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Numerical Mathematics II

Exercise Problems 01 (Repetition)

The solutions have to be presented in the tutorial by participants of the course. In order to fulfill the tutorial requirements, each student has to present two correct solutions (depending on the number of subproblems, a 'solution' might cover only a part of the subproblems) and obtain a total of 4 points from their presentations. A fully correct solution is awarded 2 points, a partially correct solution is awarded 1 point, and an incorrect solution is awarded 0 points.
Prepare these presentations! All statements have to be proved, auxiliary calculations have to be presented. Statements given in the lectures can be used without proof.

1. *Types of integrable first order ordinary differential equations.* Solve the following ordinary differential equations.

   a) $y'(x) + y^2(x) = 1,$
   b) $y'(x) + y(x)\cos x = 0,$
   c) $y'(x) = xy^7(x),$
   d) $2y(x)y'(x) = x^2.$

2. *Solution by substitution.* Determine the general solution of

   $$y'(x) = (x - y(x))^2 + 1.$$

   Hint: Find a suitable substitution.

3. *Initial value problem with multiple solutions.* Demonstrate that solutions of the initial value problem

   $$y'(x) = \sqrt{|y(x)|}, \quad y(0) = 0$$

   are not unique.

4. *Modeling using a ordinary differential equation.* A motor boat moves over water at rest with constant velocity of 20 $^{\mathrm{km}}/_{\mathrm{h}}$. At this speed the motor of the boat stops and the velocities continuously drops to 7 $^{\mathrm{km}}/_{\mathrm{h}}$ within a period of 30 seconds. The water is assumed to decelerate the boat proportional the velocity of the latter. Compute the velocity of the boat 3 minutes after the motor stops. How far does the boat move after 2 minutes after the motor stops.

5. *Integrable classes of ordinary differential equations of first order.*

   (a) Determine the general solution for the following differential equation

   $$xy'(x) - 4y(x) - x^2\sqrt{y(x)} = 0.$$

(b) Solve the following initial value problem

$$y'(x) = \frac{-x+2}{x(1-x)} y(x) + \frac{1}{x^2(x-1)} y^2(x), \quad y(2) = a, \ a \in \mathbb{R}, \ a > 0.$$

6. *Vector norms.* Solve the following problems.

(a) Let $\underline{x} \in \mathbb{R}^n$. Show that

$$\lim_{p \to \infty} \|\underline{x}\|_p = \|\underline{x}\|_\infty .$$

(b) Show that the Euclidean vector norm and the Frobenius matrix norm are compatible, i.e.,

$$\|A\underline{x}\|_2 \leq \|A\|_F \, \|\underline{x}\|_2 \quad \forall \ A \in \mathbb{R}^{m \times n}, \underline{x} \in \mathbb{R}^n.$$

7. *Orthogonal matrices.* Let

$$\mathbb{O}_m(\mathbb{R}) := \left\{ Q \in \mathbb{R}^{m \times m} \ : \ Q^T = Q^{-1} \right\}$$

be the set of orthogonal matrices in $\mathbb{R}^{m \times m}$. Prove the following statements.

i) Let $Q \in \mathbb{O}_m(\mathbb{R})$, then $Q^T \in \mathbb{O}_m(\mathbb{R})$.
ii) Let $Q \in \mathbb{O}_m(\mathbb{R})$, then $|\det(Q)| = 1$.
iii) Let $Q_1, Q_2 \in \mathbb{O}_m(\mathbb{R})$, then $Q_1 Q_2 \in \mathbb{O}_m(\mathbb{R})$.
iv) $\|Q\mathbf{x}\|_2 = \|\mathbf{x}\|_2$ is satisfied by every $\mathbf{x} \in \mathbb{R}^m$.
v) For every matrix $A \in \mathbb{R}^{m \times m}$ we have $\|A\|_2 = \|QA\|_2 = \|AQ\|_2$.
vi) For every matrix $A \in \mathbb{R}^{m \times m}$ we have $\kappa_2(A) = \kappa_2(QA) = \kappa_2(AQ)$.
vii) Each $Q \in \mathbb{O}_m(\mathbb{R})$ satisfies $\kappa_2(Q) = 1$.

8. *Projection.* Let V be an inner product space and $P \ : \ V \to V$ a linear operator. Prove the following statements to be equivalent.

(a) $(x - Px, y) = 0$ for every $x \in V$ and every $y \in \mathrm{im}(P) = \{Pz \ : \ z \in V\}$.

(b) $P^2 = P$ and $(Px, y) = (x, Py)$ for every $x, y \in V$.

The exercise problems will be discussed at the turorial on **Thursday, April 23, 2026, 12-14**.