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Numerical Mathematics II

Exercise Problems 04

The solutions have to be presented in the tutorial by participants of the course. In order to fulfill the tutorial requirements, each student has to present two correct solutions (depending on the number of subproblems, a 'solution' might cover only a part of the subproblems) and obtain a total of 4 points from their presentations. A fully correct solution is awarded 2 points, a partially correct solution is awarded 1 point, and an incorrect solution is awarded 0 points.

Prepare these presentations! All statements have to be proved, auxiliary calculations have to be presented. Statements given in the lectures can be used without proof.

1. *Consistency conditions for a 3rd order Runge–Kutta scheme with three stages.* Consider an autonomous initial value problem
$$y'(x) = f(y(x)), \quad y(x_0) = y_0.$$
This problem shall be solved on an equidistant grid with an explicit 3-stage Runge–Kutta method. Derive the conditions for this method of being of third order that are given in the course, where for the coefficients of the Runge–Kutta scheme it shall hold
$$c_2 = a_{21}, \quad c_3 = a_{31} + a_{32}.$$
Hint: Use the same approach as for the second order method which was presented in the course.

2. *Local error of a one-step method.* Consider the interval $[a, b]$ together with an uniform mesh $x_k = a + kh$, $k = 0, ..., N$, where the step size h is given by $h = (b - a)/N$. Suppose that the solution of the initial value problem
$$y'(x) = f(x, y(x)), \quad y(a) = y_0,$$
is approximated by an explicit one-step method, in which the approximations y_k to the values $y(x_k)$ are determined by using the recurrence relation
$$y_{k+1} = y_k + h(1-\delta)f(x_k, y_k) + h\delta f\left(x_k + \frac{h}{2\delta}, y_k + \frac{h}{2\delta}f(x_k, y_k)\right), \quad \delta \in \mathbb{R}_+.$$
   a) Assuming that y is sufficiently smooth, show that the local error $\mathrm{le}(x_k, \delta)$ is given by
$$\mathrm{le}(x_k, \delta) = \frac{h^3}{8\delta}\left[\left(\frac{4}{3}\delta - 1\right)y'''(x_k) + y''(x_k)\frac{\partial f}{\partial y}(x_k, y(x_k))\right] + \mathcal{O}\left(h^4\right).$$
   b) Consider the initial value problem
$$y'(t) = -y(t)^\alpha,\ y(0) = 1,$$
   to which the numerical method gets applied and where α is a positive integer. Show that for $\alpha = 1$ one has $\mathrm{le}(x_k, \delta) = \mathcal{O}(h^3)$ and that for $\alpha \geq 2$ there exists $\delta_0 \in \mathbb{R}_+$ such that $\mathrm{le}(x_k, \delta_0) = \mathcal{O}(h^4)$.

3. *Iteration methods for a linear system of equations.* Consider the linear system of equations
$$\begin{aligned} 3x + y &= 7, \\ 2x + 4y + z &= 10, \\ x + 2z &= 6, \end{aligned}$$
for which we want to approximate the solution using iteration methods. Compute, by hand, the first two iterates using

i) the Jacobi method and

ii) the Gauss–Seidel method

with the initial values $(x^{(0)}, y^{(0)}, z^{(0)}) = (0,0,0)^T$.

4. *Gauss–Seidel method.* Let A be a symmetric positive definite matrix and let B be a matrix that satisfies
$$(A\underline{x}, \underline{x}) < 2\,(B\underline{x}, \underline{x}) \quad \forall\ \underline{x} \in \mathbb{R}^d.$$

   i) Verify that $B + B^T - A$ is a symmetric positive definite matrix.

   ii) Show that the method
   $$\underline{x}^{(k+1)} = \underline{x}^{(k)} + B^{-1}\left(\underline{b} - A\underline{x}^{(k)}\right)$$
   converges.

   iii) Using ii), prove that the Gauss–Seidel method converges for any $\underline{x}^{(0)}$, if the matrix A is a symmetric positive definite matrix.

5. *Convergence of damped Jacobi method.* Proof the following statement: If the Jacobi method converges for each initial iterate, then also the damped Jacobi method with $0 < \omega \le 1$ converges for each initial iterate.

6. *Programming problem: damped Jacobi and SOR method.* Continue Problem 4 from Exercise Sheet 03. A sample code for Problem 4 from Exercise Sheet 03, also including an implementation of the Jacobi method, is available on the homepage of the course. If you do not have your own code, you can use this sample code as basis for solving the problems given below.

   (a) Compute the estimates of the spectral condition number for the matrices that are obtained with $h \in \{1/8, 1/16, 1/32, 1/64, 1/128, 1/256\}$ with the MATLAB command `condest` or a corresponding command in other languages. What can be observed? **1 point**

   (b) Implement the damped Jacobi method for solving the linear systems of equations that are obtained for $h \in \{1/8, 1/16, 1/32, 1/64, 1/128, 1/256\}$. Use the damping factors $\omega \in \{0.1, 0.2, \ldots, 1, 1.1\}$. Use as starting iterate the zero vector and stop the iteration if the Euclidean norm of the residual $\|A\underline{u} - \underline{f}\|_2$ is less than 10^{-10} or after 100 000 iterations were performed. Count the number of iterations. What can be observed? **4 points**

   (c) Implement the SOR method for solving the linear systems of equations that are obtained for $h \in \{1/8, 1/16, 1/32, 1/64, 1/128, 1/256\}$. Use the damping factors $\omega \in \{0.1, 0.2, \ldots, 1.8, 1.9, 2.0\}$. Use as starting iterate the zero vector and stop the iteration if the Euclidean norm of the residual $\|A\underline{u} - \underline{f}\|_2$ is less than 10^{-10} or after 10 000 iterations were performed. Count the number of iterations. What can be observed? Give for each h the best relaxation factor (least number of iterations)!

   The simulations may take a while. The best way is to write a loop that performs one simulation after the other.

The exercise problems will be discussed at the turorial on **Thursday, May 21, 2026, 12-14**. Notice that Thursday, May 14, 2026, is a public holiday in Germany.