European Conference on Computational Fluid Dynamics
ECCOMAS CFD 2006
P. Wesseling, E. Oñate and J. Périaux (Eds)
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ON FINITE ELEMENT VARIATIONAL MULTISCALE METHODS FOR INCOMPRESSIBLE TURBULENT FLOWS

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Key words: Navier–Stokes Equations, incompressible turbulent flow, variational multiscale method, finite element method

Abstract. *Two realizations of finite element variational multiscale (VMS) methods for the simulation of incompressible turbulent flows are studied. The difference between the two approaches consists in the way the spaces for the large scales and the resolved small scales are chosen. The paper addresses issues of the implementation of these methods, the treatment of the additional terms and equations in the temporal discretization, and the additional costs of these methods.*

1 INTRODUCTION

Consider the incompressible Navier–Stokes equations (NSE) modeling the flow of an incompressible fluid:

$$\begin{aligned} \mathbf{u}_t - 2\nu\nabla\cdot\mathbb{D}(\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } (0,\,T)\times\Omega, \\ \nabla\cdot\mathbf{u} &= 0, \quad \text{in } [0,\,T]\times\Omega, \\ \mathbf{u}(0,\,\mathbf{x}) &= \mathbf{u}_0, \quad \text{in } \Omega, \end{aligned} \tag{1}$$

where $\mathbf{u}(t,\,\mathbf{x})$ denotes the fluid velocity, $\mathbb{D}(\mathbf{u}) = (\nabla\mathbf{u}+\nabla\mathbf{u}^T)/2$ is the velocity deformation tensor, $p(t,\,\mathbf{x})$ the fluid pressure, T the simulation time, and $\Omega \subset \mathbb{R}^3$ a bounded spatial domain. The parameter $\nu > 0$ is the kinematic viscosity, and $\mathbf{f}(t,\,\mathbf{x})$ the external body forces (e.g. gravity). The initial velocity field $\mathbf{u}_0$ is assumed to be divergence–free. For simplicity of presentation, it is assumed that

$$\mathbf{u} = \mathbf{0}, \quad \text{on } [0,\,T]\times\partial\Omega.$$

Mathematically, whether the flow is laminar or turbulent depends on the Reynolds number associated with the flow, $Re = \mathcal{O}(\nu^{-1})$, with turbulent flows occurring for high Reynolds numbers. Physically, turbulent flows are highly unsteady, possessing a multitude of different sizes for the flow structures (eddies, scales). Due to this broad range of scales, simulating high or even moderately high Reynolds number flows by a direct numerical simulation (DNS) is in most of the cases not feasible with the currently available computer hardware, as a DNS attempts a complete resolution of all flow scales with no unresolved scales left over. Hence there is the necessity of using a turbulence model, namely modeling the effect of the (small) unresolved scales onto the resolved ones.

A popular approach of turbulence modeling is the large eddy simulation (LES).[1,12,18] The strategy of LES consists in the decomposition of the turbulent flow into large scales and small scales, simulating only the large ones. In classical LES, a filter is used to separate the scales: the large scales are defined by an average in space through a convolution with an appropriate filter function. The derivation of equations for the large scales, however, requires the commutation of filtering and differentiation, which does not hold in bounded domains, leading to commutation errors shown to be not negligible, especially near the boundary.[2,4,12] A second unresolved problem of classical LES is, again for bounded domains, the definition of appropriate boundary conditions for the large scales. The latter problem is treated often by using some form of wall laws, in applications often physically motivated, or, alternatively, by solving simpler equations near the boundary.[17]

A promising idea to avoid the above problems of the classical LES is the separation of scale groups through variational projections into appropriate spaces, constituting the central strategy of variational multiscale methods (VMS) (or variational multiscale LES). The VMS approach is based on ideas developed in[8,9]. In VMS methods, the number of separated scale groups is equal to three, the three scales accounting for "(resolved) large scales", "resolved small scales", and "unresolved (small) scales". Assuming that the unresolved scales do not influence the large scales directly, and thus the direct influence of the unresolved scales is confined to the resolved small scales, the influence of the unresolved scales onto the resolved small scales is modeled with a turbulence model. There is no direct turbulence modeling for the large scales. However, the large scales are still influenced indirectly by the unresolved small scales due to the coupling of all three scales.

The present paper studies the advantages, drawbacks and difficulties of several possible realizations of VMS methods in the framework of finite element discretizations. The key of the realization is the definition of the finite dimensional spaces which represent the different scales.

2 THE VMS METHOD

The starting point of VMS methods is the variational formulation of the Navier–Stokes equations. Let $V = (H_0^1(\Omega))^d$ and $Q = L_0^2(\Omega)$. A variational formulation of (1) reads:

find $\mathbf{u} : [0, T] \to V$, $p : (0, T] \to Q$ satisfying

$$\begin{aligned} (\mathbf{u}_t,\, \mathbf{v}) + (2\nu\mathbb{D}(\mathbf{u}), \mathbb{D}(\mathbf{u})) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (p,\, \nabla\cdot\mathbf{v}) &= (\mathbf{f},\, \mathbf{v}),\ \forall\, \mathbf{v} \in V, \quad (2)\\ (q,\, \nabla\cdot\mathbf{u}) &= 0, \qquad \forall\, q \in Q, \quad (3) \end{aligned}$$

with $\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \in V$ and $b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = ((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w})$. The equations (2)–(3) can be written in short form as

$$A(\mathbf{u};\, (\mathbf{u},\, p),\, (\mathbf{v}, q)) = F(\mathbf{v}).$$

The form $A(\mathbf{u};\, (\mathbf{u},\, p),\, (\mathbf{v}, q))$ is linear with respect to the test functions. Decompose the ansatz and the test spaces into

$$V = \bar{V} \oplus \tilde{V} \oplus \hat{V}, \quad Q = \bar{Q} \oplus \tilde{Q} \oplus \hat{Q},$$

with $\bar{(\cdot)}$ denoting the large scales, $\tilde{(\cdot)}$, the resolved small scales, and $\hat{(\cdot)}$ the unresolved small scales. Considering also the three scale separation

$$\mathbf{u} = \bar{\mathbf{u}} + \tilde{\mathbf{u}} + \hat{\mathbf{u}}, \quad p = \bar{p} + \tilde{p} + \hat{p},$$

for the solution, and

$$\mathbf{v} = \bar{\mathbf{v}} + \tilde{\mathbf{v}} + \hat{\mathbf{v}}, \quad q = \bar{q} + \tilde{q} + \hat{q},$$

for the test functions, (2) can be written as a system of three variational equations: find $\mathbf{u} = \bar{\mathbf{u}} + \tilde{\mathbf{u}} + \hat{\mathbf{u}} : [0, T] \to V = \bar{V} \oplus \tilde{V} \oplus \hat{V}$, $p = \bar{p} + \tilde{p} + \hat{p} : (0, T] \to Q = \bar{Q} \oplus \tilde{Q} \oplus \hat{Q}$ satisfying

$$\begin{aligned} A(\mathbf{u};\, (\bar{\mathbf{u}},\, \bar{p}),\, (\bar{\mathbf{v}}, \bar{q})) + A(\mathbf{u};\, (\tilde{\mathbf{u}},\, \tilde{p}),\, (\bar{\mathbf{v}}, \bar{q})) + A(\mathbf{u};\, (\hat{\mathbf{u}},\, \hat{p}),\, (\bar{\mathbf{v}}, \bar{q})) &= F(\bar{\mathbf{v}}),\\ A(\mathbf{u};\, (\bar{\mathbf{u}},\, \bar{p}),\, (\tilde{\mathbf{v}}, \tilde{q})) + A(\mathbf{u};\, (\tilde{\mathbf{u}},\, \tilde{p}),\, (\tilde{\mathbf{v}}, \tilde{q})) + A(\mathbf{u};\, (\hat{\mathbf{u}},\, \hat{p}),\, (\tilde{\mathbf{v}}, \tilde{q})) &= F(\tilde{\mathbf{v}}),\\ A(\mathbf{u};\, (\bar{\mathbf{u}},\, \bar{p}),\, (\hat{\mathbf{v}}, \hat{q})) + A(\mathbf{u};\, (\tilde{\mathbf{u}},\, \tilde{p}),\, (\hat{\mathbf{v}}, \hat{q})) + A(\mathbf{u};\, (\hat{\mathbf{u}},\, \hat{p}),\, (\hat{\mathbf{v}}, \hat{q})) &= F(\hat{\mathbf{v}}), \end{aligned}$$

for all $(\mathbf{v},\, q) \in V \times Q$.

As it is not intended to explicitly resolve the quantities termed as "unresolved", the equation for the unresolved scales is neglected. As already discussed, it is assumed that the unresolved scales do not influence the large scales directly and thus

$$A(\mathbf{u};\, (\hat{\mathbf{u}},\, \hat{p}),\, (\bar{\mathbf{v}}, \bar{q})) = 0. \quad (4)$$

The term $A(\mathbf{u};\, (\hat{\mathbf{u}},\, \hat{p}),\, (\tilde{\mathbf{v}}, \tilde{q}))$ may be identified as the influence of the unresolved scales onto the resolved small scales.

Because of (4), the influence of the unresolved scales onto the resolved small scales is the only one that needs to be modeled

$$A(\mathbf{u};\, (\hat{\mathbf{u}},\, \hat{p}),\, (\tilde{\mathbf{v}}, \tilde{q})) \approx B(\mathbf{u};\, (\bar{\mathbf{u}},\, \bar{p}),\, (\tilde{\mathbf{u}},\, \tilde{p}),\, (\tilde{\mathbf{v}}, \tilde{q})). \quad (5)$$

From the numerical point of view, the model introduces additional viscosity which acts as stabilization. Note that the reason for introducing a model term here is different than in the classical LES - in the classical LES, the model term appears due to the modeling of the Reynolds stress tensor.

As a result of neglecting the unresolved scale equation and of considering the modeled form of the small scale equation, the new system reads: find $\bar{\mathbf{u}} + \tilde{\mathbf{u}} : [0, T] \to \bar{V} \oplus \tilde{V}$, $\bar{p} + \tilde{p} : (0, T] \to \bar{Q} \oplus \tilde{Q}$ such that

$$\begin{aligned} A(\bar{\mathbf{u}}+\tilde{\mathbf{u}};\, (\bar{\mathbf{u}}, \bar{p}), (\bar{\mathbf{v}}, \bar{q})) + A(\bar{\mathbf{u}}+\tilde{\mathbf{u}};\, (\tilde{\mathbf{u}}, \tilde{p}), (\bar{\mathbf{v}}, \bar{q})) &= F(\bar{\mathbf{v}}), \\ A(\bar{\mathbf{u}}+\tilde{\mathbf{u}};\, (\bar{\mathbf{u}}, \bar{p}), (\tilde{\mathbf{v}}, \tilde{q})) + A(\bar{\mathbf{u}}+\tilde{\mathbf{u}};\, (\tilde{\mathbf{u}}, \tilde{p}), (\tilde{\mathbf{v}}, \tilde{q})) & \\ +B(\bar{\mathbf{u}}+\tilde{\mathbf{u}};\, (\bar{\mathbf{u}}, \bar{p}), (\tilde{\mathbf{u}}, \tilde{p}), (\tilde{\mathbf{v}}, \tilde{q})) &= F(\tilde{\mathbf{v}}), \end{aligned} \tag{6}$$

for all $(\bar{\mathbf{v}}, \tilde{\mathbf{v}}, \bar{q}, \tilde{q}) \in \bar{V} \times \tilde{V} \times \bar{Q} \times \tilde{Q}$, with the model $B(\bar{\mathbf{u}}+\tilde{\mathbf{u}};\, (\bar{\mathbf{u}}, \bar{p}), (\tilde{\mathbf{u}}, \tilde{p}), (\tilde{\mathbf{v}}, \tilde{q}))$ acting directly only on the resolved small scales. An indirect influence on the large scales however, is insured through the coupling of the two (non–linear) variational equations.

A widely used way of accounting for the effect of the unresolved scales in the numerical simulation of turbulent flows is the Smagorinsky model and variations of it. This model is used in general also within VMS methods. One version is the so–called "small–small" model

$$B(\mathbf{u};\, (\bar{\mathbf{u}}, \bar{p}), (\tilde{\mathbf{u}}, \tilde{p}), (\tilde{\mathbf{v}}, \tilde{q})) = (\nu_T \mathbb{D}(\tilde{\mathbf{u}}), \mathbb{D}(\tilde{\mathbf{v}})), \quad \nu_T = C_S \delta^2 \|\mathbb{D}(\tilde{\mathbf{u}})\|_F,$$

with C_S a constant, δ is a scaling factor related to the mesh width (in LES, δ is the filter width), and $\|\cdot\|_F$, the Frobenius norm of a tensor. Another version is the "large–small" model

$$B(\mathbf{u};\, (\bar{\mathbf{u}}, \bar{p}), (\tilde{\mathbf{u}}, \tilde{p}), (\tilde{\mathbf{v}}, \tilde{q})) = (C_S \delta^2 \|\mathbb{D}(\bar{\mathbf{u}})\|_F \mathbb{D}(\bar{\mathbf{u}}), \mathbb{D}(\tilde{\mathbf{v}})).$$

Instead of a constant, also a dynamical model[5,16] can be used where $C_S = C_S(t, \mathbf{x})$.

The available numerical studies show that there are neither large differences between using the "small–small" and the "large–small" model[10,11], nor between using a constant and a dynamical C_S.[7] Thus, the actual choice of the turbulence model 5) seems to be not a main issue in VMS methods. Simple models, like the Smagorinsky model, work already well.

The crucial point of a VMS method is the definition of appropriate spaces for the large scales and the resolved small scales. In the context of finite element methods (FEM), two different strategies are presented in the following sections: in the bubble VMS methods described in Section 3, standard finite element spaces are taken for the large scale spaces $\bar{V} \times \bar{Q}$, while mesh cell bubble functions are used for the resolved small scale spaces $\tilde{V} \times \tilde{Q}$. The advantage of this strategy is that the resolved small scales equation in (6) is localized to the mesh cells. However, since the bubble functions vanish on the faces of the mesh cells, the resolved small scales are restricted only to the interior of the mesh cells without crossing the mesh cell boundaries, a property which does not reflect the physical behavior of the resolved small scales. A second strategy, without the above restriction, are coarse space projection–based VMS methods, presented in Section 4: standard finite element spaces are chosen for all resolved scales with an additional large scale space. The large scales are given by a projection into this large scale space. The separation of scales is still observable due to the turbulent viscosity modeling the effect of the unresolved small scales and still acting directly only on the resolved small scales, which in turn, are no longer restricted to the mesh cells.

3 BUBBLE VMS METHODS

This section follows the presentation in[6]. Bubble VMS methods are based directly on the form (6) of the coupled equations. Having chosen a pair of finite element spaces $(\bar{V}^h, \bar{Q}^h)$ for the large scales, a higher resolution is needed for the spaces $(\tilde{V}, \tilde{Q})$ of the small scales. This can be achieved in two ways: by introducing higher order finite elements or a refined mesh. Herein, a combination of both ways will be considered.

3.1 The residual free bubble (RFB) VMS method

In order to obtain an efficient method, the solution of the resolved small scale equation should not be too expensive. Therefore, the finite element functions are decoupled in order to obtain local problems. For localizing the solution of the small scale equation, residual free bubble (RFB) functions are used. To this extent, for the small scale part of the solution, zero Dirichlet boundary conditions are assumed at the boundaries of each mesh cell K of a fixed regular triangulation $\mathcal{T}^h$ of the domain Ω. Furthermore, the small scale bubble part of the solution solves the strong form of the Navier–Stokes equations in each mesh cell.

Choose standard finite element spaces for the large scales $(\bar{V}^h, \bar{Q}^h)$, and denote with $(\tilde{V}, \tilde{Q})$ the spaces for the resolved small scales, with bubble functions for the velocity

$$\tilde{V} = \{\mathbf{w} \ : \ \mathbf{w} \in (H_0^1(\Omega))^3, \ \mathbf{w}|_{\partial K} = 0, \text{ for all } K \in \mathcal{T}^h\},$$

with the following direct sum decomposition

$$V \times Q = (\bar{V}^h \times \bar{Q}^h) \oplus (\tilde{V} \times \tilde{Q}).$$

The space $\tilde{Q}$ will not be used in the bubble VMS approach described below. Note that $\tilde{V}$ is an infinite dimensional space. Thus, each function of V, in particular the velocity $\mathbf{u}$ corresponding to the solution of the Navier–Stokes equations (1), can be written uniquely as a linear combination of a large scale part from $\bar{V}^h$ and an RFB part from $\tilde{V}$. The idea of the RFB method consists in constructing a basis of $\tilde{V}$ such that $\mathbf{u}$ can be represented by using only the finite number of unknown coefficients needed in the representation of the large scale part $\bar{\mathbf{u}}$ of $\mathbf{u}$. If the basis of $\tilde{V}$ is known, these coefficients can be computed by solving a finite dimensional linear system of equations. However, in general, the appropriate basis of $\tilde{V}$ cannot be computed analytically and a finite element approximation of this basis will be used in practice. These finite element spaces represent the resolved small scales and the infinite dimensional remainder of $\tilde{V}$ the unresolved scales. In this way, the three scale decomposition of the velocity field is recovered.

For a given $K \in \mathcal{T}^h$, $\tilde{\mathbf{u}} \in \tilde{V}$ satisfies the strong form corresponding to the weak small scale equation in (6), which after an appropriate linearization reads:

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} + (\mathbf{u}^{\text{old}} \cdot \nabla)\tilde{\mathbf{u}} + \nabla \tilde{p} - 2(\nu + \tilde{\nu}_{T,k})\nabla \cdot \mathbb{D}(\tilde{\mathbf{u}}) = \tag{7}$$
$$= \frac{\partial \bar{\mathbf{u}}^h}{\partial t} - (\mathbf{u}^{\text{old}} \cdot \nabla)\bar{\mathbf{u}}^h - \nabla \bar{p}^h + 2\nu \nabla \cdot \mathbb{D}(\bar{\mathbf{u}}^h) + \mathbf{f}, \quad \text{in } K \times (0, T),$$

$$\nabla \cdot \tilde{\mathbf{u}} = -\nabla \cdot \bar{\mathbf{u}}^h, \quad \text{in } K \times (0, T), \tag{8}$$

$$\tilde{\mathbf{u}} = \mathbf{0}, \quad \text{on } \partial K \times (0, T), \tag{9}$$

where $\mathbf{u}^{\text{old}}$ is a currently available approximation of the velocity field, e.g., $\mathbf{u}^{\text{old}} = \mathbf{u}^{h,\text{old}}$ being the large scale part. The subgrid viscosity term on the left–hand side of equation (7) models the effect of the unresolved scales onto the resolved small scales with $\tilde{\nu}_{T,k}$ being an elementwise subgrid turbulent viscosity.

In the bubble VMS method from[6], the term $\nabla \cdot \tilde{p}$ is neglected in (7) and the small scale pressure is treated separately, see below. After having neglected $\tilde{p}$, (7) becomes an equation for $\tilde{\mathbf{u}}$ only and it takes the form

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} + (\mathbf{u}^{\text{old}} \cdot \nabla)\tilde{\mathbf{u}} - 2(\nu + \tilde{\nu}_{T,k})\nabla \cdot \mathbb{D}(\tilde{\mathbf{u}}) = \tag{10}$$
$$= -\frac{\partial \bar{\mathbf{u}}^h}{\partial t} - (\mathbf{u}^{\text{old}} \cdot \nabla)\bar{\mathbf{u}}^h - \nabla \bar{p}^h + 2\nu \nabla \cdot \mathbb{D}(\bar{\mathbf{u}}^h) + \mathbf{f}, \quad \text{in } K \times (0, T).$$

Equation (10) gives $\tilde{\mathbf{u}}$ in terms of $\bar{\mathbf{u}}^h$, $\bar{p}^h$, and $\mathbf{f}$ in each mesh cell. Together with $\tilde{p} = 0$, through a substitution into the large scale equation in (6), a closed form for $(\bar{\mathbf{u}}, \bar{p})$ is obtained, being the only equation to be solved globally, for details on the final matrix system, see[6].

Consider the time discretization

$$\frac{1}{\Delta t_k}\tilde{\mathbf{u}} + (\mathbf{u}^{\text{old}} \cdot \nabla)\tilde{\mathbf{u}} - 2(\nu + \tilde{\nu}_{T,k})\nabla \cdot \mathbb{D}(\tilde{\mathbf{u}}) = \tag{11}$$
$$= -\frac{1}{\Delta t_k}\bar{\mathbf{u}} - (\mathbf{u}^{\text{old}} \cdot \nabla)\bar{\mathbf{u}} - \nabla \bar{p} + 2\nu \nabla \cdot \mathbb{D}(\bar{\mathbf{u}}) + (\mathbf{f} + \frac{\mathbf{u}^k}{\Delta t_k}), \quad \text{in } K$$

of (10), with Δt_k the chosen time step, k the previous time level subject to $k = 0, \ldots, \frac{T}{\Delta t_k} - 1$. Next, let $\{\boldsymbol{\varphi}^h_{i,j}\}_{i=1,2,3;\, j=1,\ldots,N_u}$ and $\{\psi^h_j\}_{j=1,\ldots,N_p}$ be the basis of $\bar{V}^h \times \bar{Q}^h$, and let K be a given mesh cell. For (11), the only basis functions of importance are the ones that do not vanish on K. For simplicity of presentation, let these be the first $N_u(K)$ velocity basis functions, and, respectively, the first $N_p(K)$ pressure basis functions. Elementwise extensions for the large scale velocity and pressure part of the solution then read:

$$\bar{\mathbf{u}}^h|_K = \sum_{i=1}^{3} \sum_{j=1}^{N_u(K)} u_{i,j}\, \boldsymbol{\varphi}_{i,j}^h, \quad \bar{p}^h|_K = \sum_{j=1}^{N_p(K)} p_j\, \psi_j^h \tag{12}$$

with $u_{i,j}$ and p_j the unknown velocity, respectively pressure coefficients.

By inserting (12) into (11) and decomposing into the basic components, the following independent equations are obtained for $i = 1, 2, 3$, $j = 1, \ldots, N_u(K)$, and $l = 1, \ldots, N_p(K)$, with $\boldsymbol{\varphi}^b$ denoting a so–called velocity bubble basis function, ψ^b a pressure bubble basis function, and f^b a right–hand side bubble basis function:

$$\begin{aligned}
\Big(\frac{1}{\triangle t_k} + (\bar{\mathbf{u}}^{\mathrm{old}} \cdot \nabla)\Big)\, \boldsymbol{\varphi}_{i,j}^b - (\nu + \tilde{\nu}_{T,k})\Delta \boldsymbol{\varphi}_{i,j}^b &= -(\bar{\mathbf{u}}^{\mathrm{old}} \cdot \nabla)\boldsymbol{\varphi}_{i,j}^h + \nu \Delta \boldsymbol{\varphi}_{i,j}^h \\
&\quad - \frac{1}{\Delta t_k}\boldsymbol{\varphi}_{i,j}^h, \quad \text{in } K, \qquad (13)\\
\Big(\frac{1}{\triangle t_k} + (\bar{\mathbf{u}}^{\mathrm{old}} \cdot \nabla)\Big)\, \psi_{i,k}^b \mathbf{e}_i - (\nu + \tilde{\nu}_{T,k})\Delta \psi_{i,k}^b \mathbf{e}_i &= -\frac{\partial \psi_k}{\partial x_i}\mathbf{e}_i, \quad \text{in } K, \qquad (14)\\
\Big(\frac{1}{\triangle t_k} + (\bar{\mathbf{u}}^{\mathrm{old}} \cdot \nabla)\Big)\, f_i^b \mathbf{e}_i - (\nu + \tilde{\nu}_{T,k})\Delta f_i^b \mathbf{e}_i &= f_i \mathbf{e}_i + \frac{\mathbf{u}^k}{\Delta t_k}\mathbf{e}_i, \quad \text{in } K, \qquad (15)
\end{aligned}$$

where $\mathbf{e}_i$ denotes the unit vector in the spatial direction of i, and f_i, the i-th component of $\mathbf{f}$. All of the above $N\,(3\,(N_u(K) + N_p(K) + 1))$ equations (N being the number of mesh cells $K \in \mathcal{T}^h$) satisfy zero Dirichlet boundary conditions on the boundary of the corresponding mesh cell. Note that equations (13)–(15) have identical left–hand sides

$$\Big(\frac{1}{\triangle t_k} + (\bar{\mathbf{u}}^{\mathrm{old}} \cdot \nabla)\Big)\, \phi^b - (\nu + \tilde{\nu}_{T,k})\Delta \phi^b, \tag{16}$$

with ϕ^b standing for the residual free bubble functions $\boldsymbol{\varphi}^b$, ψ^b, and f^b.

From the linearity of (11), it follows that the solution $\tilde{\mathbf{u}}$ of (11) can be written as a linear combination of the bubble basis functions solving (13)–(15) with the same coefficients $\{u_{i,j}\}_{i=1,2,3;\,j=1,\ldots,N_u(K)}$ and $\{p_j\}_{j=1,\ldots,N_p(K)}$ as the large scale part of the solution given in (12):

$$\tilde{\mathbf{u}}|_K = \sum_{i=1}^{3} \sum_{j=1}^{N_u(K)} u_{i,j}\, \boldsymbol{\varphi}_{i,j}^b + \sum_{j=1}^{N_p(K)} p_j \sum_{i=1}^{3} \psi_{i,j}^b\, \mathbf{e}_i + \sum_{j=1}^{3} f_i^b\, \mathbf{e}_i. \tag{17}$$

Since an analytical solution of (13)–(15) is not feasible, a finite element method is applied. On each mesh cell K of the triangulation, a submesh is introduced on which the bubble basis functions are approximated and then substituted into $\tilde{\mathbf{u}}|_K$ in place of the exact bubble basis functions. For solving (13)–(15), finite element spaces of different (higher) order than $\bar{V}^h$ can be used.

In[6], it is proposed to take the effect of the small scale continuity equation and the small scale pressure into account via a stabilizing term in the form of a so–called "bulk–viscosity" term. This is achieved by the approximation

$$\tilde{p} \approx \tau_C (\nabla \cdot \bar{\mathbf{u}}^h),$$

with $\{\tau_C\}$ a family of stabilization parameters which are defined elementwise. The stabilizing term in the large scale equation takes the form

$$\sum_{K \in \mathcal{T}^h} \tau_C (\nabla \cdot \bar{\mathbf{u}}^h, \nabla \cdot \bar{\mathbf{v}}^h).$$

Such terms are well known in the stabilization of convection dominated discrete Navier–Stokes equations, the so–called div–div stabilization, e.g., see[3].

3.2 Additional costs of the RFB VMS method

To summarize, the bubble VMS method consists of two steps: one at the global mesh level, while the bubble part of the solution is unknown, and the other at the submesh level. In order to compute the bubble part of the solution, the decomposition (13)–(15) is used. The bubble basis functions are calculated approximately on a submesh defined in each global mesh cell.

From the point of view of the time discretization, there are two options. First, an implicit approach solves (6) iteratively, thereby switching forth and back between these two equations. An explicit treatment of the small scale equation solves this equation only once at the beginning in each discrete time t_k, using the velocity field from the previous time. After having computed the solution of the small scale equation, it is inserted into the large scale equation. The large scale equation is solved without updating the small scale solution in the present discrete time again.

The additional costs which are introduced by the bubble VMS method consist in solving the local equations (13)–(15) with the same left–hand side (16) on each mesh cell independently and in the assembling of the right–hand side of the large scale equation. This has to be done once in each discrete time if the small scale equation is treated explicitly and in general more than once for an implicit treatment of the small scale equation. The left–hand side (16) is different in each discrete time.

3.3 Alternative bubble VMS methods

An alternative strategy to the RFB approach consists in considering the variational form of (7)–(9) from the beginning in a finite dimensional (bubble) space $\tilde{V}^h \times \tilde{Q}^h$ and solving the coupled system. This approach is the subject of ongoing research.

4 COARSE SPACES PROJECTION–BASED VMS METHODS

In projection–based VMS methods, the spaces for the large scales and the resolved small scales are chosen in a different way than in bubble VMS methods: a standard finite element space models all resolved scales, with an additional space for the large scales. The large scales are defined by an appropriate projection into this space. In this section, different choices for the large scale spaces will be studied and the efficiency of the fully implicit and the semi–implicit projection–based VMS method will be discussed.

4.1 The general approach

Choose a standard pair of conforming finite element spaces $V^h \times Q^h$ for the velocity and pressure which fulfill the inf–sup condition. In addition, let L^H be a finite dimensional space of symmetric $d \times d$ tensor–valued functions representing a coarse or large scale space, and a non–negative function $\nu_T((\mathbf{u}^h, p^h), h)$, depending on the mesh size and the solution, representing the turbulent viscosity. The semi–discrete coarse space projection–based VMS method with parameters ν_T and L^H then seeks $\mathbf{u}^h : [0, T] \to V^h$, $p^h : (0, T] \to Q^h$, and $\mathbb{G}^H : [0, T] \to L^H$ such that

$$\begin{array}{rcll} (\mathbf{u}_t^h, \mathbf{v}^h) + (2\nu\mathbb{D}(\mathbf{u}^h), \mathbb{D}(\mathbf{v}^h)) + ((\mathbf{u}^h \cdot \nabla)\mathbf{u}^h, \mathbf{v}^h) & & & \\ -(p^h, \nabla \cdot \mathbf{v}^h) + (\nu_T(\mathbb{D}(\mathbf{u}^h) - \mathbb{G}^H), \mathbb{D}(\mathbf{v}^h)) & = & (\mathbf{f}, \mathbf{v}^h), & \forall \mathbf{v}^h \in V^h, \\ (q^h, \nabla \cdot \mathbf{u}^h) & = & 0, & \forall q^h \in Q^h, \\ (\mathbb{G}^H - \mathbb{D}(\mathbf{u}^h), \mathbb{L}^H) & = & 0, & \forall \mathbb{L}^H \in L^H. \end{array} \qquad (18)$$

From the last equation in (18), it follows that $\mathbb{G}^H$ is the L^2–projection of $\mathbb{D}(\mathbf{u}^h)$ into the large scale space L^H, and thus $\mathbb{G}^H$ represents the large scales of $\mathbb{D}(\mathbf{u}^h)$. Consequently, $\mathbb{D}(\mathbf{u}^h) - \mathbb{G}^H$ represents the resolved small scales. Therefore, the additional term introduced by the projection–based VMS methods in the momentum equation $(\nu_T(\mathbb{D}(\mathbf{u}^h) - \mathbb{G}^H), \mathbb{D}(\mathbf{v}^h))$ is a viscous term acting directly only on the resolved small scales. The main feature of a VMS method is hence recovered again. Moreover, in[13,14], it is shown that (18) can be transformed into the form (6) of a VMS method.

A crucial point for the efficiency of a projection–based VMS method of the form (18) is the choice of the large scale space L^H. It is first demanded that $L^H \subseteq \{\mathbb{D}(\mathbf{v}^h) \ : \ \mathbf{v}^h \in V^h\}$. At the same time, since it has been distinguished between small scales and large scales, with L^H representing the large scales, L^H must be in some sense a coarse finite element space. Depending on $V^h \times Q^h$, there are two possibilities:

- On the one hand, if $V^h \times Q^h$ are higher order finite element spaces, one can choose for L^H a lower order finite element space on the same grid as $V^h \times Q^h$. This approach leads to the so–called one–level projection–based VMS method.
- On the other hand, in the case of low order finite elements for $V^h \times Q^h$, the large scale space L^H can be defined on a coarser grid, leading to the two–level projection–based VMS method.

The first strategy was discussed in[14], whereas the second strategy was studied in[15] for convection–dominated convection–diffusion equations.

Define L^H on some grid, either on the fine one or a coarser one, and equip it with a basis of tensor–valued functions:

$$L^H = \mathrm{span}\left\{ \begin{pmatrix} l_j^H & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & l_j^H & 0 \\ l_j^H & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 0 & l_j^H \\ 0 & 0 & 0 \\ l_j^H & 0 & 0 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & l_j^H & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & l_j^H \\ 0 & l_j^H & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & l_j^H \end{pmatrix} \; : \; j = 1, \ldots, n_L \right\}.$$

In the following, implementational issues of projection–based VMS methods will be studied, for an implicit and explicit treatment of the projection, respectively.

4.2 The fully implicit projection–based VMS method

After an implicit discretization in time using a θ–scheme, e.g., the Crank–Nicolson scheme, the fully discrete equation in the discrete time t_k reads:

$$\begin{aligned}
&(\mathbf{u}_k^h, \mathbf{v}^h) + \theta_1 \Delta t_k \Big[((2\nu + \nu_{T,k})\mathbb{D}(\mathbf{u}_k^h), \mathbb{D}(\mathbf{v}^h)) + ((\mathbf{u}_k^h \cdot \nabla)\mathbf{u}_k^h, \mathbf{v}^h) - (\nu_{T,k}\mathbb{G}_k^H, \mathbb{D}(\mathbf{v}^h)) \Big] \\
&\qquad -(p_k, \nabla \cdot \mathbf{v}^h) \qquad\qquad (19)\\
&\quad = (\mathbf{u}_{k-1}^h, \mathbf{v}^h) + \theta_2 \Delta t_k \Big[((2\nu + \nu_{T,k-1})\mathbb{D}(\mathbf{u}_{k-1}^h), \mathbb{D}(\mathbf{v}^h)) + ((\mathbf{u}_{k-1}^h \cdot \nabla)\mathbf{u}_{k-1}^h, \mathbf{v}^h) \\
&\qquad -(\nu_{T,k-1}\mathbb{G}_{k-1}^H, \mathbb{D}(\mathbf{v}^h)) \Big] + \theta_3 \Delta t_k (\mathbf{f}_{k-1}, \mathbf{v}^h) + \theta_4 \Delta t_k (\mathbf{f}_k, \mathbf{v}^h), \quad \forall \mathbf{v}^h \in V^h, \\
&0 = (q^h, \nabla \cdot \mathbf{u}_k^h), \quad \forall q^h \in Q^h, \\
&0 = (\mathbb{G}_k^H - \mathbb{D}(\mathbf{u}_k^h), \mathbb{L}^H), \quad \forall \mathbb{L}^H \in L^H,
\end{aligned}$$

with $\Delta t_k = t_k - t_{k-1}$. With an appropriate linearization of the convective term in the current time step, one obtains the algebraic representation of (19) as a large coupled system of equations of the form:

$$\begin{pmatrix}
A_{11} & A_{12} & A_{13} & B_1^T & \tilde{G}_{11} & \tilde{G}_{12} & \tilde{G}_{13} & 0 & 0 & 0 \\
A_{21} & A_{22} & A_{23} & B_2^T & 0 & \tilde{G}_{22} & 0 & \tilde{G}_{24} & \tilde{G}_{25} & 0 \\
A_{31} & A_{32} & A_{33} & B_3^T & 0 & 0 & \tilde{G}_{33} & 0 & \tilde{G}_{35} & \tilde{G}_{36} \\
B_1 & B_2 & B_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
G_{11} & 0 & 0 & 0 & M & 0 & 0 & 0 & 0 & 0 \\
G_{21} & G_{22} & 0 & 0 & 0 & \frac{M}{2} & 0 & 0 & 0 & 0 \\
G_{31} & 0 & G_{33} & 0 & 0 & 0 & \frac{M}{2} & 0 & 0 & 0 \\
0 & G_{42} & 0 & 0 & 0 & 0 & 0 & M & 0 & 0 \\
0 & G_{52} & G_{53} & 0 & 0 & 0 & 0 & 0 & \frac{M}{2} & 0 \\
0 & 0 & G_{63} & 0 & 0 & 0 & 0 & 0 & 0 & M
\end{pmatrix}
\begin{pmatrix} u_1^h \\ u_2^h \\ u_2^h \\ p^h \\ g_{11}^H \\ g_{12}^H \\ g_{13}^H \\ g_{22}^H \\ g_{23}^H \\ g_{33}^H \end{pmatrix}
=
\begin{pmatrix} f_1^h \\ f_2^h \\ f_3^h \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad (20)$$

where

$$\mathbb{G}_k^H = \begin{pmatrix} g_{11}^H & g_{12}^H & g_{13}^H \\ g_{12}^H & g_{22}^H & g_{23}^H \\ g_{13}^H & g_{23}^H & g_{33}^H \end{pmatrix},$$

M is the mass matrix of L^H: $M_{ij} = (l_j^H, l_i^H)$ and the vector representation of the solution and the right–hand side is used. The entries of G_{ij} and $\tilde{G}_{ij}$ can be computed using the bases of V^h and L^H (see[14]) and one obtains

$$\begin{array}{lll} G_{21} = G_{53} = \frac{1}{2}G_{42}, & G_{22} = G_{33} = \frac{1}{2}G_{11}, & G_{31} = G_{52} = \frac{1}{2}G_{63}, \\ \tilde{G}_{12} = \tilde{G}_{35} = \frac{1}{2}\tilde{G}_{24}, & \tilde{G}_{13} = \tilde{G}_{25} = \frac{1}{2}\tilde{G}_{36}, & \tilde{G}_{22} = \tilde{G}_{33} = \frac{1}{2}\tilde{G}_{11}, \end{array}$$

leaving, in comparison to the Navier–Stokes equations, altogether seven additional sparse matrices. The matrices G_{ij} and M for the L^2-projection need to be assembled only once if the spaces V^h and L^H are time–independent. Since the matrices $\tilde{G}_{ij}$ depend on ν_T, which in turn depends in general on $\mathbf{u}^h$, $\tilde{G}_{ij}$ need to be assembled each time $\mathbf{u}^h$ changes, i.e., at each discrete time t_k in each step of the iteration solving the non–linearity of (19). To solve the coupled system (20) has been found to be an inefficient approach (see[15] for the case of scalar convection–diffusion equations). A more efficient strategy consists in eliminating the equations describing the L^2–projection leading to a saddle point problem for $(\mathbf{u}^h, p^h)$ of the form

$$\begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} & B_1^T \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & B_2^T \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{13} & B_3^T \\ B_1 & B_2 & B_3 & 0 \end{pmatrix} \begin{pmatrix} u_1^h \\ u_2^h \\ u_3^h \\ p^h \end{pmatrix} = \begin{pmatrix} f_1^h \\ f_2^h \\ f_3^h \\ 0 \end{pmatrix}, \tag{21}$$

with

$$\begin{aligned} \tilde{A}_{11} &= A_{11} - \tilde{G}_{11}M^{-1}G_{11} - \frac{1}{2}\tilde{G}_{24}M^{-1}G_{42} - \frac{1}{2}\tilde{G}_{36}M^{-1}G_{63}, \\ &\vdots \\ \tilde{A}_{33} &= A_{33} - \tilde{G}_{36}M^{-1}G_{63} - \frac{1}{2}\tilde{G}_{11}M^{-1}G_{11} - \frac{1}{2}\tilde{G}_{24}M^{-1}G_{42}. \end{aligned} \tag{22}$$

The sparsity pattern of the matrices A_{ij} is determined by the standard finite element space V^h, and from (22), the sparsity pattern of $\tilde{A}_{ij}$ is in general at least the sparsity pattern of A_{ij}. In order to obtain an efficient method, the sparsity pattern of $\tilde{A}_{ij}$ should not be larger than the one of A_{ij}. Otherwise, matrix–vector products become more expensive and it is unclear whether iterative methods for solving (21) still work well. Moreover, in many codes, it comes to internal difficulties when using non–standard sparsity patterns. In[14], two conditions have been found to ensure the same sparsity pattern of A_{ij} and $\tilde{A}_{ij}$:

- First, the support of a basis function of L^H should only be one mesh cell on the finest grid.
- Second, the basis of L^H should be L^2–orthogonal, which ensures that M is a diagonal matrix and thus M^{-1} can be computed easily.

The crucial condition is the first one as it can be fulfilled only if L^H is defined on the fine grid, e.g., by discontinuous finite element spaces. When L^H is defined on a coarser grid than V^h, i.e., in the two–level approach, the first of the above two conditions cannot be fulfilled. Even if on the coarse grid, for a basis function of L^H the support is only one mesh cell, it will be several mesh cells on the finer grids. Thus, an efficient implementation of the fully implicit two–level projection–based VMS seems not to be possible.

To summarize, the fully implicit projection–based VMS method might be efficient only if L^H is defined on the finest grid, with the properties given above. The treatment of the term and the equation introduced by the VMS method requires the following operations:

- The matrices M, G_{11}, G_{42}, and G_{63} have to be assembled only once in a pre–processing step at the initial time.
- The matrices $\tilde{G}_{11}$, $\tilde{G}_{24}$, and $\tilde{G}_{36}$ have to be assembled each time $\mathbf{u}^h$ changes, at each discrete time in each step of the non–linear iteration.
- The sparse matrix–matrix products $\tilde{G}_{ij}M^{-1}G_{mn}$ have to be computed in order to obtain the $\tilde{A}_{ij}$ matrices, and this again, at each discrete time, at each step of the non–linear iteration.

Numerical studies with the fully implicit one–level projection–based VMS method can be found in[14], using $V^h = Q_2, Q^h = P_1^{\mathrm{disc}}$ and $L^H = P_0$ or $L^H = P_1^{\mathrm{disc}}$, respectively.

4.3 The semi–implicit projection–based VMS method

In order to avoid the extra equations for $g_{11}^H, \ldots, g_{33}^H$ as well as the extra matrix fill–ins in the two–level projection–based VMS method, a semi–implicit version of (19) can be used:

$$
\begin{aligned}
&(\mathbf{u}_k^h, \mathbf{v}^h) + \theta_1 \Delta t_k \Big[((2\nu + \nu_{T,k})\mathbb{D}(\mathbf{u}_k^h), \mathbb{D}(\mathbf{v}^h)) + ((\mathbf{u}_k^h \cdot \nabla)\mathbf{u}_k^h, \mathbf{v}^h) \Big] - (p_k, \nabla \cdot \mathbf{v}^h) \\
&\quad = (\mathbf{u}_{k-1}^h, \mathbf{v}^h) + \theta_2 \Delta t_k \Big[((2\nu + \nu_{T,k-1})\mathbb{D}(\mathbf{u}_{k-1}^h), \mathbb{D}(\mathbf{v}^h)) + ((\mathbf{u}_{k-1}^h \cdot \nabla)\mathbf{u}_{k-1}^h, \mathbf{v}^h) \Big] \\
&\qquad + \Delta t_k (\nu_{T,k-1} \mathbb{G}_{k-1}^H, \mathbb{D}(\mathbf{v}^h)) + \theta_3 \Delta t_k (\mathbf{f}_{k-1}, \mathbf{v}^h) + \theta_4 \Delta t_k (\mathbf{f}_k, \mathbf{v}^h), \quad \forall \mathbf{v}^h \in V^h, \\
&0 = (q^h, \nabla \cdot \mathbf{u}_k^h), \quad \forall q^h \in Q^h, \\
&0 = (\mathbb{G}_{k-1}^H - \mathbb{D}(\mathbf{u}_{k-1}^h), \mathbb{L}^H), \quad \forall \mathbb{L}^H \in L^H,
\end{aligned} \tag{23}
$$

where the subtraction of the additional viscosity in the large scales is treated explicitly. In comparison to (19), $\mathbb{G}^H$ does not appear on the left–hand side of (23) and the scaling of the $\mathbb{G}^H$ term on the right–hand side is Δt_k instead of $\theta \Delta t_k$. The left–hand side of the first equation in (23) is in general a stable discretization. If, e.g., $\nu_{T,k}$ is chosen to be the Smagorinsky model, one obtains the same left–hand side as in the Smagorinsky LES model.

Using the matrix notations introduced in Section 4.2, one obtains on the right–hand side additional terms of the form, e.g. for the first component

$$
\begin{aligned}
\Delta t_k \Big(\tilde{G}_{11} M^{-1} G_{11} u_{1,k-1}^h + \frac{1}{2} \tilde{G}_{24} M^{-1} (G_{42} u_{1,k-1}^h + G_{11} u_{2,k-1}^h) \\
+ \frac{1}{2} \tilde{G}_{36} M^{-1} (G_{63} u_{1,k-1}^h + G_{11} u_{3,k-1}^h) \Big).
\end{aligned} \tag{24}
$$

An easy evaluation of the products in (24) is possible if the mass matrix M of L^H is diagonal, i.e., if L^H is equipped with a L^2–orthogonal basis.

In contrast to the fully implicit approach, an efficient implementation of the semi–implicit approach is possible for both the one–level and the two–level projection–based VMS method.

The additional terms introduced by the semi–implicit projection–based VMS method require the following operations:

- the assembling of the matrices M and G_{ij} is done in a pre–processing step,
- the matrices $\tilde{G}_{ij}$ are assembled only once in each discrete time,
- the sparse matrix–vector products $\tilde{G}_{ij}M^{-1}G_{mn}u^h_{l,k-1}$ are computed once in each discrete time.

These are fewer operations than in the fully implicit projection–based VMS method.

4.4 Additional costs of the projection–based VMS methods

Let $\nu_{T,k}$ be given by the Smagorinsky model. In comparison to solving the Smagorinsky LES, in both of the above strategies, i.e. fully implicit and semi–implicit, the additional costs of the projection–based VMS method consist in:

- storing and assembling the seven sparse matrices M, G_{ij}, and $\tilde{G}_{ij}$,
- computing the sparse matrix–matrix products $\tilde{G}_{ij}M^{-1}G_{mn}$ in the fully implicit approach, respectively the matrix–vector products $\tilde{G}_{ij}M^{-1}G_{mn}u^h_{l,k-1}$ in the semi–implicit approach.

5 SUMMARY AND ONGOING RESEARCH

The application of VMS methods within the framework of finite element discretizations requires above all the choice of spaces to separate the large and the resolved small scales. There are two approaches which are quite different in this respect – the bubble VMS method and the projection–based VMS method. This paper studied implementational aspects of these approaches. Their comparison in numerical studies of turbulent flows is still an open issue and it is the subject of ongoing research.

Acknowledgments

The research of A. Tambulea was supported by the Deutsche Forschungsgemeinschaft (DFG) by grant No. Jo 329/7–1.

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