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Numerical Mathematics II

Exercise Problems 06

The solutions have to be presented in the tutorial by participants of the course. In order to fulfill the tutorial requirements, each student has to present two correct solutions (depending on the number of subproblems, a 'solution' might cover only a part of the subproblems) and obtain a total of 4 points from their presentations. A fully correct solution is awarded 2 points, a partially correct solution is awarded 1 point, and an incorrect solution is awarded 0 points.

Prepare these presentations! All statements have to be proved, auxiliary calculations have to be presented. Statements given in the lectures can be used without proof.

1. *Radau-IA method.* A so-called Radau-IA method takes the left boundary of the interval as a node and it satisfies B$(2s-1)$ and D(s). Derive the Butcher tableau of the Radau-IA method for $s = 2$.

2. *Representation of the numerical solution of the model IVP.* Prove the following theorem. Consider a Runge–Kutta method with s stages and with the parameters $(A, \mathbf{b}, \mathbf{c})$. If $z^{-1} = (\lambda h)^{-1}$ is not an eigenvalue of A, then the Runge–Kutta scheme is well-defined for the initial value problem (2.7). In this case, it is
$$y_k = (R(h\lambda))^k, \quad k = 0, 1, 2, \ldots.$$

3. *Several methods applied to the model problem of linear stability.* Write a code that solves the model initial value problem for stability, problem (2.6), with the explicit Euler method, the implicit Euler method, and the trapezoidal rule. Take $\lambda = -10$, the interval $[0, 1]$, and meshes with 2, 4, 8, 16 intervals. Compute the error at the final point $x = 1$. Interprete the results.
Hint: Use the formulas from Example 2.11 for implementing the methods.

4. *Richarson iteration.* Continue Problem 6 from Exercise sheet 04. Solve the system now with the Richardson iteration. Use the information from Problem 5 on Exercise sheet 05 and from the lecture notes to find a suitable damping parameter (apply a safety factor of 0.9 to obtain a strictly 'lower than' relation). Perform at most 100 000 iterations. How does the damping parameter and the number of iterations behave if the mesh width varies?

5. *Second order boundary value problem with first order term.* Consider the following boundary value problem
$$-\varepsilon u''(x) + u'(x) = 1, \quad u(0) = u(1) = 0,$$
where $\varepsilon > 0$ is a parameter. The solution of this problem is
$$u(x) = x - \frac{e^{-(1-x)/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}.$$

   (a) Draw the solution in $[0, 1]$ for $\varepsilon \in \{1, 10^{-2}, 10^{-4}, 10^{-6}\}$. How does the solution change with respect to ε ?

   (b) Consider a decomposition of $[0, 1]$ by a grid as, e.g., in Problem 3, Exercise sheet 01. Show that the approximation (central finite difference)
$$u'(x_i) \approx \frac{u(x_{i+1}) - u(x_{i-1})}{2h} = u_{x,i}, \quad i = 1, \ldots, n-1,$$
$x_{i-1} = x_i - h, x_{i+1} = x_i + h$, is of second order, i.e.,
$$u_{x,i} = u'(x_i) + \mathcal{O}(h^2)$$
if $u \in C^3([0, 1])$.

(c) Modify the code of Problem 3, Exercise sheet 02, such that it applies to the differential equation given here, where the first order derivative is approximated by the central difference.

(d) Consider the grid with $h = 1/128$ and compute the solution for $\varepsilon \in \{1, 10^{-2}, 10^{-4}, 10^{-6}\}$ (solve the linear system of equations with the backslash command), compute the errors $\|u - u_h\|_{l^2}$, and draw the computed solutions. How do they change when ε becomes smaller ?

The exercise problems will be discussed at the turorial on **Thursday, June 04, 2026, 12-14**.