Finite element methods for the incompressible Stokes equations with variable viscosity

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Finite element error estimates are derived for the incompressible Stokes equations with variable viscosity. The ratio of the supremum and the infimum of the viscosity appears in the error bounds. Numerical studies show that this ratio can be observed sometimes. However, often the numerical results show a weaker dependency on the viscosity.

1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded domain with Lipschitz boundary $\partial\Omega$. Then, the incompressible Stokes equations with variable kinematic viscosity $\nu(\boldsymbol{x}) \geq \nu_{\min} > 0$ almost everywhere in Ω are given by

$$\begin{aligned} -2\nabla\cdot(\nu\mathbb{D}(\boldsymbol{u})) + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega, \end{aligned} \tag{1}$$

where $\boldsymbol{u}$ is the velocity field, $\mathbb{D}(\boldsymbol{u}) = (\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)/2$ the velocity deformation tensor, p the pressure, and $\boldsymbol{f}$ represents exterior forces acting on the fluid. In the analysis, the case of homogeneous Dirichlet boundary conditions

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega \tag{2}$$

will be considered.

Problems of type (1) appear as subproblems in several applications. Our special interest comes from models for the simulation of mantle convection, e.g., see [3, 11], where the viscosity takes large values (dynamic viscosity of order 10^{21} Pa s) and viscosity variations of several orders of magnitude do appear. Also the simulation of non-Newtonian flow problems might lead to systems of type (1), e.g., if these flows are modeled with a power law (Bingham model) or with pressure and shear-dependent viscosity, e.g., see [7, 15] and also [5], where iterative methods for the solution of the discrete systems were studied.

The goals of this note are twofold. First, it will be studied in which way the variable viscosity is reflected in finite element error estimates. Second, numerical simulations will study in which situations the dependency on the viscosity predicted by the error bounds, which are worst case estimates, can be observed. Concerning the first goal, classical approaches for deriving finite element error estimates for the Stokes equations will be applied. Despite of intensively searching the literature, we could find the presentation of finite element error analysis for incompressible flow problems with variable viscosity only in [14], where the Oseen equations are studied. The Stokes equations are just a special case of the Oseen equations. However, the results from [14] do not cover the results in the present note since both contributions have different goals and use different analytical techniques. In addition, we like to mention that a finite element error analysis for a coupled system with temperature-dependent viscosity is presented in [16].

This note is organized as follows. The weak formulation of the problem is formulated in Sect. 2. Section 3 presents the finite element error analysis and Sect. 4 contains numerical studies. The note finishes with a summary and outlook in Sect. 5.

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2 The continuous problem

Let $V = (H_0^1(\Omega))^d$ and $Q = L_0^2(\Omega)$ be the standard spaces used in the classical analysis of the Stokes equations. The subscripts mean that the functions from V posses a vanishing trace on $\partial\Omega$ and the functions from Q have integral mean value zero.

A weak formulation of (1) and (2) reads as follows: Find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$\begin{aligned} 2(\nu\mathbb{D}(\boldsymbol{u}), \mathbb{D}(\boldsymbol{v})) - (\nabla\cdot\boldsymbol{v}, p) &= \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{V',V} \quad \forall\, \boldsymbol{v} \in V, \\ -(\nabla\cdot\boldsymbol{u}, q) &= 0 \qquad\qquad\quad \forall\, q \in Q. \end{aligned} \tag{3}$$

Here, $V' = (H^{-1}(\Omega))^d$ is the dual space of V. Denote $b(\boldsymbol{v}, q) = -(\nabla\cdot\boldsymbol{v}, q)$ and let

$$V_{\mathrm{div}} = \left\{\boldsymbol{v} \in V \ :\ b(\boldsymbol{v}, q) = 0 \ \ \forall\, q \in Q\right\}$$

be the space of weakly divergence-free functions

Throughout the analysis, it will be assumed that $\nu \in L^\infty(\Omega)$. If even $\nu \in H^1(\Omega)$, then the viscous term can be reformulated in the following way

$$-2\nabla\cdot(\nu\mathbb{D}(\boldsymbol{u})) = -2\nu\nabla\cdot\mathbb{D}(\boldsymbol{u}) - 2\mathbb{D}(\boldsymbol{u})\nabla\nu = -\nu\Delta\boldsymbol{u} - 2\mathbb{D}(\boldsymbol{u})\nabla\nu.$$

Thus, compared with the standard Stokes equations, a variable viscosity introduces an additional first order term for the velocity.

From the uniform positivity and the boundedness of ν it is clear that the classical theory for linear saddle point problems [1, 2, 4] can be applied for proving that (3) possesses a unique solution. However, in performing this analysis there are two straightforward possibilities for equipping V with an appropriate norm, the standard norm

$$|\boldsymbol{v}|_1 = \|\nabla\boldsymbol{v}\|_0$$

and the norm which is induced by the bilinear form of the viscous term

$$\|\boldsymbol{v}\|_\nu = \|\nu^{1/2}\mathbb{D}(\boldsymbol{v})\|_0.$$

Lemma 2.1 (Norm equivalence.). *Set* $\nu_{\max} = \|\nu\|_{L^\infty(\Omega)}$, *then it holds*

$$\nu_{\max}^{-1/2}\|\boldsymbol{v}\|_\nu \le \|\nabla\boldsymbol{v}\|_0 \le C_K\,\nu_{\min}^{-1/2}\|\boldsymbol{v}\|_\nu, \tag{4}$$

where C_K *is the constant from Korn's inequality.*

P r o o f . Korn's inequality [6] gives the estimate

$$\|\nabla\boldsymbol{v}\|_0 \le C_K\|\mathbb{D}(\boldsymbol{v})\|_0 \quad \forall\, \boldsymbol{v} \in V.$$

Obviously it is

$$\|\mathbb{D}(\boldsymbol{v})\|_0 \le \frac{1}{2}\left(\|\nabla\boldsymbol{v}\|_0 + \|\nabla\boldsymbol{v}^T\|_0\right) = \|\nabla\boldsymbol{v}\|_0.$$

Combining these estimates yields

$$\begin{aligned} \|\boldsymbol{v}\|_\nu &= \|\nu^{1/2}\mathbb{D}(\boldsymbol{v})\|_0 \le \nu_{\max}^{1/2}\|\mathbb{D}(\boldsymbol{v})\|_0 \le \nu_{\max}^{1/2}\|\nabla\boldsymbol{v}\|_0 \\ &\le C_K\nu_{\max}^{1/2}\|\mathbb{D}(\boldsymbol{v})\|_0 \le C_K\nu_{\max}^{1/2}\nu_{\min}^{-1/2}\|\nu^{1/2}\mathbb{D}(\boldsymbol{v})\|_0 \\ &= C_K\nu_{\max}^{1/2}\nu_{\min}^{-1/2}\|\boldsymbol{v}\|_\nu, \end{aligned}$$

such that (4) follows. □

In [5], also a different choice than $L_0^2(\Omega)$ for the pressure space was considered, namely

$$L_\nu^2 = \left\{q \in L^2(\Omega) \ :\ (q, \nu^{-1}) = 0\right\}.$$

For this space, the norm $\|\cdot\|_\nu$ for the velocity space, and a sufficiently smooth viscosity, a continuous inf-sup condition was derived where the inf-sup constant depends on norms of the viscosity and the gradient of the inverse viscosity.

3 Finite element error analysis

In this note, the focus is on conforming discretizations of (3) which satisfy the discrete inf-sup condition, i.e., it holds $V^h \subset V$, $Q^h \subset Q$, and

$$\inf_{q^h \in Q^h \setminus \{0\}} \sup_{\boldsymbol{v}^h \in V^h \setminus \{\boldsymbol{0}\}} \frac{b(\boldsymbol{v}^h, q^h)}{\|\nabla \boldsymbol{v}^h\|_0 \|q^h\|_0} \geq \beta_{\text{is}} > 0. \tag{5}$$

Since $\|\boldsymbol{v}^h\|_\nu$ is an equivalent norm to $\|\nabla \boldsymbol{v}^h\|_0$ in V^h, also an inf-sup condition of the form

$$\inf_{q^h \in Q^h \setminus \{0\}} \sup_{\boldsymbol{v}^h \in V^h \setminus \{\boldsymbol{0}\}} \frac{b(\boldsymbol{v}^h, q^h)}{\|\boldsymbol{v}^h\|_\nu \|q^h\|_0} \geq \beta_{\text{is},\nu} > 0 \tag{6}$$

is satisfied. Using the norm equivalence (4), one obtains from (5)

$$\beta_{\text{is}} \|q^h\|_0 \leq \sup_{\boldsymbol{v}^h \in V^h \setminus \{\boldsymbol{0}\}} \frac{b(\boldsymbol{v}^h, q^h)}{\|\nabla \boldsymbol{v}^h\|_0} \leq \frac{1}{\nu_{\max}^{-1/2}} \sup_{\boldsymbol{v}^h \in V^h \setminus \{\boldsymbol{0}\}} \frac{b(\boldsymbol{v}^h, q^h)}{\|\boldsymbol{v}^h\|_\nu} \quad \forall\, q^h \in Q^h,$$

which shows that one can choose

$$\beta_{\text{is},\nu} = \nu_{\max}^{-1/2} \beta_{\text{is}}. \tag{7}$$

Let

$$V_{\text{div}}^h = \left\{ \boldsymbol{v}^h \in V^h \ : \ b(\boldsymbol{v}^h, q^h) = 0 \ \forall\, q^h \in Q^h \right\}$$

be the space of discretely divergence-free functions. In the finite element error analysis, an estimate of the best approximation error in V_{div}^h appears. For some pairs of inf-sup stable finite element spaces, it is possible to construct a sequence of discretely divergence-free functions which have the optimal order of convergence. If this approach is not possible, the best approximation error in V_{div}^h can be estimated with the best approximation error in V^h. Considering an arbitrary function $\boldsymbol{v} \in V_{\text{div}}$ and equipping V^h with the norm $\|\cdot\|_0$, this estimate is known to has the form [4]

$$\inf_{\boldsymbol{v}^h \in V_{\text{div}}^h} \|\nabla(\boldsymbol{v} - \boldsymbol{v}^h)\|_0 \leq \left(1 + \frac{1}{\beta_{\text{is}}}\right) \inf_{\boldsymbol{w}^h \in V^h} \|\nabla(\boldsymbol{v} - \boldsymbol{w}^h)\|_0. \tag{8}$$

It is possible to obtain an estimate for the ν-weighted norm just by using the norm equivalence (4) and (8). However, one might even derive a better estimate.

Lemma 3.1 (Estimate of the best approximation in V_{div}^h in the ν-weighted norm.)**.** *Let $\boldsymbol{v} \in V_{\text{div}}$ be arbitrary, then*

$$\inf_{\boldsymbol{v}^h \in V_{\text{div}}^h} \|\boldsymbol{v} - \boldsymbol{v}^h\|_\nu \leq \left(1 + \frac{C_K}{\beta_{\text{is},\nu}\, \nu_{\min}^{1/2}}\right) \inf_{\boldsymbol{w}^h \in V^h} \|\boldsymbol{v} - \boldsymbol{w}^h\|_\nu. \tag{9}$$

P r o o f . The proof follows the lines of [4, Chap. II, Theorem 1.1]. Let the operator $B^h \in \mathcal{L}(V, (Q^h)')$ be defined by

$$\langle B^h \boldsymbol{v}, q^h \rangle_{(Q^h)', Q^h} = b(\boldsymbol{v}, q^h), \quad \forall\, \boldsymbol{v} \in V, q^h \in Q^h.$$

The operator B^h is an isomorphism from $\left(V_{\text{div}}^h\right)^\perp$ (with respect to the inner product in V) onto $(Q^h)'$ and $\|B^h \boldsymbol{z}^h\|_{(Q^h)'} \geq \beta_{\text{is}} \|\nabla \boldsymbol{z}^h\|_0$ for all $\boldsymbol{z}^h \in \left(V_{\text{div}}^h\right)^\perp$.

Let $\boldsymbol{w}^h$ be an arbitrary element of V^h. Since $B^h(\boldsymbol{v} - \boldsymbol{w}^h) \in (Q^h)'$ then there exists a unique $\boldsymbol{z}^h \in \left(V_{\text{div}}^h\right)^\perp$ such that

$$B^h \boldsymbol{z}^h = B^h(\boldsymbol{v} - \boldsymbol{w}^h). \tag{10}$$

Using the estimate of the $L^2(\Omega)$ norm of the divergence by the same norm of the gradient, it follows that

$$\|\nabla \boldsymbol{z}^h\|_0 \leq \frac{1}{\beta_{\text{is}}} \|B^h(\boldsymbol{v} - \boldsymbol{w}^h)\|_{(Q^h)'} \leq \frac{1}{\beta_{\text{is}}} \|\nabla(\boldsymbol{v} - \boldsymbol{w}^h)\|_0.$$

Applying now the norm equivalence (4) and (7) yields

$$\|\boldsymbol{z}^h\|_\nu \leq \nu_{\max}^{1/2} \|\nabla \boldsymbol{z}^h\|_0 \leq \frac{1}{\beta_{\text{is},\nu}} \|\nabla(\boldsymbol{v} - \boldsymbol{w}^h)\|_0 \leq \frac{1}{\beta_{\text{is},\nu}} C_K \nu_{\min}^{-1/2} \|\boldsymbol{v} - \boldsymbol{w}^h\|_\nu.$$

Setting $\boldsymbol{v}^h = \boldsymbol{z}^h + \boldsymbol{w}^h$, one obtains with (10) that $b(\boldsymbol{v}^h, q^h) = b(\boldsymbol{v} - \boldsymbol{w}^h, q^h) + b(\boldsymbol{w}^h, q^h) = b(\boldsymbol{v}, q^h) = 0$ for all $q^h \in Q^h$, which implies that $\boldsymbol{v}^h \in V^h_{\mathrm{div}}$. The triangle inequality gives

$$\|\boldsymbol{v} - \boldsymbol{v}^h\|_\nu \le \|\boldsymbol{v} - \boldsymbol{w}^h\|_\nu + \|\boldsymbol{z}^h\|_\nu \le \left(1 + \frac{C_K}{\beta_{\mathrm{is},\nu}\nu_{\min}^{1/2}}\right) \|\boldsymbol{v} - \boldsymbol{w}^h\|_\nu .$$

As $\boldsymbol{w}^h$ was chosen to be arbitrary, the statement of the lemma is proved. □

Applying the simple scaling argument to (8) one gets the term $C_K(\nu_{\max}/\nu_{\min})^{1/2}$ instead of 1 in the parentheses on the right-hand side of (9).

The Galerkin finite element formulation of (3) reads as follows: Find $(\boldsymbol{u}^h, p^h) \in V^h \times Q^h$ such that

$$\begin{aligned} 2(\nu\mathbb{D}(\boldsymbol{u}^h), \mathbb{D}(\boldsymbol{v}^h)) - (\nabla\cdot\boldsymbol{v}^h, p^h) &= \langle \boldsymbol{f}, \boldsymbol{v}^h\rangle_{V',V} \quad && \forall\, \boldsymbol{v}^h \in V^h, \\ -(\nabla\cdot\boldsymbol{u}^h, q^h) &= 0 && \forall\, q^h \in Q^h. \end{aligned} \tag{11}$$

The existence and uniqueness of a solution is again a direct consequence of the general theory of linear saddle point problems, the uniform positivity of ν, and its boundedness.

Theorem 3.2 (Finite element error estimate for $\|\boldsymbol{u} - \boldsymbol{u}^h\|_\nu$.). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with polyhedral Lipschitz boundary and let $(\boldsymbol{u}, p) \in V \times Q$ be the unique solution of the Stokes problem (3). Given a discretization with inf-sup stable conforming finite element spaces $V^h \times Q^h$ and let $\boldsymbol{u}^h \in V^h_{\mathrm{div}}$ be the finite element solution for the velocity field. Then the following error estimate holds:*

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_\nu \le 2\left(1 + \frac{C_K}{\beta_{\mathrm{is},\nu}\nu_{\min}^{1/2}}\right) \inf_{\boldsymbol{v}^h\in V^h} \|\boldsymbol{u} - \boldsymbol{v}^h\|_\nu + \frac{C_K}{2\nu_{\min}^{1/2}} \inf_{q^h\in Q^h} \|p - q^h\|_0, \tag{12}$$

where $\beta_{\mathrm{is},\nu}$ depends on $\nu_{\max}$ like in (7).

Proof. The proof follows the classical way. Taking $\boldsymbol{v}^h \in V^h_{\mathrm{div}}$ as a test function in (3) and (11), observing that generally $V^h_{\mathrm{div}} \not\subset V_{\mathrm{div}}$, subtracting these equations and using $(\nabla\cdot\boldsymbol{v}^h, q^h) = 0$ for all $q^h \in Q^h$ gives

$$2(\nu\mathbb{D}(\boldsymbol{u} - \boldsymbol{u}^h), \mathbb{D}(\boldsymbol{v}^h)) - (\nabla\cdot\boldsymbol{v}^h, p - q^h) = 0 \quad \forall\, (\boldsymbol{v}^h, q^h) \in V^h_{\mathrm{div}} \times Q^h. \tag{13}$$

Now, the error is split into an approximation error in V^h_{div} and a finite element remainder

$$\boldsymbol{u} - \boldsymbol{u}^h = (\boldsymbol{u} - I^h\boldsymbol{u}) - (\boldsymbol{u}^h - I^h\boldsymbol{u}) = \boldsymbol{\eta} - \boldsymbol{\phi}^h .$$

Here $I^h\boldsymbol{u} \in V^h_{\mathrm{div}}$ is an interpolant of $\boldsymbol{u}$ in V^h_{div}. Choosing $\boldsymbol{\phi}^h \in V^h_{\mathrm{div}}$ as test function in (13) leads to the estimate

$$\|\boldsymbol{\phi}^h\|_\nu^2 \le \left|(\nu\mathbb{D}(\boldsymbol{\eta}), \mathbb{D}(\boldsymbol{\phi}^h))\right| + \frac{1}{2}\left|(\nabla\cdot\boldsymbol{\phi}^h, p - q^h)\right| .$$

Both terms are estimated with the Cauchy–Schwarz inequality and for the second term also the norm equivalence (4) is applied. One obtains

$$\left|(\nu\mathbb{D}(\boldsymbol{\eta}), \mathbb{D}(\boldsymbol{\phi}^h))\right| \le \|\nu^{1/2}\mathbb{D}(\boldsymbol{\eta})\|_0 \|\nu^{1/2}\mathbb{D}(\boldsymbol{\phi}^h)\|_0 = \|\boldsymbol{\eta}\|_\nu \|\boldsymbol{\phi}^h\|_\nu$$

and

$$\begin{aligned} \left|(\nabla\cdot\boldsymbol{\phi}^h, p - q^h)\right| &\le \|\nabla\cdot\boldsymbol{\phi}^h\|_0 \|p - q^h\|_0 \le \|\nabla\boldsymbol{\phi}^h\|_0 \|p - q^h\|_0 \\ &\le C_K \nu_{\min}^{-1/2} \|\boldsymbol{\phi}^h\|_\nu \|p - q^h\|_0 . \end{aligned}$$

Since these estimates hold for all $I^h\boldsymbol{u}$ and all q^h, one gets with the triangle inequality

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_\nu \le 2 \inf_{I^h\boldsymbol{u}\in V^h_{\mathrm{div}}} \|\boldsymbol{u} - I^h\boldsymbol{u}\|_\nu + \frac{C_K}{2\nu_{\min}^{1/2}} \inf_{q^h\in Q^h} \|p - q^h\|_0 .$$

Now, estimate (12) follows by applying estimate (9). □

To see the dependency of the error bound on the viscosity in a clearer way, one can rewrite (12), using (4) and (7), in the following form

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_\nu \le 2\nu_{\max}^{1/2}\left(1 + \frac{C_K}{\beta_{\mathrm{is}}}\left(\frac{\nu_{\max}}{\nu_{\min}}\right)^{1/2}\right) \inf_{\boldsymbol{v}^h\in V^h} \|\nabla(\boldsymbol{u} - \boldsymbol{v}^h)\|_0 + \frac{C_K}{2\nu_{\min}^{1/2}} \inf_{q^h\in Q^h} \|p - q^h\|_0 .$$

With (4), one gets immediately the estimate

$$
\begin{aligned}
&\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\|_0 \\
&\le 2C_K\left(\frac{\nu_{\max}}{\nu_{\min}}\right)^{1/2}\left(1+\frac{C_K}{\beta_{\mathrm{is}}}\left(\frac{\nu_{\max}}{\nu_{\min}}\right)^{1/2}\right)\inf_{\boldsymbol{v}^h\in V^h}\|\nabla(\boldsymbol{u}-\boldsymbol{v}^h)\|_0+\frac{C_K^2}{2\nu_{\min}}\inf_{q^h\in Q^h}\|p-q^h\|_0. \qquad (14)
\end{aligned}
$$

One can perform an analogous proof as the proof of Theorem 3.2 to obtain an error estimate for $\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\|_0$ directly. It turns out that the only change in the error bound is the factor in front of the best approximation error with respect to the velocity, which is for the direct proof

$$
\left(1+C_K^2\frac{\nu_{\max}}{\nu_{\min}}\right)\left(1+\frac{1}{\beta_{\mathrm{is}}}\right).
$$

Altogether, the error bound for $\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\|_0$ becomes large if $\nu_{\min}$ is small or if the ratio $\nu_{\max}/\nu_{\min}$ is large.

In the case $V_{\mathrm{div}}^h\subset V_{\mathrm{div}}$ the pressure term on the left-hand side of (13) vanishes such that the pressure term in the estimates (12) and (14) does not appear and only the dependency on $\nu_{\max}/\nu_{\min}$ remains in the error bound. Another approach which removes the pressure from the right-hand side of the estimate for the velocity was recently proposed in [12]. It relies on using a projection of the test function for the right-hand side of the momentum equation into an appropriate space of divergence-free functions. Whereas the approach presented in [12] is for a pair of non-conforming finite element spaces, extensions to conforming spaces can be found in [13].

Theorem 3.3 (Finite element error estimate for $\|p-p^h\|_0$.)**.** *Let the assumptions of Theorem 3.2 hold, then it is*

$$
\begin{aligned}
\|p-p^h\|_0 &\le \left(1+\frac{2\,C_K}{\beta_{\mathrm{is}}}\left(\frac{\nu_{\max}}{\nu_{\min}}\right)^{1/2}\right)\inf_{q^h\in Q^h}\|p-q^h\|_0 \\
&\quad+\frac{4\nu_{\max}}{\beta_{\mathrm{is}}}\left(1+\frac{C_K}{\beta_{\mathrm{is}}}\left(\frac{\nu_{\max}}{\nu_{\min}}\right)^{1/2}\right)\inf_{\boldsymbol{v}^h\in V^h}\|\nabla(\boldsymbol{u}-\boldsymbol{v}^h)\|_0. \qquad (15)
\end{aligned}
$$

P r o o f. It is

$$
\|p-p^h\|_0\le\|p-q^h\|_0+\|p^h-q^h\|_0,
$$

where $q^h\in Q^h$ is arbitrary. The finite element problem (11) can be rewritten as follows

$$
b(\boldsymbol{v}^h,p^h-q^h)=\langle\boldsymbol{f},\boldsymbol{v}^h\rangle_{V',V}-2(\nu\mathbb{D}(\boldsymbol{u}^h),\mathbb{D}(\boldsymbol{v}^h))-b(\boldsymbol{v}^h,q^h). \qquad (16)
$$

Since conforming finite element spaces are considered, all $\boldsymbol{v}^h\in V^h$ can be used as test function in the continuous problem (3). Replacing $\langle\boldsymbol{f},\boldsymbol{v}^h\rangle_{V',V}$ in (16) with the left-hand side of the continuous problem yields

$$
b(\boldsymbol{v}^h,p^h-q^h)=2(\nu(\mathbb{D}(\boldsymbol{u})-\mathbb{D}(\boldsymbol{u}^h)),\mathbb{D}(\boldsymbol{v}^h))+b(\boldsymbol{v}^h,p-q^h)\quad\forall\,q^h\in Q^h,\boldsymbol{v}^h\in V^h.
$$

With the discrete inf-sup condition (6), the Cauchy–Schwarz inequality, and the norm equivalence (4), one gets

$$
\begin{aligned}
\|p^h-q^h\|_0 &\le \frac{1}{\beta_{\mathrm{is},\nu}}\sup_{\boldsymbol{v}^h\in V^h\setminus\{\boldsymbol{0}\}}\frac{2(\nu(\mathbb{D}(\boldsymbol{u})-\mathbb{D}(\boldsymbol{u}^h)),\mathbb{D}(\boldsymbol{v}^h))+b(\boldsymbol{v}^h,p-q^h)}{\|\boldsymbol{v}^h\|_\nu} \\
&\le \frac{1}{\beta_{\mathrm{is},\nu}}\sup_{\boldsymbol{v}^h\in V^h\setminus\{\boldsymbol{0}\}}\frac{2\|\boldsymbol{u}-\boldsymbol{u}^h\|_\nu\|\boldsymbol{v}^h\|_\nu+\|\nabla\boldsymbol{v}^h\|_0\|p-q^h\|_0}{\|\boldsymbol{v}^h\|_\nu} \\
&\le \frac{1}{\beta_{\mathrm{is},\nu}}\sup_{\boldsymbol{v}^h\in V^h\setminus\{\boldsymbol{0}\}}\frac{2\|\boldsymbol{u}-\boldsymbol{u}^h\|_\nu\|\boldsymbol{v}^h\|_\nu+C_K\nu_{\min}^{-1/2}\|\boldsymbol{v}^h\|_\nu\|p-q^h\|_0}{\|\boldsymbol{v}^h\|_\nu} \\
&\le \frac{1}{\beta_{\mathrm{is},\nu}}\left(2\|\boldsymbol{u}-\boldsymbol{u}^h\|_\nu+C_K\nu_{\min}^{-1/2}\|p-q^h\|_0\right).
\end{aligned}
$$

Now the proof finishes by inserting (12) and using (4) and (7). □

The error bound (15) for the pressure is large for large values of $\nu_{\max}$ and large ratios $\nu_{\max}/\nu_{\min}$.

Finally, the error $\|\boldsymbol{u}-\boldsymbol{u}^h\|_0$ will be studied. To this end, consider the dual Stokes problem: Find $(\boldsymbol{\phi}_{\hat{f}}, \xi_{\hat{f}}) \in V \times Q$ such that for given $\hat{\boldsymbol{f}} \in L^2(\Omega)$

$$\begin{aligned} -2\nabla\cdot(\nu\mathbb{D}(\boldsymbol{\phi}_{\hat{f}})) + \nabla\xi_{\hat{f}} &= \hat{\boldsymbol{f}} \quad \text{in } \Omega, \\ -\nabla\cdot\boldsymbol{\phi}_{\hat{f}} &= 0 \quad \text{in } \Omega, \end{aligned} \tag{17}$$

with homogeneous Dirichlet boundary conditions and its weak form

$$\begin{aligned} 2(\nu\mathbb{D}(\boldsymbol{\phi}_{\hat{f}}), \mathbb{D}(\boldsymbol{v})) - (\nabla\cdot\boldsymbol{v}, \xi_{\hat{f}}) &= (\hat{\boldsymbol{f}}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in V, \\ -(\nabla\cdot\boldsymbol{\phi}_{\hat{f}}, q) &= 0 \qquad\quad \forall\, q \in Q. \end{aligned} \tag{18}$$

It is assumed that the mapping $(\boldsymbol{\phi}_{\hat{f}}, \xi_{\hat{f}}) \mapsto -2\nabla\cdot(\nu\mathbb{D}(\boldsymbol{\phi}_{\hat{f}})) + \nabla\xi_{\hat{f}}$ is an isomorphism from $(H^2(\Omega))^d \cap V \times H^1(\Omega) \cap Q$ to $(L^2(\Omega))^d$. Then, $(\boldsymbol{\phi}_{\hat{f}}, \xi_{\hat{f}})$ is called regular solution of (17).

Theorem 3.4 (Finite element error estimate for $\|\boldsymbol{u}-\boldsymbol{u}^h\|_0$.)**.** *With the assumptions of Theorem 3.2 and $(\boldsymbol{\phi}_{\hat{f}}, \xi_{\hat{f}})$ being the regular solution of* (17)*, the following error estimate for the $L^2(\Omega)$ norm of velocity holds*

$$\begin{aligned} &\|\boldsymbol{u}-\boldsymbol{u}^h\|_0 \\ &\le \left(2\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\|_0 + \frac{1}{\nu_{\max}} \inf_{q^h\in Q^h} \|p-q^h\|_0\right) \\ &\quad \times \sup_{\hat{f}\in L^2(\Omega)\setminus\{\boldsymbol{0}\}} \frac{1}{\|\hat{\boldsymbol{f}}\|_0} \left[\left(1+\frac{1}{\beta_{\mathrm{is}}}\right)\nu_{\max} \inf_{\boldsymbol{\phi}^h\in V^h} \|\nabla(\boldsymbol{\phi}_{\hat{f}}-\boldsymbol{\phi}^h)\|_0 + \frac{1}{2}\inf_{r^h\in Q^h}\|\xi_{\hat{f}}-r^h\|_0\right]. \end{aligned} \tag{19}$$

P r o o f . Starting point of the proof is the definition of the $L^2(\Omega)$ norm

$$\|\boldsymbol{u}-\boldsymbol{u}^h\|_0 = \sup_{\hat{f}\in L^2(\Omega)\setminus\{\boldsymbol{0}\}} \frac{(\hat{\boldsymbol{f}}, \boldsymbol{u}-\boldsymbol{u}^h)}{\|\hat{\boldsymbol{f}}\|_0}. \tag{20}$$

Choosing $\boldsymbol{v} = \boldsymbol{u}-\boldsymbol{u}^h$ in (18) gives

$$(\hat{\boldsymbol{f}}, \boldsymbol{u}-\boldsymbol{u}^h) = 2\left(\nu\mathbb{D}(\boldsymbol{\phi}_{\hat{f}}), \mathbb{D}(\boldsymbol{u}-\boldsymbol{u}^h)\right) - \left(\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}^h), \xi_{\hat{f}}\right). \tag{21}$$

Using the weak form of the Stokes problem (3) and the corresponding finite element problem (11), one finds for $\boldsymbol{\phi}^h \in V_{\mathrm{div}}^h \subset V$ and $q^h \in Q^h$ arbitrary

$$2\left(\nu\mathbb{D}(\boldsymbol{\phi}^h), \mathbb{D}(\boldsymbol{u}-\boldsymbol{u}^h)\right) = (\nabla\cdot\boldsymbol{\phi}^h, p) = (\nabla\cdot\boldsymbol{\phi}^h, p-q^h).$$

Inserting this identity into (21) and expanding (21) with some terms which are zero leads to

$$\begin{aligned} (\hat{\boldsymbol{f}}, \boldsymbol{u}-\boldsymbol{u}^h) &= 2\left(\nu\mathbb{D}(\boldsymbol{\phi}_{\hat{f}}-\boldsymbol{\phi}^h), \mathbb{D}(\boldsymbol{u}-\boldsymbol{u}^h)\right) - \left(\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}^h), \xi_{\hat{f}}-r^h\right) \\ &\quad +(\nabla\cdot(\boldsymbol{\phi}^h-\boldsymbol{\phi}_{\hat{f}}), p-q^h) \end{aligned}$$

for all $\boldsymbol{\phi}^h \in V_{\mathrm{div}}^h$ and $q^h, r^h \in Q^h$. Applying now the Cauchy–Schwarz inequality, the estimate of the norm of the divergence by the norm of the gradient, and the norm equivalence (4) leads to

$$\begin{aligned} &\left|(\hat{\boldsymbol{f}}, \boldsymbol{u}-\boldsymbol{u}^h)\right| \\ &\le 2\|\boldsymbol{\phi}_{\hat{f}}-\boldsymbol{\phi}^h\|_\nu \|\boldsymbol{u}-\boldsymbol{u}^h\|_\nu + \|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\|_0\|\xi_{\hat{f}}-r^h\|_0 + \|\nabla(\boldsymbol{\phi}_{\hat{f}}-\boldsymbol{\phi}^h)\|_0\|p-q^h\|_0 \\ &\le \left(2\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\|_0 + \frac{1}{\nu_{\max}}\|p-q^h\|_0\right)\left(\nu_{\max}\|\nabla(\boldsymbol{\phi}_{\hat{f}}-\boldsymbol{\phi}^h)\|_0 + \frac{1}{2}\|\xi_{\hat{f}}-r^h\|_0\right) \end{aligned}$$

for all $\boldsymbol{\phi}^h \in V_{\mathrm{div}}^h$ and $q^h, r^h \in Q^h$. Inserting this estimate into (20) leads in a straightforward way to estimate (19). □

A discussion of the dependency of the error bound on the right-hand side of (19) on the viscosity is only meaningful if the dependency of all norms in (19) on the viscosity is taken into account. In the discussion of the results of Theorems 3.2 and 3.3, it was assumed that $(\boldsymbol{u}, p)$ is independent of the viscosity, i.e., only $\boldsymbol{f}$ depends on the viscosity and $\boldsymbol{f}$ does not appear in the estimates. However, one of the quantities $\|\hat{\boldsymbol{f}}\|_0$, $\|\nabla(\boldsymbol{\phi}_{\hat{f}} - \boldsymbol{\phi}^h)\|_0$, and $\|\xi_{\hat{f}} - r^h\|_0$ has to depend on the viscosity. A possible discussion of this topic can assume that $\hat{\boldsymbol{f}}$ is independent of the viscosity. Using $\boldsymbol{\phi}_{\hat{f}}$ as test function in (18) gives with the Cauchy–Schwarz inequality, the Poincaré inequality, and the norm equivalence (4)

$$\|\nabla \boldsymbol{\phi}_{\hat{f}}\|_0 \le \frac{C}{\nu_{\min}} \|\hat{\boldsymbol{f}}\|_0.$$

Thus, the term $\|\nabla(\boldsymbol{\phi}_{\hat{f}} - \boldsymbol{\phi}^h)\|_0$ in (19) can be estimated to scale with $\nu_{\min}^{-1}$. Using the Helmholtz decomposition, $\hat{\boldsymbol{f}}$ can be decomposed into $\hat{\boldsymbol{f}} = \boldsymbol{w} + \nabla r$ where $\boldsymbol{w}$ is divergence-free and ∇r is orthogonal to $\boldsymbol{w}$ with respect to the $L^2(\Omega)$ inner product. Inserting this decomposition into (17) gives

$$-2\nabla \cdot (\nu \mathbb{D}(\boldsymbol{\phi}_{\hat{f}})) + \nabla \xi_{\hat{f}} = \boldsymbol{w} + \nabla r.$$

Thus, $\xi_{\hat{f}}$ is balanced by r, i.e., it is independent of ν since $\hat{\boldsymbol{f}}$ was assumed to be independent of ν. One finds that only $\|\nabla(\boldsymbol{\phi}_{\hat{f}} - \boldsymbol{\phi}^h)\|_0$ depends on ν and one can expect that the term in (19) with the dual velocity solution scales like $\nu_{\max}/\nu_{\min}$.

4 Numerical studies

The goal of the numerical studies consists in checking if the errors show the dependency on the extremal values of the viscosity as predicted by the error bounds. It should be emphasized that the error bounds are always worst case estimates such that in concrete examples the dependency might be weaker.

For the numerical studies, analytical solutions were prescribed which do not depend on the viscosity. Let $\Omega = (0, 1)^2$ and

$$\phi(x, y) = 1000x^2(1-x)^4 y^3 (1-y)^2,$$

then the velocity solution is given by $\boldsymbol{u} = (\partial_y \phi, -\partial_x \phi)^T$. Clearly, $\boldsymbol{u}$ is divergence-free and it satisfies homogeneous Dirichlet boundary conditions. For the first three examples, the pressure was chosen to be

$$p(x, y) = \pi^2 \left(xy^2 \cos(2\pi x^2 y) - x^2 y \sin(2\pi xy)\right) - \frac{1}{8}.$$

A number of different functions for the viscosity were studied:

$$\begin{aligned}
\nu_1(x, y) &= \nu_{\min} + (\nu_{\max} - \nu_{\min}) x^2 (1-x) y^2 (1-y) \cdot 721/16, \\
\nu_2(x, y) &= \nu_{\min} + (\nu_{\max} - \nu_{\min}) \exp\left(-10^{13}\left((x-0.5)^{10} + (y-0.5)^{10}\right)\right), \\
\nu_3(x, y) &= \nu_{\min} + (\nu_{\max} - \nu_{\min}) \left(1 - \exp\left(-10^{13}\left((x-0.5)^{10} + (y-0.5)^{10}\right)\right)\right),
\end{aligned}$$

see Fig. 1 for an illustration. The viscosity ν_1 is smoothly varying, whereas ν_2 and ν_3 posses steep layers between $\nu_{\min}$ and $\nu_{\max}$. In most of the domain, ν_2 takes values close to $\nu_{\min}$ and ν_3 takes values close to $\nu_{\max}$. Numerical results will be presented for the case $\nu_{\max} = 1$ and different values of $\nu_{\min}$ and the case $\nu_{\min} = 1$ and different values of $\nu_{\max}$.

Simulations were performed with the Taylor–Hood pair of finite element spaces P_2/P_1 on unstructured grids. The coarsest grid, level 0, is presented in Fig. 2. On the finest grid, level 8, there are 9 442 306 degrees of freedom for the velocity and 1 180 929 degrees of freedom for the pressure. The linear systems of equations were first solved with the sparse direct solver PARDISO [17]. Since in many cases the bad condition number of the matrices resulted in large round-off errors, an iterative post-processing of the solution was performed. As iterative method, the flexible GMRES(restart) method, with restart= 50, and with a coupled multigrid preconditioner, as described, e.g., in [8], was used. The iterations were stopped if the Euclidean norm of the residual vector was less than 10^{-13}. All simulations were performed with the code MooNMD [9].

For brevity, only a few representative results will be presented here. More results, with different analytical solutions, different viscosities, and different pairs of finite element spaces, can be found in [10].

Example 4.1. *Smoothly varying viscosity* ν_1. Results for the smoothly varying viscosity are presented in Fig. 3. One can see in the left column of pictures that small values of $\nu_{\min}$, for fixed $\nu_{\max} = 1$, might lead indeed to an increase of the velocity errors, see the results for $\nu_{\min} = 10^{-4}$ and $\nu_{\min} = 10^{-6}$. The differences of the corresponding errors for these two values of $\nu_{\min}$ are around two orders of magnitude. This behavior corresponds to the worst case predictions of the

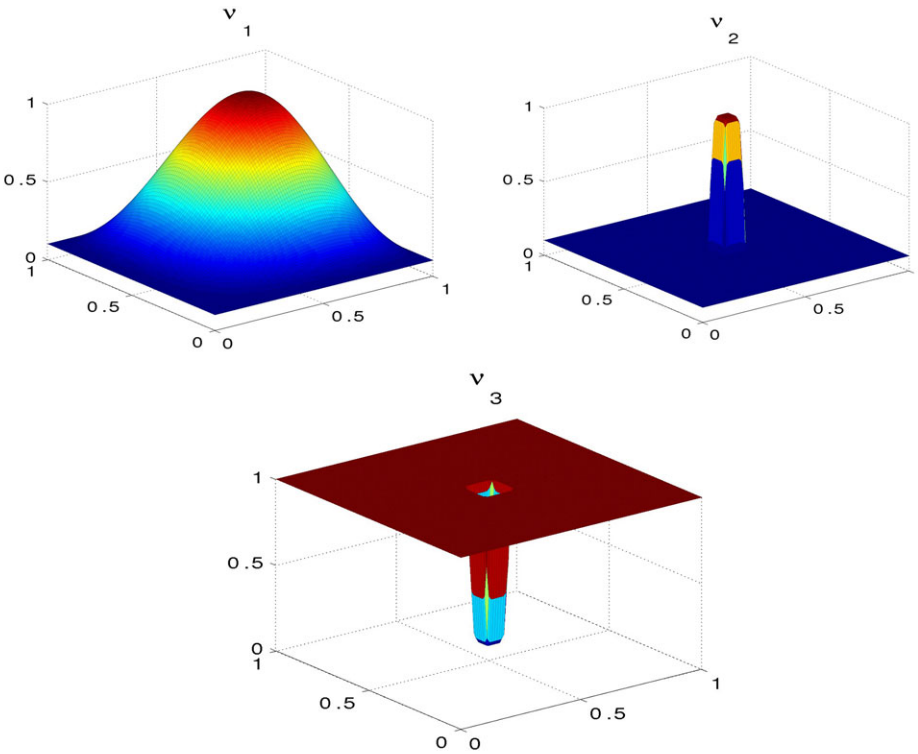

Fig. 1 The different functions for the viscosity.

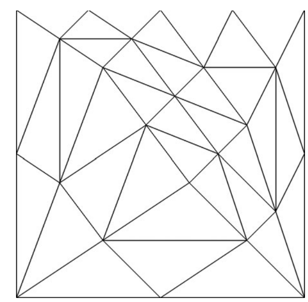

Fig. 2 Coarsest grid (level 0).

numerical analysis. The value of $\nu_{\min}$ has almost no influence on the pressure error. Large values of $\nu_{\max}$, in the case that $\nu_{\min} = 1$ is fixed, have an impact on the pressure error, as it can be expected from the error bound (15). But in this example, one cannot observe a notable impact of $\nu_{\max}$ on the velocity errors.

Example 4.2. *Viscosity with steep layer* ν_2, *mainly taking values close to* $\nu_{\min}$. Figure 4 presents the results for ν_2. Here, a dependency of the errors on the viscosity can be observed on all coarse grids. Often, the errors scale like $\nu_{\min}^{-1}$ (if $\nu_{\max} = 1$) or $\nu_{\max}$ (if $\nu_{\min} = 1$). However, on finer grids, the impact of the extremal values of the viscosity on the errors decreases. For these grids, one can draw similar conclusions as for Example 4.1. The most remarkable impact is again in the velocity errors for $\nu_{\max} = 1$, $\nu_{\min} \in \{10^{-4}, 10^{-6}\}$. Even on level 8, the errors for these values are different by two orders of magnitude, which is the difference predicted by the error bounds.

Example 4.3. *Viscosity with steep layer* ν_3, *mainly taking values close to* $\nu_{\max}$. The results for this example are presented in Fig. 5. Comparing with Example 4.2, ν_3 is large where ν_2 is small and vice versa. The part of the domain where the viscosity takes large values is much larger for ν_3 than for ν_2. One can see the impact of this situation in the velocity errors for small $\nu_{\min}$ and in the pressure error for large $\nu_{\max}$. For ν_3, the velocity errors for small $\nu_{\min}$ depend only weakly on $\nu_{\min}$ (note the different scales of the ordinate for the velocity errors in Figs. 4 and 5) and the pressure error for large $\nu_{\max}$ depends stronger on $\nu_{\max}$ than for ν_2.

Among the considered viscosities, ν_3 is closest to the situation which occurs in the simulation of mantle convection. It takes mainly large values and there are only few regions with steep gradients and with smaller values. The case of mantle convection is reflected best by the right column of Fig. 5. It can be seen that the velocity errors are mainly independent of the viscosity and the pressure error depends strongly on $\nu_{\max}$. Fortunately, only the velocity is needed in the other equations for simulating mantle convection and the pressure does not play any role. Thus, one can expect accurate finite element velocity solutions in spite of the large values and steep gradients of the viscosity.

Example 4.4. *Viscosity with steep layer* ν_2, *mainly taking values close to* $\nu_{\min}$, *and* $p = 0$. This example serves for illustrating that the ratio $\nu_{\max}/\nu_{\min}$, which appears in the error bounds for the velocity (14) and (19), can be observed in numerical simulations. To this end, the prescribed pressure in this example was $p = 0$ such that $\nu_{\max}/\nu_{\min}$ appears as factor in (14) and (19), since the best approximation error of the pressure is zero. Representative results obtained with ν_2 are presented in Tables 1 and 2. The dependency of the errors on $\nu_{\max}/\nu_{\min}$ can be clearly observed. For large ratios $\nu_{\max}/\nu_{\min}$,

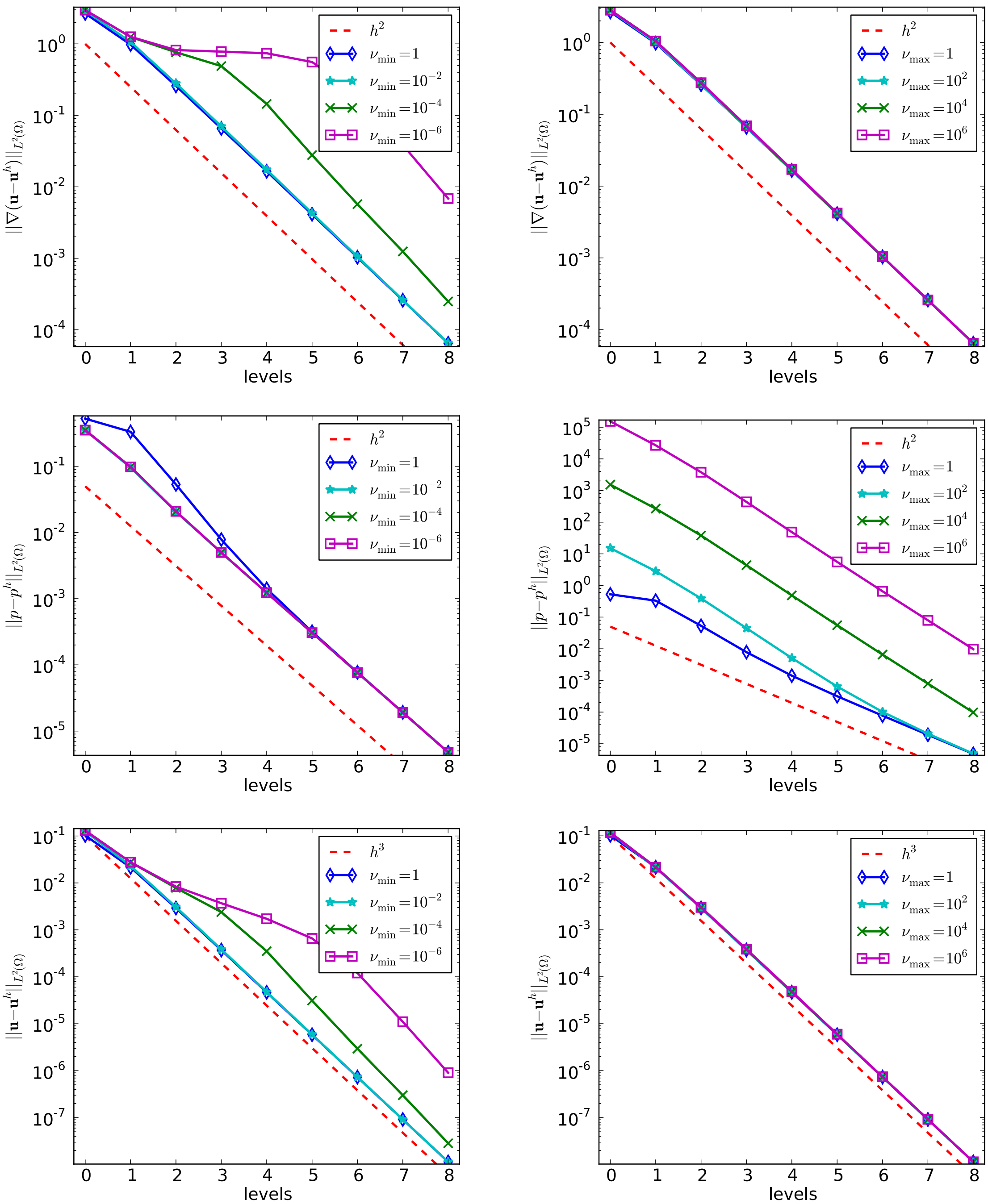

Fig. 3 Example 4.1: Smoothly varying viscosity ν_1 with $\nu_{\max} = 1$ and different values of $\nu_{\min}$ (left), with $\nu_{\min} = 1$ and different values of $\nu_{\max}$ (right).

even the linear dependency on this ratio, predicted as worst case in the analysis, can be seen. For completeness, results for the pressure error are shown in Table 3. Here, the linear dependency on $\nu_{\max}$ as predicted by (15) is visible. In addition, a slight increase of the errors for increasing ratios $\nu_{\max}/\nu_{\min}$ can be seen.

5 Summary and outlook

This note studied finite element methods for the Stokes equations with variable viscosity. Error bounds were derived which depend on the ratio $\nu_{\max}/\nu_{\min}$. Numerical studies show that this ratio might be observed sometimes. However, also

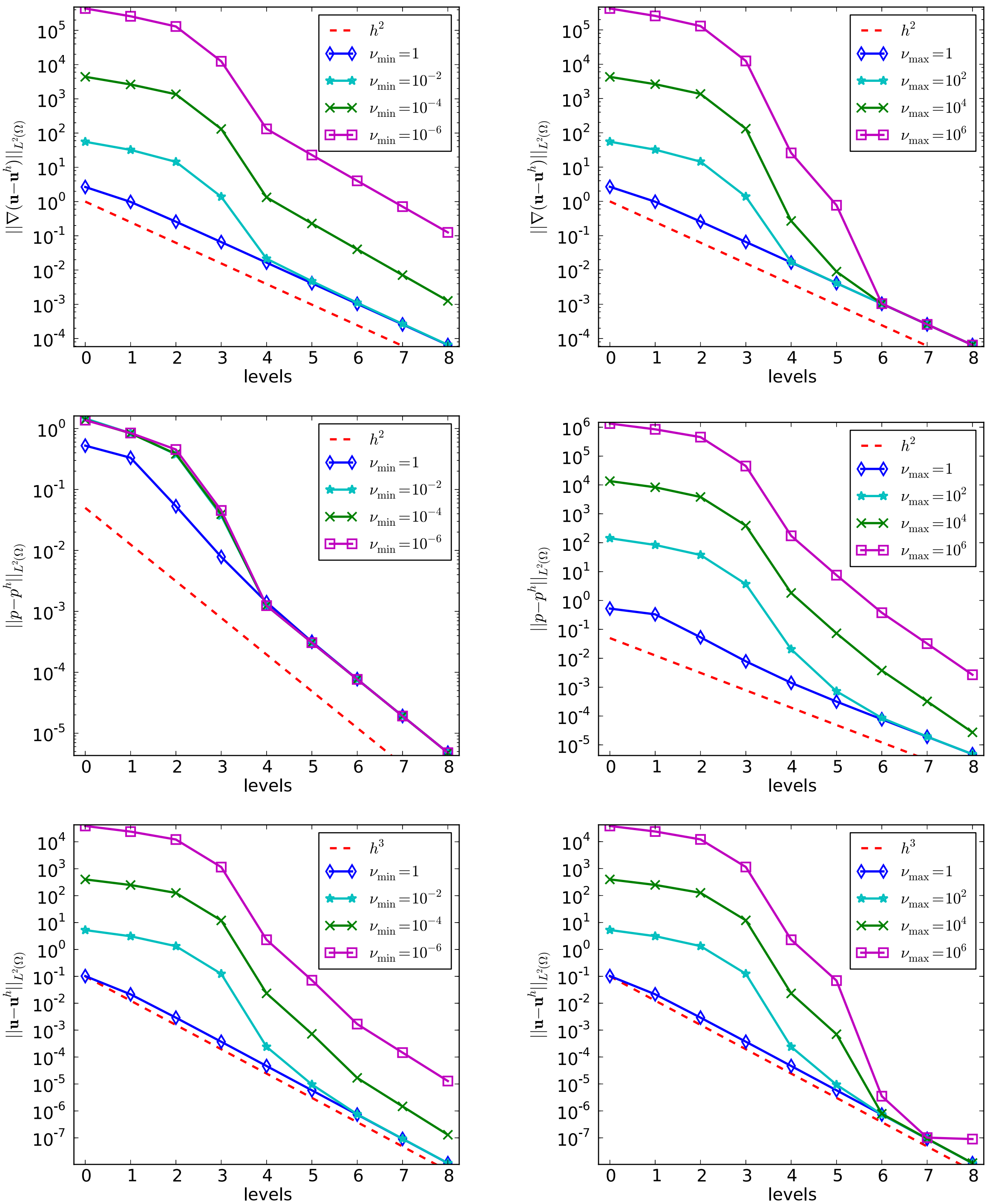

Fig. 4 Example 4.2: Viscosity with steep layer ν_2, mainly taking values close to $\nu_{\min}$ with $\nu_{\max} = 1$ and different values of $\nu_{\min}$ (left), with $\nu_{\min} = 1$ and different values of $\nu_{\max}$ (right).

depending on the actual form of the viscosity, often the numerical results show a weaker dependency on $\nu_{\max}/\nu_{\min}$, such that the derived estimates can be considered as worst case estimates. In the example whose viscosity function comes closest to the situation appearing in mantle convection, the computed velocities showed errors which are almost independent of $\nu_{\max}/\nu_{\min}$.

Future plans include the numerical analysis of models from mantle convection. Then one has to consider coupled problems where the viscosity depends in a nonlinear way on the solution of the problem.

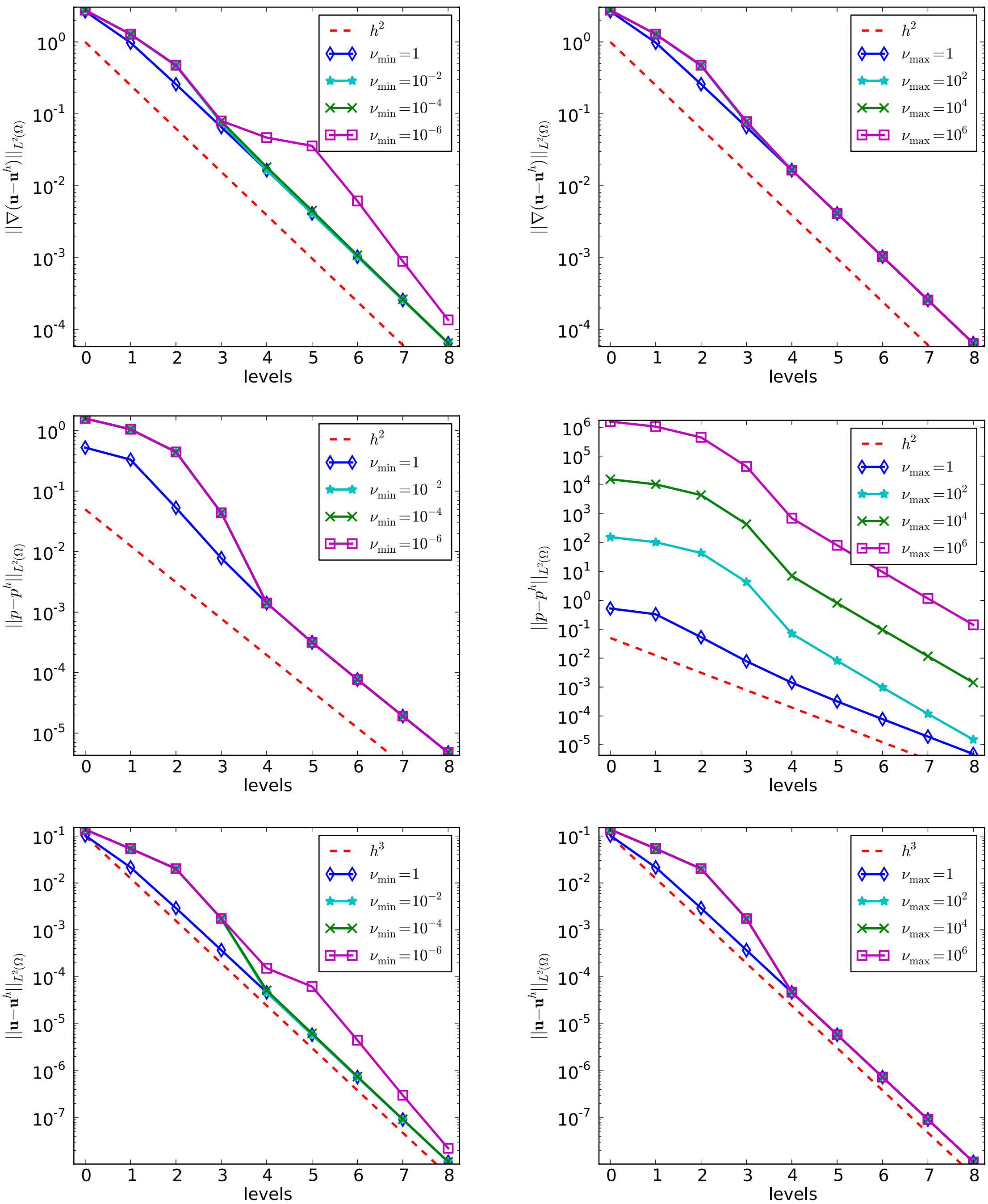

Fig. 5 Example 4.3: Viscosity with steep layer ν_3, mainly taking values close to $\nu_{\max}$ with $\nu_{\max} = 1$ and different values of $\nu_{\min}$ (left), with $\nu_{\min} = 1$ and different values of $\nu_{\max}$ (right).

Table 1 Example 4.4: Viscosity with steep layer ν_2, $p = 0$, $\|\nabla(\boldsymbol{u} - \boldsymbol{u}^h)\|_0$, level 5.

$\nu_{\min} \mid \nu_{\max}$	1	10^2	10^4	10^6
1	4.126828e-3	4.130606e-3	8.942451e-3	7.706366e-1
10^{-2}	4.130606e-3	8.942451e-3	7.706366e-1	7.569407e+1
10^{-4}	8.942451e-3	7.706366e-1	7.569399e+1	7.465854e+3
10^{-6}	7.706366e-1	7.569399e+1	7.465874e+3	7.407314e+5

Table 2 Example 4.4: Viscosity with steep layer ν_2, $p = 0$, $\|\boldsymbol{u} - \boldsymbol{u}^h\|_0$, level 5.

$\nu_{\min} \mid \nu_{\max}$	1	10^2	10^4	10^6
1	5.837456e-6	9.401190e-6	7.040169e-4	6.922841e-2
10^{-2}	9.401190e-6	7.040170e-4	6.922841e-2	6.831881e+0
10^{-4}	7.040170e-4	6.922841e-2	6.831875e+0	6.764145e+2
10^{-6}	6.922841e-2	6.831874e+0	6.764165e+2	6.718501e+4

Table 3 Example 4.4: Viscosity with steep layer ν_2, $p = 0$, $\|p - p^h\|_0$, level 5.

$\nu_{\min} \mid \nu_{\max}$	1	10^2	10^4	10^6
1	8.097410e-5	6.521591e-4	7.316504e-2	7.526082e+0
10^{-2}	6.521591e-6	7.316504e-4	7.526082e-2	7.564355e+0
10^{-4}	7.316504e-6	7.526082e-4	7.564354e-2	7.614313e+0
10^{-6}	7.526082e-6	7.564354e-4	7.614344e-2	7.925653e+0

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References

[1] I. Babuška, Error-bounds for finite element method, Numer. Math. **16**, 322–333 (1970/1971).
[2] F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge **8**(R-2), 129–151 (1974).
[3] C. Burstedde, O. Ghattas, G. Stadler, T. Tu, and L. C. Wilcox, Parallel scalable adjoint-based adaptive solution of variable-viscosity stokes flow problems, Comput. Methods App. Mech. Eng. **198**(21–26), 1691–1700 (2009). Advances in Simulation-Based Engineering Sciences Honoring J. Tinsley Oden.
[4] V. Girault and P.-A. Raviart, Finite Element Methods for Navier-Stokes Equations, volume 5 of Springer Series in Computational Mathematics Theory and Algorithms (Springer-Verlag, Berlin, 1986).
[5] P. P. Grinevich and M. A. Olshanskii, An iterative method for the Stokes-type problem with variable viscosity, SIAM J. Sci. Comput. **31**(5), 3959–3978 (2009).
[6] C. O. Horgan, Korn's inequalities and their applications in continuum mechanics, SIAM Rev. **37**(4), 491–511 (1995).
[7] J. Hron, J. Málek, J. Nečas, and K. R. Rajagopal, Numerical simulations and global existence of solutions of two-dimensional flows of fluids with pressure- and shear-dependent viscosities, Math. Comput. Simulation **61**(3–6), 297–315 (2003). MODELLING 2001 (Pilsen).
[8] V. John, Higher order finite element methods and multigrid solvers in a benchmark problem for the 3D Navier-Stokes equations, Internat. J. Numer. Methods Fluids **40**(6), 775–798 (2002).
[9] V. John and G. Matthies, MooNMD—a program package based on mapped finite element methods, Comput. Vis. Sci. **6**(2-3), 163–169 (2004).
[10] K. Kaiser, Finite Element Methods for the Incompressible Stokes Equations with Non-Constant Viscosity Master's Thesis (Freie Universität Berlin, 2014).
[11] M. Kronbichler, T. Heister, and W. Bangerth, High accuracy mantle convection simulation through modern numerical methods, Geophys. J. Int. **191**(1), 12–29 (2012).
[12] A. Linke, On the role of the Helmholtz decomposition in mixed methods for incompressible flows and a new variational crime, Comput. Methods Appl. Mech. Engrg. **268**, 782–800 (2014).
[13] A. Linke, G. Matthies, and L. Tobiska, Robust Arbitrary Order Mixed Finite Element Methods for the Incompressible Stokes Equations Preprint 2027 (Weierstrass Institute for Applied Analysis and Stochastics, 2014).
[14] G. Lube and G. Rapin, Residual-based stabilized higher-order FEM for a generalized Oseen problem, Math. Models Methods Appl. Sci. **16**(7), 949–966 (2006).
[15] J. Málek, J. Nečas, and K. R. Rajagopal, Global existence of solutions for flows of fluids with pressure and shear dependent viscosities, Appl. Math. Lett. **15**(8), 961–967 (2002).
[16] C. E. Pérez, J.-M. Thomas, S. Blancher, and R. Creff, The steady Navier-Stokes/energy system with temperature-dependent viscosity. II. The discrete problem and numerical experiments, Internat. J. Numer. Methods Fluids **56**(1), 91–114 (2008).
[17] O. Schenk, M. Bollhöfer, and R. A. Römer, On large-scale diagonalization techniques for the anderson model of localization, SIAM Rev. **50**(1), 91–112 (2008).