Chapter 1
Explicit One-Step Methods

Remark 1.1. Contents. This course presents methods for the numerical solution of explicit systems of initial value problems for ordinary differential equations of first order

$$\boldsymbol{y}'(x) = \boldsymbol{f}(x, \boldsymbol{y}(x)), \quad \boldsymbol{y}(x_0) = \boldsymbol{y}_0.$$

For the most part, only initial value problems for scalar ordinary differential equations of first order

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0, \tag{1.1}$$

are considered, for simplicity of presentation. The extension of the results and the methods to systems is generally straightforward.

It will be always assumed that there is a unique solution of the initial value problem in a neighborhood of the initial value. In applications, the independent variable is often the time. □

1.1 Consistency and Convergence

Definition 1.2. Grid, step size. A grid is a decomposition I_h of the interval $I = [x_0, x_e]$

$$I_h = \{x_0, x_1, \ldots, x_N = x_e\}$$

with $x_0 < x_1 < \ldots < x_N$. The differences between neighboring grid points $h_k = x_{k+1} - x_k$ are called step sizes. For an equidistant grid, the notation $h = h_k$ will be used for the step size, see Figure 1.1. □

Remark 1.3. Explicit and implicit methods. Let $y(x_k)$ denote the solution of (1.1) in the node x_k and y_k a numerical approximation of $y(x_k)$. A numerical method for the solution of (1.1) on a grid I_h is called explicit, if an approximation y_{k+1} in x_{k+1} can be calculated directly by inserting already computed values y_i, $i \leq k$, in some formula(s). Otherwise, the method is called implicit method. Implicit methods require in each step the solution of a generally nonlinear equation for computing y_{k+1}. □

Fig. 1.1 Equidistant grid.

Definition 1.4. One-step method, incremental function. A one-step method for the computation of an approximation y_{k+1} of the solution of (1.1) on a grid I_h has the form

$$y_{k+1} = y_k + h_k \Phi\left(x, y, h_k\right), \ k = 0, 1, \ldots, \quad x \in [x_k, x_{k+1}], \quad y_0 = y(x_0). \tag{1.2}$$

Here, $\Phi(\cdot, \cdot, \cdot)$ is called incremental function of the one-step method. □

Example 1.5. One-step methods, incremental functions. The explicit or forward Euler method

$$y_{k+1} = y_k + h_k f\left(x_k, y_k\right), \ k = 0, 1, 2, \ldots, \quad y_0 = y(x_0),$$

is an explicit one-step method with the incremental function

$$\Phi\left(x, y, h_k\right) = f\left(x_k, y_k\right).$$

The computation of y_{k+1} requires only the substitution of already computed values in the function $f(x, y)$ from the initial value problem (1.1).

The implicit or backward Euler method

$$y_{k+1} = y_k + h_k f\left(x_{k+1}, y_{k+1}\right), \ k = 0, 1, 2, \ldots, \quad y_0 = y(x_0),$$

is an implicit one-step method with the incremental function

$$\Phi\left(x, y, h_k\right) = f\left(x_{k+1}, y_{k+1}\right).$$

One has to solve an equation for computing y_{k+1}. The complexity of this step depends on $f(x, y)$. □

Remark 1.6. Representation of implicit one-step methods. Explicit one-step methods require only that known values are inserted in the incremental function. Hence, their incremental function can be written finally in the form $\Phi\left(x, y, h_k\right) = \Phi\left(x_k, y_k, h_k\right)$. For the considerations in this section, one can adopt the point of view that also implicit one-step methods can be written as explicit one-step methods, because the data for the nonlinear equation are x_k, y_k, and h_k. However, generally one does not know the concrete form of the incremental function. □

Example 1.7. Incremental function of the implicit Euler method. The incremental function of the implicit Euler method on an equidistant grid can be written in the form

$$\Phi\left(x, y, h\right) = f\left(x + h, y + h\Phi\left(x, y, h\right)\right),$$

which allows formally the representation of this method as explicit one-step scheme. □

Definition 1.8. Local error. Let $\hat{y}_{k+1}$ be the result of one step of an explicit one-step method (1.2) with the initial value $y(x_k)$, i.e.,

$$\hat{y}_{k+1} = y(x_k) + h_k \Phi\left(x_k, y(x_k), h_k\right).$$

Then,

$$\begin{aligned} \mathrm{le}\left(x_{k+1}\right) = \mathrm{le}_{k+1} &= y\left(x_{k+1}\right) - \hat{y}_{k+1} \\ &= y\left(x_{k+1}\right) - \left(y(x_k) + h_k \Phi\left(x_k, y(x_k), h_k\right)\right) \end{aligned} \tag{1.3}$$

is called local error, see Figure 1.2 □

Remark 1.9. The local error. In the literature, sometimes

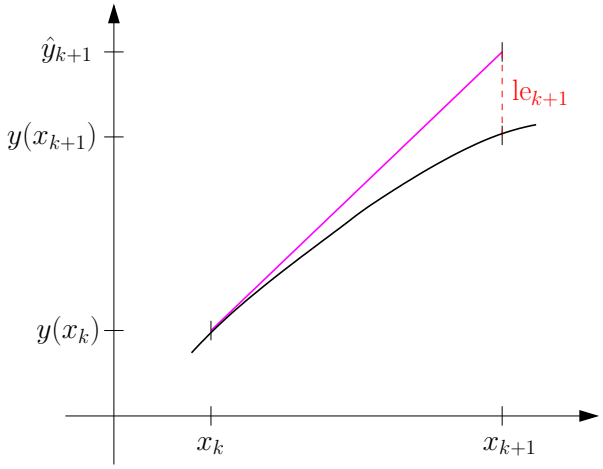

Fig. 1.2 The local error.

$$\frac{y\left(x_{k+1}\right)-y(x_k)}{h_k}-\Phi\left(x_k,y(x_k),h_k\right)$$

is defined to be the local error.

For the local error, one starts from the solution of the initial value problem and considers the error after one step of the numerical method.

One should require for a reasonable method that the local error is small in an appropriate sense. □

Definition 1.10. Consistent method. Let $y(x)$ be the solution of the initial value problem (1.1), $h_{\max} = \max_k h_k$, and

$$S := \{(x,y) \ : \ x \in [x_0, x_e], \ y \in \mathbb{R}\}.$$

The one-step method (1.2) is said to be consistent, if for all $f \in C(S)$, which satisfy in S a Lipschitz condition with respect to y, it holds

$$\lim_{h_{\max}\to 0}\left(\max_{x_k\in I_h}\frac{|\mathrm{le}\left(x_{k+1}\right)|}{h_k}\right)=0$$

or

$$\lim_{h_{\max}\to 0}\left(\max_{x_k\in I_h}\left|f(x_k,y(x_k))-\Phi\left(x_k,y(x_k),h_k\right)\right|\right)=0. \tag{1.4}$$

Both conditions are equivalent, compare Remark 1.11. □

Remark 1.11. Approximation of the derivative with the incremental function. For bounded incremental functions, it is obvious that the local error converges to zero if $h_{\max} \to 0$, because in this case it holds $h_k \to 0$ and $y(x_{k+1}) \to y(x_k)$, such that this statement follows from (1.3). Consistency requires more, namely that the incremental function approximates the derivative of the solution sufficiently well. Applying (1.3) and (1.1) yields

$$\begin{aligned}\frac{\mathrm{le}\left(x_{k+1}\right)}{h_k} &= \frac{y\left(x_{k+1}\right)-y(x_k)}{h_k}-\Phi\left(x_k,y(x_k),h_k\right)\\ &\approx y'(x_k)-\Phi\left(x_k,y(x_k),h_k\right)\\ &= f\left(x_k,y_k\right)-\Phi\left(x_k,y(x_k),h_k\right),\end{aligned}$$

compare (1.4). □

Example 1.12. Consistency of the explicit Euler method. For the explicit Euler method, it is $\Phi(x_k, y(x_k), h_k) = f(x_k, y(x_k))$. Hence, condition (1.4) from Definition 1.10 is satisfied and the method is consistent. □

Remark 1.13. Quality of the approximation of the incremental function. For practical purposes, not only the consistency itself but the quality of the approximation of the derivative by the incremental function is essential. The quality allows a comparison of different one-step methods. For simplicity of presentation, let $h_k = h$ for all k. □

Definition 1.14. Order of consistency. A one-step method (1.2) has the consistency order $p \in \mathbb{N}$, if p is the largest natural number such that for all functions $f \in C(S)$, which satisfy a Lipschitz condition with respect to y, it holds

$$|\mathrm{le}(x_k + h)| \leq Ch^{p+1}$$

for all $x_k \in I_h$, for all I_h with $h \in (0, H]$, and with the constant $C > 0$ being independent of h. The constant C might depend on derivatives of $y(x)$, on $f(x, y)$, and on partial derivatives of $f(x, y)$. □

Example 1.15. Order of consistency of the explicit Euler method. Consider the explicit Euler method and assume that the function $y(x)$ is two times continously differentiable. Then, it follows with Taylor series expansion and using the differential equation that

$$\begin{aligned}
|\mathrm{le}(x_k + h)| &= |y(x_k + h) - \hat{y}_{k+1}| \\
&= |y(x_k) + hy'(x_k) + \frac{h^2}{2} y''(x_k + \theta h) - y(x_k) - h \underbrace{f(x_k, y(x_k))}_{=y'(x_k)}| \\
&= \frac{h^2}{2} |y''(x_k + \theta h)| \leq \frac{h^2}{2} \|y\|_{C^2([x_0, x_e])},
\end{aligned}$$

with $\theta \in (0, 1)$. Since there is no way to replace the term on the right-hand side by a term with a larger power of h, the method has consistency order 1. □

Remark 1.16. Consistency and convergence. The consistency is a local property of a one-step method. For practical purposes, it is important that the computed solution converges to the analytic solution if the grid becomes finer and finer. Of course, the order of convergence is of importance, too.

It will be shown that, under certain conditions, the convergence of a one-step method follows from its consistency and that the order of convergence equals the consistency order. □

Definition 1.17. Convergent method, order of convergence. A one-step method (1.2) converges for the initial value problem (1.1) on the interval $I = [x_0, x_e]$, if for each sequence of grids $\{I_h\}$ with $h_{\max} = \max_{h_k} h_k \to 0$ for the global error

$$e(x_k, h) = y(x_k) - y_k, \quad x_k \in I_h,$$

it follows that

$$\max_{x_k \in I_h} |e(x_k, h)| \to 0 \quad \text{for} \quad h_{\max} \to 0.$$

The one-step method has the order of convergence p^*, if p^* is the largest natural number such that for all step lengths $h_{\max} \in (0, H]$, for some $H > 0$, it holds

$$|e(x_k, h)| \leq Ch_{\max}^{p^*} \quad \forall\, x_k \in I_h,$$

where $C > 0$ is independent of $h_{\max}$. □