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Numerical Mathematics II

Exercise Problems 04

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *Embedded Runge–Kutta scheme with two stages.* Derive the embedded explicit Runge–Kutta scheme $p(q) = 1(2)$ with two stages and the condition $a_{21} = b_1$, i.e., write the Butcher tableau in terms of c_2. Which schemes are obtained in the special case $c_2 = 1$? **3 points**

2. *Convergence of damped Jacobi method.* Proof the following statement: If the Jacobi method converges for each initial iterate, then also the damped Jacobi method with $0 < \omega \le 1$ converges for each initial iterate. **3 points**

3. *Optimal relaxation parameter for SOR method.* Continue Problem 3, Exercise sheet 03. One can show that the optimal relaxation parameter for the SOR method is
$$\omega_{\text{opt}} = \frac{2}{1 + \sqrt{1 - \rho^2}},$$
where ρ is the spectral radius of the iteration matrix of the Jacobi method. In the considered example, one finds that
$$\omega_{\text{opt}} = \frac{2}{1 + \sin(\pi h)}.$$
Use this relaxation parameter in the code from Problem 3, Exercise sheet 03. How does the optimal parameter behave if h decreases? Compare the number of iterations with the numbers obtained for the other relaxation parameters from the solution of Problem 3, Exercise sheet 03. What can be observed? **4 points**

4. *Optimal damping parameter for Richardson iteration with s.p.d. matrix.* Let $A \in \mathbb{R}^{n \times n}$ be a s.p.d. matrix. Consider the Richardson iteration (in fixed-point) form
$$\underline{x}^{(k+1)} = (I - \alpha A)\,\underline{x}^{(k)} + \alpha \underline{b},$$
with $\alpha \in \mathbb{R}$, for the iterative solution of $A\underline{x} = \underline{b}$. Compute the damping factor α which is optimal in the sense that it minimizes the spectral radius of the iteration matrix and show that the iteration converges for any initial iterate with this parameter. **4 points**

The exercise problems should be solved in groups of four students. The solutions have to be submitted until **Monday, Nov. 11, 2024, 10:00 a.m.** via the whiteboard.