upwind finite difference)

   $$u'(x_i) \approx \frac{u(x_i) - u(x_{i-1})}{h} = u_{\bar{x},i}, \quad i = 1, \ldots, n-1,$$

   is of first order, i.e.,

   $$u_{\bar{x},i} = u'(x_i) + \mathcal{O}(h)$$

   if $u \in C^2([0,1])$. **1 point**

   (b) Modify the code of Problem 3 from Exercise Sheet 06 such that it applies to the differential equation given in this problem and such that the first order derivative is approximated by the backward difference. **2 points**

   (c) Consider the grids with $h \in \{1/8, 1/16, 1/32, 1/64, 1/128, 1/256\}$ and compute the solution for $\varepsilon \in \{1, 10^{-2}, 10^{-4}, 10^{-6}\}$ (solve the linear system of equations with the backslash command) and compute the errors $\|u - u_h\|_{l^2}$. How does the error behaves with respect to the size ε ? **2 points**

   (d) Consider the same situations as in Problem 4c. Solve the arising linear systems of equations with the Jacobi method, the Gauss–Seidel method, and GMRES. In the used programming language, a provided GMRES routine can be used (MATLAB: read the documentation carefully). GMRES should be used without restart, with $tol = 1e - 10$, and the maximal number of iterations should coincide with the dimension of the problem. Give the number of iterations for all methods ! How do they change with respect to ε and to h ? **2 points**

The exercise problems will be discussed at the turorial on **Thursday, June 11, 2026, 12-14**.