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Numerical Mathematics II

Exercise Problems 04

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. Consider the course on Numerical Methods for ODEs, Lemma 2.15.
   (a) Write the proof of part i) in detail for $n = 2, 3$.
   (b) Prove part iv).

2. Continue Problem 4 from Exercise Problems 03. One can show that the optimal relaxation parameter for the SOR method is
$$\omega_{\text{opt}} = \frac{2}{1+\sqrt{1-\rho^2}},$$
where ρ is the spectral radius of the iteration matrix of the Jacobi method. In the considered example, one finds that
$$\omega_{\text{opt}} = \frac{2}{1+\sin(\pi h)}.$$
Use this relaxation parameter in the code from Problem 4 from Exercise Problems 03. How does the optimal parameter behave if h decreases? Compare the number of iterations with the numbers obtained for the other relaxation parameters. What can be observed?

3. Continue Problem 2 from Exercise Problems 01. The matrix that has to be assembled in this problem has the form
$$A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & & \cdots & & 0 \\ -1 & 2 & -1 & 0 & & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & & 0 & -1 & 2 & -1 \\ 0 & & \cdots & & 0 & -1 & 2 \end{pmatrix} \in \mathbb{R}^{(n-1)\times(n-1)}.$$
   (a) Show with the help of the Definition 2.10 that this matrix is positive definite.
   (b) Show that the eigenvalues of this matrix are
$$\lambda_k = \frac{4}{h^2}\sin^2\left(\frac{k\pi}{2n}\right), \quad k = 1,\ldots,n-1, \tag{1}$$
and the corresponding eigenvectors are $\mathbf{v}_k = (v_{k,1}, v_{k,2}, \ldots, v_{k,n-1})^T$ with
$$v_{k,j} = \sin\left(\frac{jk\pi}{n}\right), \quad k,j = 1,\ldots,n-1.$$

(c) Continue Problem 4 from Exercise Problems 03. Solve the system now with the Richardson method. Use the information from (1) and from the lecture notes to find a suitable damping parameter (apply a safety factor of 0.9 to obtain a strictly lower than relation). Perform at most 100 000 iterations. How does the damping parameter and the number of iterations behave if the mesh width varies?

The exercise problems should be solved in groups of two students. The written parts have to be submitted until **Tuesday, Nov. 20, 2012** either before one of the lectures or directly at the office of Mrs. Hardering. The executable codes have to be send by email to Mrs. Hardering.