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Numerical Mathematics II

Exercise Problems 05

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *Trapezoidal rule.* Consider the initial value problem

$$y'(x) = f(x,y), \quad y(x_0) = y_0.$$

Derive a formula for one step of the trapezoidal rule for solving this initial value problem from the Butcher tableau of the trapezoidal rule. **2 points**

2. *Properties of the matrix obtained with the finite difference discretization.* Continue Problem 3 from Exercise sheet 01. The matrix that has to be assembled in this problem has the form

$$A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & & \cdots & & 0 \\ -1 & 2 & -1 & 0 & & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & & 0 & -1 & 2 & -1 \\ 0 & & \cdots & & 0 & -1 & 2 \end{pmatrix} \in \mathbb{R}^{(n-1)\times(n-1)}.$$

(a) Show with the help of the Definition 2.10 that this matrix is positive definite. **3 points**

(b) Show that the eigenvalues of this matrix are

$$\lambda_k = \frac{4}{h^2}\sin^2\left(\frac{k\pi}{2n}\right), \quad k = 1,\ldots,n-1, \tag{1}$$

and the corresponding eigenvectors are $\underline{v}_k = (v_{k,1}, v_{k,2},\ldots,v_{k,n-1})^T$ with

$$v_{k,j} = \sin\left(\frac{jk\pi}{n}\right), \quad k,j = 1,\ldots,n-1.$$

3 points

3. *Richarson iteration.* Continue Problem 3 from Exercise sheet 03. Solve the system now with the Richardson iteration. Use the information from Problem 2 and from the lecture notes to find a suitable damping parameter (apply a safety factor of 0.9 to obtain a strictly ‘lower than’ relation). Perform at most 100 000 iterations. How does the damping parameter and the number of iterations behave if the mesh width varies? **4 points**

The exercise problems should be solved in groups of four to five students. The solutions have to be submitted until **Monday, Nov. 21, 2022, 10:00 a.m.** via the whiteboard.