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# Numerical Mathematics II

## Exercise Problems 02

The solutions have to be presented in the tutorial by participants of the course. In order to fulfill the tutorial requirements, each student has to present two correct solutions (depending on the number of subproblems, a 'solution' might cover only a part of the subproblems) and obtain a total of 4 points from their presentations. A fully correct solution is awarded 2 points, a partially correct solution is awarded 1 point, and an incorrect solution is awarded 0 points.
**Prepare these presentations!** All statements have to be proved, auxiliary calculations have to be presented. Statements given in the lectures can be used without proof.

1. *Forward Euler method.* Consider the initial value problem

$$(1+x)y'(x) + y(x) = \frac{1}{1+x}, \quad y(0) = 1.$$

   (a) Compute an approximation of the solution with the forward Euler method in $[0,1]$ with the step lengths $h_1 = 0.2$ and $h_2 = 0.1$.
   Hint: write a short code.

   (b) Compute the error to the analytical solution

$$y(x) = \frac{\ln(x+1)+1}{1+x}$$

   at $x = 1$.

   (c) Discuss the results briefly.

2. *Estimate for a sequence of real numbers.* Assume that for real numbers $x_n$, $n = 0, 1, \ldots$, the inequality

$$|x_{n+1}| \le (1+\delta)\,|x_n| + \beta$$

   holds with constants $\delta > 0$, $\beta \ge 0$. Then, it holds that

$$|x_n| \le e^{n\delta}\,|x_0| + \frac{e^{n\delta}-1}{\delta}\beta, \quad n = 0, 1, \ldots.$$

3. *Some linear algebra problems.*

   (a) Let $A \in \mathbb{R}^{n\times n}$ be a s.p.d. matrix. Consider for $1 \le p \le n$ the upper $p \times p$ block of $A$, which is denoted by $A_{11}$:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A_{11} \in \mathbb{R}^{p\times p}.$$

   Show that $A_{11}$ is also a s.p.d. matrix.

   (b) Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Show that for the spectral condition number

$$\kappa_2\left(A^T A\right) = \left(\kappa_2(A)\right)^2.$$

   (c) Let $Q \in \mathbb{R}^{n\times n}$ be an orthogonal matrix, i.e., it holds $Q^T = Q^{-1}$, and $A \in \mathbb{R}^{n\times m}$. Show that

$$\|QA\|_2 = \|A\|_2.$$

(d) Let $A \in \mathbb{R}^{n \times n}$. Show that $\rho(A) \leq \|A\|$ for any matrix norm that is compatible with the vector norm, where $\rho(A)$ is the spectral radius of $A$.

4. *Finite Difference matrix for the second order derivative.* Consider the differential equation (Poisson equation, boundary value problem)

$$\begin{aligned} -u'' &= f \quad \text{in } (0,1), \\ u(0) &= a, \\ u(1) &= b. \end{aligned}$$

Use an equidistant grid

$$0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1, \quad h = x_i - x_{i-1},\ i = 1, \ldots, n,$$

for the discretization of the second derivative.

(a) Show that the approximation (finite difference)

$$u''(x_i) \approx \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} =: u_{xx,i}, \quad i = 1, \ldots, n-1$$

is of second order consistent, i.e.,

$$u_{xx,i} = u''(x_i) + \mathcal{O}(h^2)$$

if $u \in C^4([0,1])$.

(b) Insert the approximation of the second order derivative in the differential equation and derive a linear system of equations for the values $u_i = u(x_i)$, $i = 0, \ldots, n$. Set the boundary conditions and use as approximation for the right-hand side $f_i = f(x_i)$, $i = 1, \ldots, n-1$.

(c) Reduce this linear system of equations to a system for $u_i$, $i = 1, \ldots, n-1$, by eliminating the equations for the boundary conditions.

(d) Given $n$. Write a code that computes the matrix of the reduced system. If the programming language supports a `sparse` format, then store this matrix in this format.

**This matrix is needed for further exercise problems!**

The exercise problems will be discussed at the turorial on **Thursday, April 30, 2026, 12-14**.