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Numerical Mathematics II

Exercise Problems 05

The solutions have to be presented in the tutorial by participants of the course. In order to fulfill the tutorial requirements, each student has to present two correct solutions (depending on the number of subproblems, a 'solution' might cover only a part of the subproblems) and obtain a total of 4 points from their presentations. A fully correct solution is awarded 2 points, a partially correct solution is awarded 1 point, and an incorrect solution is awarded 0 points.
Prepare these presentations! All statements have to be proved, auxiliary calculations have to be presented. Statements given in the lectures can be used without proof.

1. *Explicit vs. implicit Euler schemes.*
   a) Consider the IVP
   $$y'(x) = -x(y(x))^2, \quad y(x_0) = y_0 > 0.$$
   Using the step size h, compute analytically one step with the explicit and implicit Euler schemes.
   b) In problem 2 of exercise sheet 03, the IVP
   $$y'(x) = y^{1/5}, \quad y(0) = 0$$
   was considered and it was shown that the explicit Euler method fails to approximate the solution $y(x) = \left({}^{4x}/_{5}\right)^{5/4}$.
   Now consider the approximation of the same problem using the implicit Euler method. Show that there is a solution of the form
   $$y_k = (C_k h)^{5/4}, \quad k \geq 0,$$
   with $C_0 = 0$, $C_1 = 1$, and $C_k > 1$ for all $k \geq 2$.
2. *Embedded Runge–Kutta scheme with two stages.* Derive the embedded explicit Runge–Kutta scheme $p(q) = 1(2)$ with two stages and the condition $a_{21} = b_1$, i.e., write the Butcher tableau in terms of c_2. Which schemes are obtained in the special case $c_2 = 1$?
3. *Trapezoidal rule.* Consider the initial value problem
   $$y'(x) = f(x, y), \quad y(x_0) = y_0.$$
   Derive a formula for one step of the trapezoidal rule for solving this initial value problem from the Butcher tableau of the trapezoidal rule
   $$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}.$$
4. *Optimal relaxation parameter for SOR method.* Continue Problem 3, Exercise sheet 03. One can show that the optimal relaxation parameter for the SOR method is
   $$\omega_{\text{opt}} = \frac{2}{1 + \sqrt{1 - \rho^2}},$$

where ρ is the spectral radius of the iteration matrix of the Jacobi method. In the considered example, one finds that

$$\omega_{\text{opt}} = \frac{2}{1+\sin(\pi h)}.$$

Use this relaxation parameter in the code from Problem 3, Exercise sheet 03. How does the optimal parameter behave if h decreases? Compare the number of iterations with the numbers obtained for the other relaxation parameters from the solution of Problem 3, Exercise sheet 03. What can be observed?

5. *Properties of the matrix obtained with the finite difference discretization.* Continue Problem 4 from Exercise sheet 02. The matrix that has to be assembled in this problem has the form

$$A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & & \cdots & & 0 \\ -1 & 2 & -1 & 0 & & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & & 0 & -1 & 2 & -1 \\ 0 & & \cdots & & 0 & -1 & 2 \end{pmatrix} \in \mathbb{R}^{(n-1)\times(n-1)}.$$

   (a) Show with the help of the Definition 2.10 that this matrix is positive definite.

   (b) Show that the eigenvalues of this matrix are

$$\lambda_k = \frac{4}{h^2}\sin^2\left(\frac{k\pi}{2n}\right), \quad k = 1,\ldots,n-1, \tag{1}$$

   and the corresponding eigenvectors are $\underline{v}_k = (v_{k,1}, v_{k,2}, \ldots, v_{k,n-1})^T$ with

$$v_{k,j} = \sin\left(\frac{jk\pi}{n}\right), \quad k,j = 1,\ldots,n-1.$$

6. *Optimal damping parameter for Richarson iteration with s.p.d. matrix.* Let $A \in \mathbb{R}^{n\times n}$ be a s.p.d. matrix. Consider the Richarson iteration (in fixed-point) form

$$\underline{x}^{(k+1)} = (I - \alpha A)\,\underline{x}^{(k)} + \alpha\underline{b},$$

with $\alpha \in \mathbb{R}$, for the iterative solution of $A\underline{x} = \underline{b}$. Compute the damping factor α which is optimal in the sense that it minimizes the spectral radius of the iteration matrix and show that the iteration converges for any initial iterate with this parameter.

The exercise problems will be discussed at the turorial on **Thursday, May 28, 2026, 12-14**.