

Numerical Mathematics II

Exercise Problems 01

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *Forward Euler method.* Consider the initial value problem

$$(1+x)y'(x) + y(x) = \frac{1}{1+x}, \quad y(0) = 1.$$

- (a) Compute an approximation of the solution with the forward Euler method in $[0, 1]$ with the step lengths $h_1 = 0.2$ and $h_2 = 0.1$.
Hint: write a short code.

- (b) Compute the error to the analytical solution

$$y(x) = \frac{\ln(x+1) + 1}{1+x}$$

at $x = 1$.

- (c) Discuss the results briefly.

4 points

2. *Vector norms.* Solve the following problems.

- (a) Let $\underline{x} \in \mathbb{R}^n$. Show that

$$\lim_{p \rightarrow \infty} \|\underline{x}\|_p = \|\underline{x}\|_\infty.$$

- (b) Show that the Euclidean vector norm and the Frobenius matrix norm are compatible, i.e.,

$$\|A\underline{x}\|_2 \leq \|A\|_F \|\underline{x}\|_2 \quad \forall A \in \mathbb{R}^{m \times n}, \underline{x} \in \mathbb{R}^n.$$

4 points

3. *Finite Difference matrix for the second order derivative.* Consider the differential equation (Poisson equation, boundary value problem)

$$\begin{aligned} -u'' &= f && \text{in } (0, 1), \\ u(0) &= a, \\ u(1) &= b. \end{aligned}$$

Use an equidistant grid

$$0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1, \quad h = x_i - x_{i-1}, \quad i = 1, \dots, n,$$

for the discretization of the second derivative.

- (a) Show that the approximation (finite difference)

$$u''(x_i) \approx \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1}))}{h^2} =: u_{xx,i}, \quad i = 1, \dots, n-1$$

is of second order consistent, i.e.,

$$u_{xx,i} = u''(x_i) + \mathcal{O}(h^2)$$

if $u \in C^4([0, 1])$.

- (b) Insert the approximation of the second order derivative in the differential equation and derive a linear system of equations for the values $u_i = u(x_i)$, $i = 0, \dots, n$. Set the boundary conditions and use as approximation for the right-hand side $f_i = f(x_i)$, $i = 1, \dots, n-1$.
- (c) Reduce this linear system of equations to a system for u_i , $i = 1, \dots, n-1$, by eliminating the equations for the boundary conditions.
- (d) Given n . Write a code that computes the matrix of the reduced system. If the programming language supports a `sparse` format, then store this matrix in this format.

This matrix is needed for further exercise problems! 4 points

The exercise problems should be solved in groups of four students. The solutions have to be submitted until **Monday, Oct. 24, 2022, 10:00 a.m.** via the whiteboard.