1. Solve the following problems.

(a) Let $G_{GS}$ be the iteration matrix of the Gauss–Seidel method. Show that
\[ G_{GS} = -D^{-1} (LG_{GS} + U). \]

(b) Consider the SOR method and verify the following identities
\[
D + \omega L = \left(1 - \frac{\omega}{2}\right) D + \frac{\omega}{2} A + \frac{\omega}{2} (L - U),
\]
\[
(1 - \omega) D - \omega U = \left(1 - \frac{\omega}{2}\right) D - \frac{\omega}{2} A + \frac{\omega}{2} (L - U).
\]

2. Consider Problem 2 from the Exercise Problems 01.

(a) Solve the boundary value problem given there for
\[ f(x) = -6\pi \cos(3\pi x) + 9\pi^2 x \sin(3\pi x), \]
and $a = b = 0$.

(b) Solve this problem numerically using the discretization described in Exercise Problems 01 with $h \in \{1/8, 1/16, 1/32, 1/64, 1/128, 1/256\}$. One can use the backslash command in MATLAB. Give the error of the computed solution $u_h$ to the analytic solution $u$ in the following norm
\[ \|u - u_h\|_2 = \left( \frac{1}{N-1} \sum_{i=1}^{N-1} (u(x_i) - u_i)^2 \right)^{1/2}, \]
where $N$ is the number of nodes.

(c) Use the following ansatz of the convergence order
\[ \|u - u_h\|_2 = ch^\alpha. \]
Compute $\alpha$ by using the results on the two finest grids.

(d) Replace the direct solver of the linear system of equations by the Jacobi method. Use as starting iterate the zero vector and stop the iteration if the Euclidean norm of the residual $\|Au - f\|_2$ is less than $1e - 10$. Count the number of iterations. What can be observed?