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The Commutation Error of the Space Averaged Navier–Stokes Equations on a Bounded Domain

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Abstract. In Large Eddy Simulation of turbulent flows, the Navier–Stokes equations are convolved with a filter and differentiation and convolution are interchanged, introducing an extra commutation error term, which is nearly universally dropped from the resulting equations. We show that the commutation error is asymptotically negligible in $L^p(\mathbb{R}^d)$ (i.e., it vanishes as the averaging radius $\delta \to 0$) if and only if the fluid and the boundary exert exactly zero force on each other. Next, we show that the commutation error tends to zero in $H^{-1}(\Omega)$ as $\delta \to 0$. Convergence is proven also for a weak form of the commutation error. The order of convergence is studied in both cases. Last, we study the influence of the commutation error on the energy balance of the filtered equations.

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1. Introduction

The space averaged Navier–Stokes equations for the space averaged fluid velocity $\overline{\mathbf{u}}$ and pressure $\overline{p}$ are the basic equations for large eddy simulation (LES) of turbulent flows. They are derived in many papers and in nearly every book on turbulence modeling, e.g. Aldama [2], Lesieur [20], Pope [21] or Sagaut [23], from the Navier–Stokes equations as follows:

1. One chooses a filter $g(\mathbf{x})$ and an averaging radius $\delta > 0$. The large eddies $\overline{\mathbf{u}}$ (of size $\geq O(\delta)$) are defined by filtering the underlying fluid velocity $\mathbf{u}$:

$$\overline{\mathbf{u}} := g * \mathbf{u}.$$

2. To derive the equations for $\overline{\mathbf{u}}$, the Navier–Stokes equations are convolved with $g(\cdot)$.

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3. Ignoring boundaries and commuting convolution and differentiation leads to the space averaged Navier–Stokes equations, given by
$$\overline{\mathbf{u}}_t - \nabla\cdot\mathbb{S}(\overline{\mathbf{u}},\overline{p}) + \nabla\cdot(\overline{\mathbf{u}\mathbf{u}^T}) = \overline{\mathbf{f}}, \quad \nabla\cdot\overline{\mathbf{u}} = 0, \tag{1}$$
where the stress tensor associated with the velocity and pressure averages $(\overline{\mathbf{u}},\overline{p})$ is given by
$$\mathbb{S}(\overline{\mathbf{u}},\overline{p}) := 2\nu\mathbb{D}\left(\overline{\mathbf{u}}\right) - \overline{p}\mathbb{I} \ \text{ where } \ \mathbb{D}(\overline{\mathbf{u}}) = \frac{\nabla\overline{\mathbf{u}} + \nabla\overline{\mathbf{u}}^T}{2} \tag{2}$$
is the velocity deformation tensor.

One central problem in LES is the closure problem of modeling $\nabla\cdot(\overline{\mathbf{u}\mathbf{u}^T})$ in terms of $\overline{\mathbf{u}}$, see, e.g., Sagaut [23]. We shall show herein that there is in fact another possibly serious closure problem in steps 2 and 3 above leading to the incorrect space filtered equations (1).

It is often reported in the LES literature that difficulties exist for simulating turbulence driven by interaction of flows with boundaries. In this report, we will show one reason: when the flow is given in a bounded domain with typical no-slip boundary conditions and the strong form of the space averaged Navier–Stokes equations is used, steps 2 and 3 lead to an $O(1)$ error near the boundary. A correct derivation of (1) (Section 2) reveals that an extra commutation error term $A_\delta(\mathbb{S}(\mathbf{u},p))$, see Definition 2.1, must be included in (1). We show, Proposition 4.2, that $\|A_\delta(\mathbb{S}(\mathbf{u},p))\|_{L^p(\mathbb{R}^d)} \to 0$ as $\delta\to 0$ if and only if the traction or Cauchy stress vector of the underlying flow is identically zero on the boundary of the domain! In other words, the equations (1) are reasonable only for flows in which the domain's boundary exerts no influence on the flow.

An inspection of the proof of Proposition 4.3 reveals that the commutation error $A_\delta(\mathbb{S}(\mathbf{u},p))$ is largest at the boundary and decays rapidly as one moves away from the boundary.

If the commutation error term is simply dropped and then the strong form of the space averaged Navier–Stokes equations is discretized, as by, e.g., a finite difference method, the results of Section 4 show that the error committed is $O(1)$. On the other hand, variational methods, such as finite element, spectral or spectral element methods, discretize the weak form of the relevant equations. These methods are known to depend on the size of the H^{-1}-norm of any omitted terms. We show in Section 5 that variational methods are possible since the $H^{-1}(\Omega)$-norm of the dropped commutation error does approach zero as $\delta\to 0$, Proposition 5.1.

Section 6 studies the weak form of the commutation error, $(A_\delta(\mathbb{S}(\mathbf{u},p)),\mathbf{v})$ for $\mathbf{v}$ fixed. The third main result, Proposition 6.1, is that the weak form of the commutation error tends to zero as $\delta\to 0$. The order of convergence in two dimensions is at least $O(\delta^{1-\varepsilon})$ with arbitrary $\epsilon > 0$.

The issue of the commutation error has appeared occasionally in the engineering community, e.g. see Fureby and Tabor [9], Ghosal and Moin [12], or Vasilyev et al. [25]. Its critical importance is beginning to be realized, see Das and Moser [6]. One approach, [12, 25], has been to shrink the averaging radius $\delta(\mathbf{x})$ as $\mathbf{x}$

tends to the boundary of the domain; the correct boundary conditions are then clear: $\overline{\mathbf{u}} = \mathbf{0}$. This approach requires extra resolution and another commutation error due to the non-constant filter width occurs. This other commutation error is usually ignored in the engineering literature on the basis of a one-dimensional Taylor series estimation of it for very smooth functions. Interesting and important mathematical challenges remain for this approach as well.

Other special treatments of the near wall regions, such as near wall models, see [23, Section 9.2.2] for an overview, are common in LES to attempt to correct for the error. Recently, there are new approaches to LES without modeling, such as post processing [16] and the variational multiscale method by Hughes and co-workers [15].

2. The space averaged Navier–Stokes equations in a bounded domain

To derive the correct space averaged Navier–Stokes equations in a bounded domain, we will extend all functions to $\mathbb{R}^d$ and derive the equations satisfied by these extensions. Then, the new equations will be convolved.

We will always use standard notations for Sobolev and Lebesgue spaces, e.g. see Adams [1]. For vectors and tensors (matrices), we use standard matrix-vector notations.

Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, with Lipschitz boundary $\partial\Omega$ with outward pointing unit normal $\mathbf{n}$ and $(d-1)$-dimensional measure $|\partial\Omega| < \infty$. We consider the incompressible Navier–Stokes equations with homogeneous Dirichlet boundary conditions

$$\begin{aligned} \mathbf{u}_t - 2\nu\nabla\cdot\mathbb{D}(\mathbf{u}) + \nabla\cdot(\mathbf{u}\mathbf{u}^T) + \nabla p &= \mathbf{f} \quad \text{in } (0,T)\times\Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } [0,T]\times\Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{in } [0,T]\times\partial\Omega, \\ \mathbf{u}\,|_{t=0} &= \mathbf{u}_0 \ \text{ in } \Omega, \\ \textstyle\int_\Omega p \,\mathrm{d}\mathbf{x} &= 0 \quad \text{in } (0,T], \end{aligned} \tag{3}$$

where ν is the constant kinematic viscosity.

It will be helpful to recall that the stress tensor $\mathbb{S}(\mathbf{u}, p)$ is given by

$$\mathbb{S}(\mathbf{u}, p) := 2\nu\mathbb{D}(\mathbf{u}) - p\mathbb{I},$$

where $\mathbb{I}$ is the unit tensor, and that the normal stress / Cauchy stress / traction vector on $\partial\Omega$ is defined by $\mathbb{S}(\mathbf{u}, p)\mathbf{n}$.

Our analysis will require that solutions $(\mathbf{u}, p)$ of (3) are regular enough such that the normal stress has a well defined trace on the $\partial\Omega$ which belongs to some

Lebesgue space defined on $\partial\Omega$. We assume that

$$\begin{aligned} \mathbf{u} \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right)^d, \quad p \in H^1(\Omega) \cap \mathrm{L}_0^2(\Omega) \text{ for a.e. } t \in [0,T], \\ \mathbf{u} \in \left(H^1((0,T))\right)^d \text{ for a.e. } \mathbf{x} \in \overline{\Omega}. \end{aligned} \tag{4}$$

Lemma 2.1. *If* (4) *holds then* $\mathbb{S}(\mathbf{u},p)\mathbf{n}$ *belongs to* $\left(H^{1/2}(\partial\Omega)\right)^d$. *In particular, for a.e.* $t \in (0,T]$, $\mathbb{S}(\mathbf{u},p)\mathbf{n} \in (L^q(\partial\Omega))^d$ *with* $1 \le q < \infty$ *if* $d = 2$ *and* $1 \le q \le 4$ *if* $d = 3$ *and*

$$\|\mathbb{S}(\mathbf{u},p)\mathbf{n}\|_{(L^q(\partial\Omega))^d} \le C\left(\nu\|\mathbf{u}\|_{(H^2(\Omega))^d} + \|p\|_{H^1(\Omega)}\right). \tag{5}$$

Proof. This follows from the usual trace theorem and embedding theorems, e.g., see Galdi [10, Chapter II, Theorem 3.1]. □

Remark 2.1. The result that $\mathbb{S}(\mathbf{u},p)\mathbf{n} \in (L^q(\partial\Omega))^d$ for $1 \le q < 4$ suffices for our purposes but it can be sharpened considerably. For example, Giga and Sohr [13, Theorem 3.1, p. 84] show that provided $\mathbf{f}$ is smooth enough and the initial condition $\mathbf{u}_0 \in \left(W^{2-2/s,s}(\Omega)\right)^d$, $s > 0$, holds, then for a.e. $t > 0$, $\mathbf{u}_t$ and $\nabla \cdot (\mathbf{u}\mathbf{u}^T)$ belong to $(L^q(\Omega))^d$ and further $\mathbb{S}(\mathbf{u},p)\mathbf{n} \in (L^q(\partial\Omega))$ for a.e. $t > 0$ when $3/q + 2/s = 4$.

In writing down an equation like (1), $\mathbf{f}$ must be extended off Ω and then $(\mathbf{u},p)$ must be extended compatible with the extension of $\mathbf{f}$. For $\overline{\mathbf{f}}$ to be computable, $\mathbf{f}$ is extended by zero off Ω. Thus, $(\mathbf{u},p)$ must be extended by zero off Ω, too. This extension is reasonable since $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$. An extension of $\mathbf{u}$ off Ω as an $\left(H^2(\mathbb{R}^d)\right)^d$ function exists but is unknown, in particular since $\mathbf{u}$ is not known. Using this extension, instead of $\mathbf{u} \equiv \mathbf{0}$ on $\mathbb{R}^d \setminus \Omega$, would make the extension of $\mathbf{f}$ unknowable and hence $\overline{\mathbf{f}}$ uncomputable in (1). Thus, define

$$\mathbf{u} = \mathbf{0}, \quad \mathbf{u}_0 = \mathbf{0}, \quad p = 0 \quad \mathbf{f} = \mathbf{0} \quad \text{if } \mathbf{x} \notin \overline{\Omega}.$$

The extended functions posses the following regularities

$$\begin{aligned} \mathbf{u} \in \left(H_0^1(\mathbb{R}^d)\right)^d, \quad p \in \mathrm{L}_0^2(\mathbb{R}^d) \text{ for a.e. } t \in [0,T], \\ \mathbf{u} \in \left(H^1((0,T))\right)^d \text{ for a.e. } \mathbf{x} \in \mathbb{R}^d. \end{aligned} \tag{6}$$

From (4) and (6) follow that the first order weak derivatives of the extended velocity $\mathbf{u}_t$, $\nabla\mathbf{u}$,$\nabla \cdot \mathbf{u}$ and $\nabla \cdot (\mathbf{u}\mathbf{u}^T)$ are well defined on $\mathbb{R}^d$, taking their indicated values in Ω and being identically zero off Ω.

Since $\mathbf{u} \notin \left(H^2(\mathbb{R}^d)\right)^d$, $p \notin H^1(\mathbb{R}^d)$, the terms $\nabla \cdot \mathbb{D}(\mathbf{u})$ and ∇p must be defined in the sense of distributions. To this end, let $\varphi \in C_0^\infty(\mathbb{R}^d)$. Since $p \equiv 0$ on $\mathbb{R}^d \setminus \Omega$, we get

$$(\nabla p)(\varphi) := -\int_{\mathbb{R}^d} p(\mathbf{x})\nabla\varphi(\mathbf{x})\mathrm{d}\mathbf{x} = \int_\Omega \varphi(\mathbf{x})\nabla p(\mathbf{x})\mathrm{d}\mathbf{x} - \int_{\partial\Omega}\varphi(\mathbf{s})p(\mathbf{s})\mathbf{n}(\mathbf{s})\mathrm{d}\mathbf{s}. \quad (7)$$

In the same way, one obtains

$$\begin{aligned}\nabla\cdot\mathbb{D}(\mathbf{u})(\varphi) &:= -\int_{\mathbb{R}^d}\mathbb{D}(\mathbf{u})(\mathbf{x})\nabla\varphi(\mathbf{x})\mathrm{d}\mathbf{x} \quad (8)\\ &= \int_\Omega\varphi(\mathbf{x})\nabla\cdot\mathbb{D}(\mathbf{u})(\mathbf{x})\mathrm{d}\mathbf{x} - \int_{\partial\Omega}\varphi(\mathbf{s})\mathbb{D}(\mathbf{u})(\mathbf{s})\mathbf{n}(\mathbf{s})\mathrm{d}\mathbf{s}.\end{aligned}$$

Both distributions have compact support. From (7) and (8) it follows that the extended functions $(\mathbf{u}, p)$ fulfill the following distributional form of the momentum equation

$$\mathbf{u}_t - 2\nu\nabla\cdot\mathbb{D}(\mathbf{u}) + \nabla\cdot(\mathbf{u}\mathbf{u}^T) + \nabla p = \mathbf{f} + \int_{\partial\Omega}\Big(2\nu\mathbb{D}(\mathbf{u})(\mathbf{s})\mathbf{n}(\mathbf{s}) - p(\mathbf{s})\mathbf{n}(\mathbf{s})\Big)\varphi(\mathbf{s})\mathrm{d}\mathbf{s}. \quad (9)$$

The correct space averaged Navier–Stokes equations are now derived by convolving (9) with a filter function $g(\mathbf{x}) \in \mathrm{C}^\infty(\mathbb{R}^d)$. Let $H(\varphi)$ be a distribution with compact support which has the form

$$H(\varphi) = -\int_{\mathbb{R}^d} f(\mathbf{x})\partial_\alpha\varphi(\mathbf{x})\mathrm{d}\mathbf{x},$$

where ∂_α is the derivative of φ with the multi-index α. Then, $H * g \in C^\infty(\mathbb{R}^d)$, see Rudin [22, Theorem 6.35], where

$$\overline{H}(\mathbf{x}) = (H * g)(\mathbf{x}) := H(g(\mathbf{x} - \cdot)) = -\int_{\mathbb{R}^d} f(\mathbf{y})\partial_\alpha g(\mathbf{x} - \mathbf{y})\mathrm{d}\mathbf{y}. \quad (10)$$

Applying the convolution with g to (9), using the fact that convolution and differentiation commute on $\mathbb{R}^d$, Hörmander [14, Theorem 4.1.1], and convolving the extra term on the right hand side accordingly to (10), we obtain the space averaged momentum equation

$$\begin{aligned}&\overline{\mathbf{u}}_t - 2\nu\nabla\cdot\mathbb{D}(\overline{\mathbf{u}}) + \nabla\cdot(\overline{\mathbf{u}\mathbf{u}^T}) + \nabla\overline{p}\\ &= \overline{\mathbf{f}} + \int_{\partial\Omega} g(\mathbf{x} - \mathbf{s})\left[2\nu\mathbb{D}(\mathbf{u})(\mathbf{s})\mathbf{n}(\mathbf{s}) - p(\mathbf{s})\mathbf{n}(\mathbf{s})\right]d\mathbf{s} \quad \text{in } (0, T] \times \mathbb{R}^d. \quad (11)\end{aligned}$$

Remark 2.2. If the viscous term in the Navier–Stokes equations is written as $\nu\Delta\mathbf{u}$ instead of $2\nu\nabla\cdot\mathbb{D}(\mathbf{u})$, the resulting space averaged equation is given by replacing $2\nu\mathbb{D}(\overline{\mathbf{u}})$ in (11) by $\nu\nabla\overline{\mathbf{u}}$.

Definition 2.1. The commutation error $A_\delta(\mathbb{S}(\mathbf{u}, p))$ in the space averaged Navier–Stokes equations is defined to be

$$A_\delta(\mathbb{S}(\mathbf{u}, p)) := \int_{\partial\Omega} g(\mathbf{x} - \mathbf{s})(\mathbb{S}(\mathbf{u}, p)\mathbf{n})(\mathbf{s})d\mathbf{s}.$$

The correct space averaged Navier–Stokes equations arising from the Navier–Stokes equations on a bounded domain thus posses an extra boundary integral, $A_\delta(\mathbb{S}(\mathbf{u}, p))$. Omitting this integral results in a commutation error. Including this integral in (1) introduces a new modeling question since it depends on the unknown normal stress on $\partial\Omega$ of $(\mathbf{u}, p)$ and not of $(\overline{\mathbf{u}}, \overline{p})$.

3. The Gaussian filter

We will present the results in the following sections for the Gaussian filter. This filter fits into the framework of Section 2. We shall briefly present the filter's properties that are used in the subsequent analysis in this section.

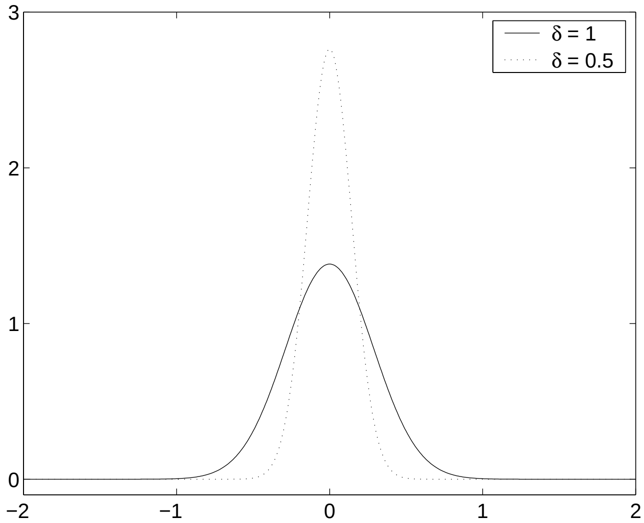

FIG. 1. The Gaussian filter function in one dimension for different δ

The Gaussian filter function has the form

$$g_\delta(\mathbf{x}) = \left(\frac{6}{\delta^2\pi}\right)^{d/2} \exp\left(-\frac{6}{\delta^2}\|\mathbf{x}\|_2^2\right),$$

see Figure 1, where $\|\cdot\|_2$ denotes the Euclidean norm of $\mathbf{x} \in \mathbb{R}^d$ and δ is a user-chosen positive length scale. The Gaussian filter has the following properties, which are easy to verify:

– regularity: $g_\delta \in C^\infty(\mathbb{R}^d)$,
– positivity: $0 < g_\delta(\mathbf{x}) \le \left(\frac{6}{\delta^2\pi}\right)^{\frac{d}{2}}$,
– integrability: $\|g_\delta\|_{L^p(\mathbb{R}^d)} < \infty$, $1 \le p \le \infty$, $\|g_\delta\|_{L^1(\mathbb{R}^d)} = 1$,

– symmetry: $g_\delta(\mathbf{x}) = g_\delta(-\mathbf{x})$,
– monotonicity: $g_\delta(\mathbf{x}) \geq g_\delta(\mathbf{y})$ if $\|\mathbf{x}\|_2 \leq \|\mathbf{y}\|_2$.

Lemma 3.1.

i) *Let $\varphi \in L^p(\mathbb{R}^d)$, then for $1 \leq p < \infty$*

$$\lim_{\delta \to 0} \|g_\delta * \varphi - \varphi\|_{L^p(\mathbb{R}^d)} = 0.$$

ii) *Let $\varphi \in L^\infty(\mathbb{R}^d)$ and if φ is uniformly continuous on a set ω, then $g_\delta * \varphi \to \varphi$ uniformly on ω as $\delta \to 0$.*

iii) *If $\varphi \in C_0^\infty(\mathbb{R}^d)$, then for $1 \leq p < \infty$, $0 \leq r < \infty$*

$$\lim_{\delta \to 0} \|g_\delta * \varphi - \varphi\|_{W^{r,p}(\mathbb{R}^d)} = 0.$$

Proof. The proof of the first two statements can be found, e.g. in Folland [8, Theorem 0.13]. The third statement is an immediate consequence of the first one. □

For convenience, the Gaussian filter function with a scalar argument x is understood in the following to be

$$g_\delta(x) := \left(\frac{6}{\delta^2 \pi}\right)^{\frac{d}{2}} \exp\left(-\frac{6x^2}{\delta^2}\right).$$

4. Error estimates in the $(L^p(\mathbb{R}^d))^d$-norm of the commutation error term

In this section, it is shown that the commutation error $A_\delta(\mathbb{S}(\mathbf{u}, p))$ belongs to $(L^p(\mathbb{R}^d))^d$. We show that $A_\delta(\mathbb{S}(\mathbf{u}, p))$ vanishes as $\delta \to 0$ if and only if the normal stress is identically zero a.e. on $\partial\Omega$. As noted earlier, this condition means the wall have zero influence on the wall-bounded turbulent flow. Thus, it is not expected to be satisfied in any interesting flow problem!

In view of Definition 2.1 and Lemma 2.1, it is necessary to study terms of the form

$$\int_{\partial\Omega} g_\delta(\mathbf{x} - \mathbf{s}) \psi(\mathbf{s}) \mathrm{d}\mathbf{s} \tag{12}$$

with $\psi \in L^q(\partial\Omega)$, $1 \leq q \leq \infty$. We will first show, that (12) belongs to $L^p(\mathbb{R}^d), 1 \leq p \leq \infty$.

Proposition 4.1. *Let $\psi \in L^q(\partial\Omega)$, $1 \leq q \leq \infty$, then (12) belongs to $L^p(\mathbb{R}^d), 1 \leq p \leq \infty$.*

Proof. By the Cauchy–Schwarz inequality, one obtains with $r^{-1}+q^{-1}=1$, $q>1$,

$$\left|\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})\mathrm{d}\mathbf{s}\right| \le \left(\int_{\partial\Omega} g_\delta^r(\mathbf{x}-\mathbf{s})\mathrm{d}\mathbf{s}\right)^{1/r} \|\psi\|_{L^q(\partial\Omega)}$$
$$= \left(\int_{\partial\Omega} \left(\frac{6}{\delta^2\pi}\right)^{rd/2} \exp\left(-\frac{6r}{\delta^2}\|\mathbf{x}-\mathbf{s}\|_2^2\right)\mathrm{d}\mathbf{s}\right)^{1/r} \|\psi\|_{L^q(\partial\Omega)}.$$

As $2\|\mathbf{x}-\mathbf{s}\|_2^2 \ge \|\mathbf{x}\|_2^2 - 2\|\mathbf{s}\|_2^2$, it follows that

$$\exp\left(-\frac{6r\|\mathbf{x}-\mathbf{s}\|_2^2}{\delta^2}\right) \le \exp\left(3r\frac{-\|\mathbf{x}\|_2^2+2\|\mathbf{s}\|_2^2}{\delta^2}\right),$$

and

$$\begin{aligned}&\left|\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})\mathrm{d}\mathbf{s}\right| \\ &\le \left(\frac{6}{\delta^2\pi}\right)^{d/2} \|\psi\|_{L^q(\partial\Omega)} \left(\int_{\partial\Omega}\exp\left(\frac{6r\|\mathbf{s}\|_2^2}{\delta^2}\right)\mathrm{d}\mathbf{s}\right)^{1/r} \exp\left(-\frac{3\|\mathbf{x}\|_2^2}{\delta^2}\right) \\ &< \infty,\end{aligned} \tag{13}$$

since $\partial\Omega$ is compact and the exponential is a bounded function. This proves the statement for $L^\infty(\mathbb{R}^d)$. The proof for $p\in[1,\infty)$ is obtained by raising both sides of (13) to the power p, integrating on $\mathbb{R}^d$ and using

$$\int_{\mathbb{R}^d} \exp\left(-\frac{3p\|\mathbf{x}\|_2^2}{\delta^2}\right)\mathrm{d}\mathbf{x} < \infty.$$

If $q=1$, we have for $1\le p<\infty$

$$\int_{\mathbb{R}^d}\left|\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})\mathrm{d}\mathbf{s}\right|^p \mathrm{d}\mathbf{x} \le \int_{\mathbb{R}^d}\sup_{\mathbf{s}\in\partial\Omega} g_\delta^p(\mathbf{x}-\mathbf{s})\mathrm{d}\mathbf{x}\ \|\psi\|_{L^1(\partial\Omega)}^p$$
$$= \int_{\mathbb{R}^d} g_\delta^p(d(\mathbf{x},\partial\Omega))\mathrm{d}\mathbf{x}\ \|\psi\|_{L^1(\partial\Omega)}^p.$$

We choose a ball $B(\mathbf{0},R)$ with radius R such that $d(\mathbf{x},\partial\Omega) > \|\mathbf{x}\|_2/2$ for all $\mathbf{x}\notin B(\mathbf{0},R)$. Then, the integral on $\mathbb{R}^d$ is split into a sum of two integrals. The first integral is computed on $B(\mathbf{0},R)$. This is finite since the integrand is a continuous function on $\overline{B}(\mathbf{0},R)$. The second integral on $\mathbb{R}^d\setminus B(\mathbf{0},R)$ is also finite because

$$\int_{\mathbb{R}^d\setminus B(\mathbf{0},R)} g_\delta^p(d(\mathbf{x},\partial\Omega))\mathrm{d}\mathbf{x} \le \int_{\mathbb{R}^d} g_\delta^p\left(\frac{\|\mathbf{x}\|_2}{2}\right)\mathrm{d}\mathbf{x}$$

and the integrability of the Gaussian filter. This concludes the proof for $p<\infty$. For $p=\infty$, we have

$$\begin{aligned}&\operatorname*{ess\,sup}_{\mathbf{x}\in\mathbb{R}^d}\left|\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})\mathrm{d}\mathbf{s}\right| \\ &\le \operatorname*{ess\,sup}_{\mathbf{x}\in\mathbb{R}^d}\operatorname*{ess\,sup}_{\mathbf{s}\in\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\|\psi\|_{L^1(\partial\Omega)} \le g_\delta(\mathbf{0})\|\psi\|_{L^1(\partial\Omega)} < \infty.\end{aligned}$$

□

In the next proposition, we study the behaviour of the $L^p(\mathbb{R}^d)$-norm of (12) for $\delta \to 0$.

Proposition 4.2. *Let $\psi \in L^p(\partial\Omega)$, $1 \le p \le \infty$. A necessary and sufficient condition for*

$$\lim_{\delta\to 0} \left\| \int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})\mathrm{d}\mathbf{s} \right\|_{L^p(\mathbb{R}^d)} = 0, \tag{14}$$

$1 \le p \le \infty$, is that ψ vanishes almost everywhere on $\partial\Omega$.

Proof. It is obvious that the condition is sufficient.

Let (14) hold. From Hölder's inequality, we obtain for an arbitrary function $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned}&\lim_{\delta\to 0} \left| \int_{\mathbb{R}^d} \varphi(\mathbf{x}) \left(\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})\mathrm{d}\mathbf{s} \right) \mathrm{d}\mathbf{x} \right| \\ &\quad \le \lim_{\delta\to 0} \|\varphi\|_{L^q(\mathbb{R}^d)} \left\| \int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})\mathrm{d}\mathbf{s} \right\|_{L^p(\mathbb{R}^d)} = 0 \end{aligned}\tag{15}$$

where $p^{-1} + q^{-1} = 1$. By Fubini's theorem and the symmetry of the Gaussian filter, we have

$$\begin{aligned}&\lim_{\delta\to 0} \int_{\mathbb{R}^d} \varphi(\mathbf{x}) \left(\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})\mathrm{d}\mathbf{s} \right) \mathrm{d}\mathbf{x} \\ &= \lim_{\delta\to 0} \int_{\partial\Omega} \psi(\mathbf{s}) \left(\int_{\mathbb{R}^d} g_\delta(\mathbf{x}-\mathbf{s})\varphi(\mathbf{x})\mathrm{d}\mathbf{x} \right) \mathrm{d}\mathbf{s} = \int_{\partial\Omega} \psi(\mathbf{s})\varphi(\mathbf{s})\mathrm{d}\mathbf{s}.\end{aligned}$$

The last step is a consequence of Lemma 3.1 since $\varphi \in L^\infty(\mathbb{R}^d)$ and φ is uniformly continuous on the compact set $\partial\Omega$. Thus, from (15) follows

$$0 = \left| \int_{\partial\Omega} \psi(\mathbf{s})\varphi(\mathbf{s})\mathrm{d}\mathbf{s} \right|$$

for every $\varphi \in C_0^\infty(\mathbb{R}^d)$. This is true if and only if $\psi(\mathbf{s})$ vanishes almost everywhere on $\partial\Omega$. □

We will now bound the $L^p(\mathbb{R}^d)$-norm of (12) in terms of δ. The next lemma proves a geometric property which is needed later.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^d$, $d = 2,3$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Then there exists a constant $C > 0$ such that*

$$\left|\{\mathbf{x} \in \mathbb{R}^d | d(\mathbf{x}, \partial\Omega) \le y\}\right| \le C(y + y^d) \tag{16}$$

for every $y \ge 0$, where $|\cdot|$ denotes the measure in $\mathbb{R}^d$.

Proof. For simplicity, we present the proof for Ω being a simply connected domain. The analysis can be extended to the case that $\partial\Omega$ consists of a finite number of non-connected parts.

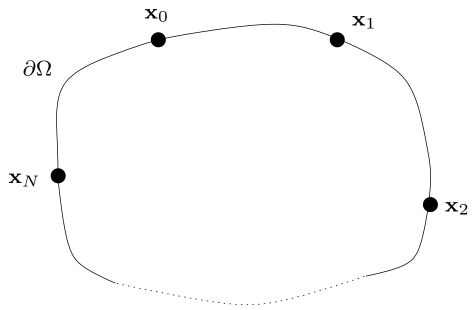

FIG. 2. Mesh on $\partial\Omega$ for $d = 2$

We will start with the case $d = 2$. We fix a point $\mathbf{x}_0$ on $\partial\Omega$ and an orientation of the boundary. Next, we construct $\mathbf{x}_1$ such that the length of the curve between $\mathbf{x}_0$ and $\mathbf{x}_1$ is y. Continuing this construction, we obtain a sequence $(\mathbf{x}_i)_{0\le i\le N}$ such that for every $0 \le i < N$ the length of curve between $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ is y. The length of the curve between $\mathbf{x}_N$ and $\mathbf{x}_0$ is less or equal than y, see Figure 2. The number of intervals is $N + 1$ with $N < |\partial\Omega|/y \le N + 1$. Obviously, we have

$$\{\mathbf{x} \in \mathbb{R}^d | d(\mathbf{x}, \partial\Omega) \le y\} = \bigcup_{\mathbf{x}\in\partial\Omega} \overline{B}(\mathbf{x}, y).$$

But for every $\mathbf{x}$ in $\partial\Omega$, there exists an i such that $\mathbf{x}$ is on the part of the curve from $\mathbf{x}_i$ to $\mathbf{x}_{i+1}$ or from $\mathbf{x}_N$ to $\mathbf{x}_0$. By the triangle inequality, this implies $\overline{B}(\mathbf{x}, y) \subset \overline{B}(\mathbf{x}_i, 2y)$. Thus

$$\{\mathbf{x} \in \mathbb{R}^d | d(x, \partial\Omega) \le y\} \subset \bigcup_{0\le i\le N} \overline{B}(\mathbf{x}_i, 2y),$$

from which

$$\begin{aligned} |\{\mathbf{x} \in \mathbb{R}^d | d(\mathbf{x}, \partial\Omega) \le y\}| &\le \sum_{i=0}^{N} |\overline{B}(\mathbf{x}_i, 2y)| < \left(\frac{|\partial\Omega|}{y} + 1\right) 4\pi y^2 \\ &= 4\pi|\partial\Omega|y + 4\pi y^2 \end{aligned}$$

follows.

In the case $d = 3$, $\partial\Omega$ is a compact manifold. Then, for every $\mathbf{x} \in \partial\Omega$, there exists a neighborhood $U_{\mathbf{x}} \subset \partial\Omega$ such that its closure $\overline{U}_{\mathbf{x}}$ is homeomorphic to a closed square $\overline{V}_{\mathbf{x}} \subset \mathbb{R}^2$ through the homeomorphism $\phi_{\mathbf{x}} : \overline{V}_{\mathbf{x}} \to \overline{U}_{\mathbf{x}}$. The homeomorphism is Lipschitz continuous with a constant L. We cover the manifold by

$$\partial\Omega = \bigcup_{\mathbf{x}\in\partial\Omega} U_{\mathbf{x}}$$

and, because $\partial\Omega$ is compact, we can choose a finite cover $(U_{\mathbf{x}_i})_{0\le i\le N}$ which will be fixed. Let the length of the sides of $\overline{V}_{\mathbf{x}_i}$ be equal to a_i. We create a mesh

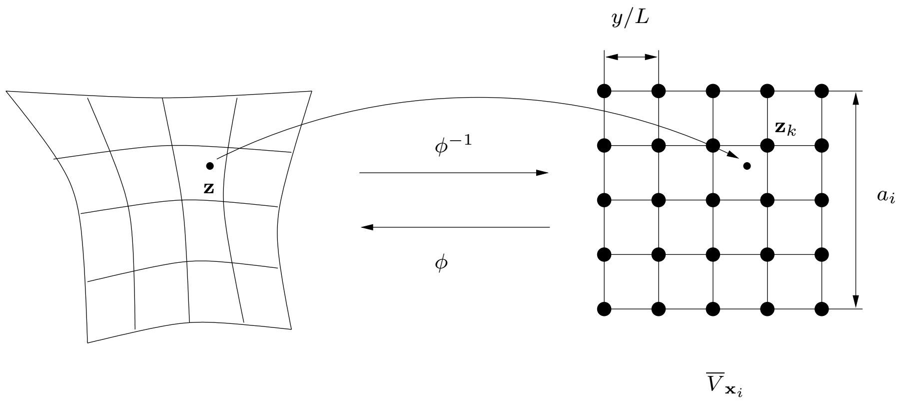

FIG. 3. Homeomorphic map to the square $\overline{V}_{\mathbf{x}_i}$, $d = 3$

over on $\overline{V}_{\mathbf{x}_i}$ of cells of size y/L (or smaller). On this mesh, there are less than $(a_iL/y+2)^2$ vertices and we denote them by $(\mathbf{z}_j)_{0\le j\le P_i}$ where $P_i < (a_iL/y+2)^2$. The order of the vertices is not important. Next, we consider $\mathbf{z} \in U_{\mathbf{x}_i}$. Then, we find the closest vertex on the mesh to $\phi^{-1}(\mathbf{z})$ and denote it by $\mathbf{z}_k$. It is easy to see that

$$||\mathbf{z}_k - \phi^{-1}(\mathbf{z})||_2 \le \frac{y}{L}$$

and the Lipschitz continuity of ϕ gives

$$||\phi(\mathbf{z}_k) - \mathbf{z}||_2 \le L||\mathbf{z}_k - \phi^{-1}(\mathbf{z})||_2 \le y.$$

By the triangle inequality follows now

$$B(\mathbf{z}, y) \subset B(\phi(\mathbf{z}_k), 2y). \tag{17}$$

Because $\mathbf{z} \in U_{\mathbf{x}_i}$ was chosen arbitrary, for every $\mathbf{z} \in U_{\mathbf{x}_i}$ there exists $\mathbf{z}_k \in \overline{V}_{\mathbf{x}_i}$ such that (17) holds. Combining (17) for $U_{\mathbf{x}_i}$, $0 \le i \le N$, gives

$$\{\mathbf{x} \in \mathbb{R}^3 | d(\mathbf{x}, \partial\Omega) \le y\} \subset \bigcup_{0\le i\le N} \bigcup_{0\le k\le P_i} \overline{B}(\phi(\mathbf{z}_k), 2y).$$

By the sub-additivity and monotonicity of Lebesgue measure, we obtain

$$\begin{aligned}|\{\mathbf{x} \in \mathbb{R}^3 | d(\mathbf{x}, \partial\Omega) \le y\}| &\le \sum_{i=0}^{N}\sum_{k=0}^{P_i} |\overline{B}(\phi(z_k), 2y)| \le \sum_{i=0}^{N} \left(\frac{a_iL}{y} + 2\right)^2 \frac{4}{3}\pi y^3 \\ &\le C(y^3 + y)\end{aligned}$$

for an appropriately chosen positive constant C. Note, the quadratic term in y can be absorbed into the linear term for $y \le 1$ and into the cubic term for $y > 1$. $\square$

Proposition 4.3. *Let Ω be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$, $\psi \in L^p(\partial\Omega)$ for some $p > 1$ and $p^{-1} + q^{-1} = 1$. Then for every $\alpha \in (0,1)$ and $k \in (0,\infty)$ there exist constants $C > 0$ and $\epsilon > 0$ such that*

$$\int_{\mathbb{R}^d} \left| \int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})\mathrm{d}\mathbf{s} \right|^k \mathrm{d}\mathbf{x} \le C\delta^{1+k\left(\frac{(d-1)\alpha}{q}-d\right)} \|\psi\|^k_{L^p(\partial\Omega)} \tag{18}$$

for every $\delta \in (0,\epsilon)$ where C and ϵ depend on α, k and $|\partial\Omega|$.

Proof. We fix an $\alpha \in (0,1)$. From Hölder's inequality, we obtain

$$\int_{\mathbb{R}^d} \left| \int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})\mathrm{d}\mathbf{s} \right|^k \mathrm{d}\mathbf{x} \le \int_{\mathbb{R}^d} \left(\int_{\partial\Omega} g_\delta^q(\mathbf{x}-\mathbf{s})\mathrm{d}\mathbf{s} \right)^{k/q} \mathrm{d}\mathbf{x}\ \|\psi\|^k_{L^p(\partial\Omega)}.$$

Let $B(\mathbf{x}, \delta^\alpha)$ be the ball centered at $\mathbf{x} \in \mathbb{R}^d$ and with radius δ^α. Then, the term containing the Gaussian filter function can be estimated by the triangle inequality

$$\int_{\mathbb{R}^d} \left(\int_{\partial\Omega} g_\delta^q(\mathbf{x}-\mathbf{s})\mathrm{d}\mathbf{s} \right)^{k/q} \mathrm{d}\mathbf{x} \le C(k) \left(\int_{\mathbb{R}^d} B_\delta^k(\mathbf{x})\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d} C_\delta^k(\mathbf{x})\mathrm{d}\mathbf{x} \right) \tag{19}$$

where

$$B_\delta(\mathbf{x}) = \left(\int_{\partial\Omega\cap B(\mathbf{x},\delta^\alpha)} g_\delta^q(\mathbf{x}-\mathbf{s})\mathrm{d}\mathbf{s} \right)^{1/q}, C_\delta(\mathbf{x}) = \left(\int_{\partial\Omega\setminus B(\mathbf{x},\delta^\alpha)} g_\delta^q(\mathbf{x}-\mathbf{s})\mathrm{d}\mathbf{s} \right)^{1/q}$$

with the constant $C(k)$ depending only on k. We estimate the terms in (19) separately.

Using the monotonicity of the Gaussian filter, one can obtain the following inequality

$$C_\delta^k(\mathbf{x}) \le C \begin{cases} g_\delta^k(\delta^\alpha) & \text{if } d(\mathbf{x},\partial\Omega) < \delta^\alpha, \\ g_\delta^k(d(\mathbf{x},\partial\Omega)) & \text{if } d(\mathbf{x},\partial\Omega) \ge \delta^\alpha, \end{cases}$$

where $C = C(|\partial\Omega|)$. We refer to the function behind the brace as bounding function, see Figure 4 for a sketch in a special situation.

Let $C(t) = \{(\mathbf{z}, t) | d(\mathbf{z}, \partial\Omega) \le y, t = g_\delta^k(y), \delta^\alpha \le y < \infty\}$ be the cross section of the bounding function at the function value t and $A(t) = |C(t)|$ the area of the cross section. Then

$$\int_{\mathbb{R}^d} C_\delta^k(\mathbf{x})\mathrm{d}\mathbf{x} \le C \int_0^{g_\delta^k(\delta^\alpha)} A(t)\mathrm{d}t.$$

From Lemma 4.1, we know $A(t) \le C(y^d + y)$, with C depending only on Ω. Using

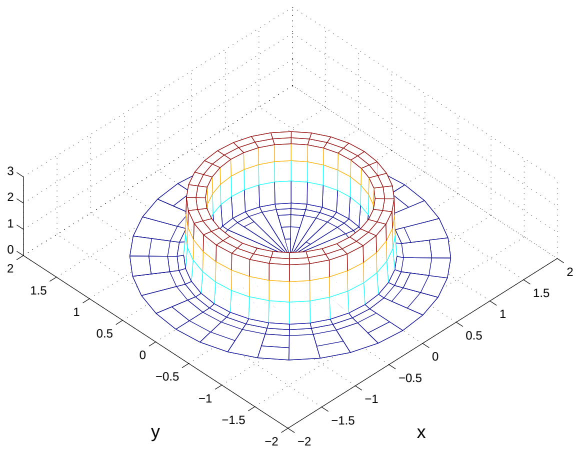

FIG. 4. Bounding function of $C_\delta^k(\mathbf{x})$, $d = 2$, $\partial\Omega = B(\mathbf{0}, 1)$, $\delta = 0.1$, $\alpha = 0.99$, $k = 1$, $C = 2\pi$

$g_\delta^k(y) = t$, changing variables and integrating by parts yield

$$\int_0^{g_\delta^k(\delta^\alpha)} A(t)\mathrm{d}t \le C \int_0^{g_\delta^k(\delta^\alpha)} (y^d + y)\mathrm{d}t = C \int_\infty^{\delta^\alpha} (y^d + y)\frac{\mathrm{d}}{\mathrm{d}y}(g_\delta^k(y))\mathrm{d}y$$
$$= C\left((\delta^{d\alpha} + \delta^\alpha)g_\delta^k(\delta^\alpha) - d\int_\infty^{\delta^\alpha} y^{d-1}g_\delta^k(y)\mathrm{d}y - \int_\infty^{\delta^\alpha} g_\delta^k(y)\mathrm{d}y\right).$$

The integrals on the last line will be estimated using the change of variables $y = \delta/t$ and by monotonicity considerations of the arising integrand. For δ sufficiently small, one obtains

$$\int_{\mathbb{R}^d} C_\delta^k(\mathbf{x})\mathrm{d}\mathbf{x} \le C\left(\delta^{d(\alpha-k)} + \delta^{\alpha-kd}\right)\exp\left(-\frac{6k}{\delta^{2(1-\alpha)}}\right),$$

from what follows, since $\alpha < 1$,

$$\lim_{\delta\to 0}\int_{\mathbb{R}^d} C_\delta^k(\mathbf{x})\mathrm{d}\mathbf{x} = 0.$$

Now we will bound the second term in (19). The function $B_\delta^k(\mathbf{x})$ can be estimated from above in the following way

$$B_\delta^k(\mathbf{x}) \le \begin{cases} |\partial\Omega \cap B(\mathbf{x},\delta^\alpha)|^{\frac{k}{q}} g_\delta^k(d(\mathbf{x},\partial\Omega)) & \text{if } d(\mathbf{x},\partial\Omega) < \delta^\alpha, \\ 0 & \text{if } d(\mathbf{x},\partial\Omega) \ge \delta^\alpha, \end{cases}$$

see Figure 5 for an illustration of the bounding function in a special situation. The bounding function is discontinuous, having a jump from the value 0 to the value $Cg_\delta^k(\delta^\alpha)$ at $\{\mathbf{x} \in \mathbb{R}^d \mid d(\mathbf{x},\partial\Omega) = \delta^\alpha\}$.

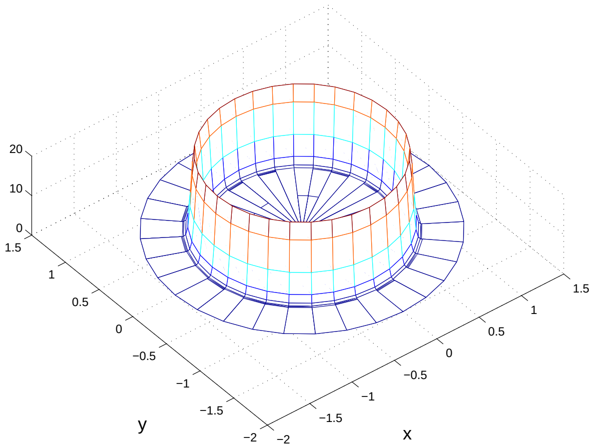

FIG. 5. Bounding function of $B_\delta^k(\mathbf{x})$, $d = 2$, $\partial\Omega = B(\mathbf{0}, 1)$, $\delta = 0.1$, $\alpha = 0.99$, $k = 1$, $C = \delta^\alpha$

Since $\partial\Omega$ is smooth, we have $|\partial\Omega \cap B(\mathbf{x}, \delta^\alpha)| \le C\delta^{(d-1)\alpha}$ if δ is small enough. It follows

$$\int_{\mathbb{R}^d} B_\delta^k(\mathbf{x})\mathrm{d}\mathbf{x} \le C \int_{\{d(\mathbf{x},\partial\Omega)<\delta^\alpha\}} \delta^{\frac{(d-1)\alpha k}{q}} g_\delta^k(d(\mathbf{x}, \partial\Omega))\mathrm{d}\mathbf{x}.$$

We will estimate the integral by integrating over the cross sections of the function in the integral. For the function values t, $0 \le t \le g_\delta^k(\delta^\alpha)$, all cross sections have the same form. For function values $t = g_\delta^k(y)$, $0 \le y < \delta^\alpha$, the cross section is $\{\mathbf{x} \in \mathbb{R}^d \mid d(\mathbf{x}, \partial\Omega) \le y\}$. We denote the area of the cross sections by $A(t)$. Integration of the areas gives

$$\begin{aligned}\int_{\{d(\mathbf{x},\partial\Omega)<\delta^\alpha\}} g_\delta^k(d(\mathbf{x}, \partial\Omega))\mathrm{d}\mathbf{x} &= \int_0^{g_\delta^k(\delta^\alpha)} A(t)\mathrm{d}t + \int_{g_\delta^k(\delta^\alpha)}^{g_\delta^k(0)} A(t)\mathrm{d}t \\ &= A(g_\delta^k(\delta^\alpha))g_\delta^k(\delta^\alpha) + \int_{g_\delta^k(\delta^\alpha)}^{g_\delta^k(0)} A(t)\mathrm{d}t.\end{aligned}$$

We will use now the estimate of the areas of the cross sections given in Lemma 4.1. If y is small enough, the term y^d can be absorbed into the term y in this estimate. Thus, if δ is small enough, we have $|\{\mathbf{x} \in \mathbb{R}^d \mid d(\mathbf{x}, \partial\Omega) \le y\}| \le Cy$, $0 \le y < \delta^\alpha$. We obtain, changing variables and applying integration by parts

$$\int_{g_\delta^k(\delta^\alpha)}^{g_\delta^k(0)} A(t)\mathrm{d}t \le -C \int_0^{\delta^\alpha} y \frac{\mathrm{d}}{\mathrm{d}y} g_\delta^k(y)\mathrm{d}y = C\left(-\delta^\alpha g_\delta^k(\delta^\alpha) + \int_0^{\delta^\alpha} g_\delta^k(y)\mathrm{d}y\right).$$

The last integral can be estimated further with the substitution $y = \delta s$

$$\int_0^{\delta^\alpha} g_\delta^k(y)\mathrm{d}y \le C\delta^{1-kd}. \tag{20}$$

with C depending on k. Collecting estimates, using $A(g_\delta(\delta^\alpha)) \leq C\delta^\alpha$, which results in the cancellation of the terms $\delta^\alpha g_\delta^k(\delta^\alpha)$, we obtain

$$\int_{\mathbb{R}^d} B_\delta^k(\mathbf{x})\mathrm{d}\mathbf{x} \leq C\delta^{1-kd+\frac{(d-1)\alpha k}{q}}.$$

The estimate for $C_\delta(\mathbf{x})$ converges exponentially for $\delta \to 0$. Thus, for δ sufficiently small, the estimate of $B_\delta(\mathbf{x})$ will dominate. This proves the proposition.

□

5. Error estimates in the $(H^{-1}(\Omega))^d$-norm of the commutation error term

The main result of this section is that the commutation error tends to zero in $H^{-1}(\Omega)$ as $\delta \to 0$, see Proposition 5.1. The order of convergence is at least $O(\delta^{1/2})$.

Lemma 5.1. *There exists a constant C, which depends only on d, such that*

$$\|\overline{v} - v\|_{H^{1/2}(\mathbb{R}^d)} \leq C\delta^{1/2}\|v\|_{H^1(\mathbb{R}^d)} \tag{21}$$

for any $v \in H^1(\mathbb{R}^d)$ *and any* $\delta > 0$.

Proof. By using the definition of $\|\cdot\|_{H^{1/2}(\mathbb{R}^d)}$, we have

$$\begin{aligned}
\|\overline{v} - v\|^2_{H^{1/2}(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} (1 + \|\mathbf{x}\|_2^2)^{1/2}|1 - \widehat{g}_\delta|^2|\widehat{v}|^2\mathrm{d}\mathbf{x} \\
&= \int_{\{\|\mathbf{x}\|_2 > \pi/\delta\}} (1 + \|\mathbf{x}\|_2^2)^{1/2}|1 - \widehat{g}_\delta|^2|\widehat{v}|^2\mathrm{d}\mathbf{x} \\
&\quad + \int_{\{\|\mathbf{x}\|_2 \leq \pi/\delta\}} (1 + \|\mathbf{x}\|_2^2)^{1/2}|1 - \widehat{g}_\delta|^2|\widehat{v}|^2\mathrm{d}\mathbf{x},
\end{aligned}$$

where $\widehat{v}$ denotes the Fourier transform of v and the Fourier transform of the Gaussian filter is given by

$$\widehat{g}_\delta(\mathbf{x}) = \exp\left(-\frac{\delta^2}{24}\|\mathbf{x}\|_2^2\right). \tag{22}$$

First, we prove a bound for the first integral. There exists a constant $C > 0$, which does not depend on δ and v, such that $(1 + \|\mathbf{x}\|_2^2)^{-1/2} < C\delta$ for $\|\mathbf{x}\|_2 > \pi/\delta$. From (22) follows the pointwise estimate $|1 - \widehat{g}_\delta(\mathbf{x})| \leq 1$ for any $\mathbf{x} \in \mathbb{R}^d$. Thus, the first integral can be bounded by

$$\begin{aligned}
&\left| \int_{\{\|\mathbf{x}\|_2 > \pi/\delta\}} (1 + \|\mathbf{x}\|_2^2)^{1/2} |1 - \widehat{g}_\delta|^2 |\widehat{v}|^2 \mathrm{d}\mathbf{x} \right| \\
&\le \int_{\{\|\mathbf{x}\|_2 > \pi/\delta\}} (1 + \|\mathbf{x}\|_2^2)(1 + \|\mathbf{x}\|_2^2)^{-1/2} |\widehat{v}|^2 \mathrm{d}\mathbf{x} \\
&\le C\delta \int_{\{\|\mathbf{x}\|_2 > \pi/\delta\}} (1 + \|\mathbf{x}\|_2^2) |\widehat{v}|^2 \mathrm{d}\mathbf{x}. \qquad (23)
\end{aligned}$$

A Taylor series expansion of (22) at $\|\mathbf{x}\|_2 = 0$ and for fixed δ gives

$$\widehat{g}_\delta(\mathbf{x}) = 1 - \frac{\delta^2 \|\mathbf{x}\|_2^2}{24} + O(\delta^4 \|\mathbf{x}\|_2^4),$$

such that we have the pointwise bound

$$|1 - \widehat{g}_\delta(\mathbf{x})|^2 \le C\delta \|\mathbf{x}\|_2$$

for any $\|\mathbf{x}\|_2 \le \pi/\delta$ where C does not depend on δ or $\mathbf{x}$. In addition, $\|\mathbf{x}\|_2 \le (1 + \|\mathbf{x}\|_2^2)^{1/2}$ and consequently the second integral can be bounded as follows

$$\left| \int_{\{\|\mathbf{x}\|_2 \le \pi/\delta\}} (1 + \|\mathbf{x}\|_2^2)^{1/2} |1 - \widehat{g}_\delta|^2 |\widehat{v}|^2 \mathrm{d}\mathbf{x} \right| \le C\delta \int_{\{\|\mathbf{x}\|_2 \le \pi/\delta\}} (1+\|\mathbf{x}\|_2^2) |\widehat{v}|^2 \mathrm{d}\mathbf{x}. \quad (24)$$

Combining (23) and (24) gives

$$\|\overline{v} - v\|_{H^{1/2}(\mathbb{R}^d)}^2 \le C\delta \int_{\mathbb{R}^d} (1 + \|\mathbf{x}\|_2^2) |\widehat{v}|^2 \mathrm{d}\mathbf{x} = C\delta \|v\|_{H^1(\mathbb{R}^d)}. \qquad \square$$

Let $H^{-1}(\Omega)$ be the dual space of $H_0^1(\Omega)$ equipped with the norm

$$\|w\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{\langle v, w \rangle}{\|v\|_{H^1(\Omega)}},$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing.

Proposition 5.1. *Let $\psi \in L^2(\partial\Omega)$, then there exists a constant $C > 0$ which depends only on Ω such that*

$$\left\| \int_{\partial\Omega} g_\delta(\mathbf{x} - \mathbf{s}) \psi(\mathbf{s}) \mathrm{d}\mathbf{s} \right\|_{H^{-1}(\Omega)} \le C\delta^{1/2} \|\psi\|_{L^2(\partial\Omega)}$$

for every $\delta > 0$.

Proof. Let $v \in H_0^1(\Omega)$. Extending v by zero outside Ω, applying Fubini's theorem, using that v vanishes on $\partial\Omega$, applying the Cauchy–Schwarz inequality, the trace theorem and Lemma 5.1, give

$$\begin{aligned}\int_\Omega \left(\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})\mathrm{d}\mathbf{s}\right) v(\mathbf{x})\ d\mathbf{x} &= \int_{\partial\Omega} \psi(\mathbf{s})\overline{v}(\mathbf{s})d\mathbf{s}\\ &= \int_{\partial\Omega} \psi(\mathbf{s})\left(\overline{v}(\mathbf{s})-v(\mathbf{s})\right)d\mathbf{s}\\ &\le \|\overline{v}-v\|_{L^2(\partial\Omega)}\|\psi\|_{L^2(\partial\Omega)}\\ &\le C\|\overline{v}-v\|_{H^{1/2}(\Omega)}\|\psi\|_{L^2(\partial\Omega)}\\ &\le C\delta^{1/2}\|v\|_{H^1(\Omega)}\|\psi\|_{L^2(\partial\Omega)}.\end{aligned}$$

Division by $\|v\|_{H^1(\Omega)}$ and using the definition of the $H^{-1}(\Omega)$ norm gives the desired result. □

Let

$$\mathcal{H} = \left\{v \in H^1(\mathbb{R}^d)\ :\ v|_{\partial\Omega} = 0\right\}$$

and let the assumption of Proposition 5.1 be fulfilled. An inspection of the proof shows that then also

$$\left\|\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})\mathrm{d}\mathbf{s}\right\|_{H^{-1}_{\mathcal{H}}(\mathbb{R}^d)} \le \sup_{v\in\mathcal{H}} \frac{\left\langle v, \int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})\mathrm{d}\mathbf{s}\right\rangle}{\|v\|_{H^1(\mathbb{R}^d)}} \le C\delta^{1/2}\|\psi\|_{L^2(\partial\Omega)}.$$

6. Error estimates for a weak form of the commutation error term

In this section, we consider a weak form of the commutation error term, $A_\delta(\mathbb{S}(\mathbf{u},p))$, multiplied with a suitable test function $\overline{v}(\mathbf{x})$ and integrated on $\mathbb{R}^d$. The following proposition shows that this weak form converges to zero as δ tends to zero for fixed $\overline{v}(\mathbf{x})$. For $d=2$, Corollary 6.1 and Remark 6.1 show that the convergence is (at least) almost of order one if ψ is sufficiently smooth.

Lemma 6.1. *Let $v \in H^1(\mathbb{R}^d)$ such that $v|_\Omega \in H^1_0(\Omega)\cap H^2(\Omega)$ and $v(\mathbf{x}) = 0$ if $\mathbf{x} \notin \overline{\Omega}$ and let $\psi \in L^p(\partial\Omega)$, $1 \le p \le \infty$. Then*

$$\lim_{\delta\to 0}\int_{\mathbb{R}^d} \overline{v}(\mathbf{x})\left(\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})\mathrm{d}\mathbf{s}\right)\mathrm{d}\mathbf{x} = 0,$$

where $\overline{v}(\mathbf{x}) = (g_\delta * v)(\mathbf{x})$.

Proof. By Fubini's theorem and the symmetry of g_δ, we obtain

$$\begin{aligned}&\lim_{\delta\to 0}\int_{\mathbb{R}^d} \overline{v}(\mathbf{x})\left(\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})\mathrm{d}\mathbf{s}\right)\mathrm{d}\mathbf{x}\\ &= \lim_{\delta\to 0}\int_{\partial\Omega}\psi(\mathbf{s})\left(\int_{\mathbb{R}^d} g_\delta(\mathbf{s}-\mathbf{x})\overline{v}(\mathbf{x})\mathrm{d}x\right)\mathrm{d}\mathbf{s}.\end{aligned}$$

By a Sobolev imbedding theorem, we get $v \in L^\infty(\Omega)$ from what follows by the construction of v that $v \in L^\infty(\mathbb{R}^d)$. Hölder's inequality for convolutions gives $\overline{v} \in L^\infty(\mathbb{R}^d)$. In addition, $\overline{v}$ is uniformly continuous on the compact set $\partial\Omega$. The same holds for v since $v \in C^0(\overline{\Omega})$ by a Sobolev imbedding theorem. Applying twice Lemma 3.1 gives

$$\lim_{\delta\to 0}\int_{\mathbb{R}^d}\overline{v}(\mathbf{x})\left(\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})\mathrm{d}\mathbf{s}\right)\mathrm{d}\mathbf{x} = \int_{\partial\Omega}\psi(\mathbf{s})v(\mathbf{s})\mathrm{d}\mathbf{s} = 0,$$

since v vanishes on $\partial\Omega$. □

With the result of Proposition 4.3, we want to study the order of convergence with respect to δ of the weak form of the commutation error term.

Proposition 6.1. *Let v and ψ be defined as in Lemma 6.1 and let the assumption of Proposition 4.3 be fulfilled. Then, there exists an $\epsilon > 0$ such that for $\delta \in (0,\epsilon)$*

$$\int_{\mathbb{R}^d}\left|\overline{v}(\mathbf{x})\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})\mathrm{d}\mathbf{s}\right|^k \mathrm{d}\mathbf{x} \le C\delta^{1+(-d+\frac{(d-1)\alpha}{q}+\beta\alpha)k}\|\psi\|^k_{L^p(\partial\Omega)}\|v\|^k_{H^2(\Omega)},$$

where $k \in [1,\infty)$, $\beta \in (0,1)$ if $d=2$ and $\beta = 1/2$ if $d=3$, $p^{-1}+q^{-1}=1$, $p>1$, and C and ϵ depend on α, k and $|\partial\Omega|$.

Proof. Analogously to the begin of the proof of Proposition 4.3, one obtains

$$\begin{aligned}&\int_{\mathbb{R}^d}\left|\overline{v}(\mathbf{x})\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})\mathrm{d}\mathbf{s}\right|^k \mathrm{d}\mathbf{x}\\ &\le C(k)\left[\int_{\mathbb{R}^d}|\overline{v}(\mathbf{x})B_\delta(\mathbf{x})|^k\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d}|\overline{v}(\mathbf{x})C_\delta(\mathbf{x})|^k\,\mathrm{d}\mathbf{x}\right]\|\psi\|^k_{L^p(\partial\Omega)},\end{aligned}$$

where $B_\delta(\mathbf{x})$ and $C_\delta(\mathbf{x})$ are defined in the proof of Proposition 4.3. The terms on the right hand side are treated separately.

In Proposition 4.3, it is proven that $C_\delta^k \in L^1(\mathbb{R}^d)$ for every $k \in (0,\infty)$. This implies

$$\left(C_\delta^k\right)^p = C_\delta^{kp} = C_\delta^{k'} \in L^1(\mathbb{R}^d),$$

since $k' \in (0,\infty)$. That means $C_\delta^k \in L^p(\mathbb{R}^d)$ for $p \in [1,\infty)$. From the bounding function of C_δ^k it is obvious that $C_\delta^k \in L^\infty(\mathbb{R}^d)$, too. Using Hölder's inequality for convolutions, see Adams [1, Theorem 4.30], and $\|g_\delta\|_{L^1(\mathbb{R}^d)} = 1$, it follows

$$\|\overline{v}\|_{L^q(\mathbb{R}^d)} \le \|g_\delta\|_{L^1(\mathbb{R}^d)}\|v\|_{L^q(\mathbb{R}^d)} = \|v\|_{L^q(\mathbb{R}^d)}$$

where $1 \le q < \infty$. With the same argument, we get for $qk \ge 1$

$$\|\overline{v}^k\|_{L^q(\mathbb{R}^d)} = \|\overline{v}\|^k_{L^{qk}(\mathbb{R}^d)} \le \|v\|^k_{L^{qk}(\mathbb{R}^d)}.$$

By the regularity assumptions on v, it follows $v \in C^0(\mathbb{R}^d)$. This implies, together with $v = 0$ outside Ω, that $v \in L^p(\mathbb{R}^d)$ for every $1 \le p \le \infty$. Consequently,

$\|v\|_{L^{qk}(\mathbb{R}^d)} < \infty$. Applying Hölder's inequality, we obtain

$$\int_{\mathbb{R}^d} |\overline{v}(\mathbf{x})C_\delta(\mathbf{x})|^k \mathrm{d}\mathbf{x} \leq \|v\|^k_{L^{qk}(\mathbb{R}^d)} \|C^k_\delta(\mathbf{x})\|_{L^p(\mathbb{R}^d)}.$$

For the second factor, we can use the bound obtained in the proof of Proposition 4.3, replacing k by kp. Thus if δ is small enough, we obtain

$$\int_{\mathbb{R}^d} |\overline{v}(\mathbf{x})C_\delta(\mathbf{x})|^k \mathrm{d}\mathbf{x} \leq C\delta^{-kd}\left(\delta^{d\alpha} + \delta^\alpha\right)^{1/p} \exp\left(-\frac{6k}{\delta^{2(1-\alpha)}}\right) \|v\|^k_{L^{qk}(\mathbb{R}^d)} \tag{25}$$

for every test function v which satisfies the regularity assumptions stated in Lemma 6.1.

The estimate of the second term starts by noting that the domain of integration can be restricted to a small neighbourhood of $\partial\Omega$

$$\begin{aligned}\int_{\mathbb{R}^d} |\overline{v}(\mathbf{x})B_\delta(\mathbf{x})|^k \mathrm{d}\mathbf{x} &= \int_{\{d(\mathbf{x},\partial\Omega)\leq\delta^\alpha\}} |\overline{v}(\mathbf{x})B_\delta(\mathbf{x})|^k \mathrm{d}\mathbf{x} \\ &\leq \|\overline{v}\|^k_{L^\infty(\{d(\mathbf{x},\partial\Omega)\leq\delta^\alpha\})} \int_{\{d(\mathbf{x},\partial\Omega)\leq\delta^\alpha\}} B^k_\delta(\mathbf{x})\mathrm{d}\mathbf{x} \\ &\leq \|\overline{v}\|^k_{L^\infty(\{d(\mathbf{x},\partial\Omega)\leq\delta^\alpha\})} \delta^{1+(-d+\frac{(d-1)\alpha}{q})k}, \end{aligned} \tag{26}$$

where $\alpha \in (0,1)$ and $p^{-1} + q^{-1} = 1$. The last estimate is taken from the proof of Proposition 4.3. It remains to estimate the norm of $\overline{v}$. By the triangle inequality, we obtain

$$\|\overline{v}\|_{L^\infty(\{d(\mathbf{x},\partial\Omega)\leq\delta^\alpha\})} \leq \|\overline{v} - v\|_{L^\infty(\{d(\mathbf{x},\partial\Omega)\leq\delta^\alpha\})} + \|v\|_{L^\infty(\{d(\mathbf{x},\partial\Omega)\leq\delta^\alpha\})}. \tag{27}$$

Since $v \in H^2(\Omega)$, we have $v \in C^{0,\beta}(\overline{\Omega})$ with $\beta \in (0,1)$ if $d = 2$ and $\beta = 1/2$ if $d = 3$. That means, there exists a constant $C_H \geq 0$ such that

$$|v(\mathbf{x}) - v(\mathbf{y})| \leq C_H \|\mathbf{x} - \mathbf{y}\|^\beta_2 \text{ for all } \mathbf{x}, \mathbf{y} \in \overline{\Omega}.$$

By the Sobolev imbedding theorem, this constant can be estimated by $C_H \leq C(\Omega)\|v\|_{H^2(\Omega)}$. We fix an arbitrary $\mathbf{x} \in \{d(\mathbf{x},\partial\Omega) \leq \delta^\alpha\}$ and we take $\mathbf{y} \in \partial\Omega$ with $\|\mathbf{x} - \mathbf{y}\|_2 = d(\mathbf{x},\mathbf{y})$. Since v vanishes on $\partial\Omega$, we obtain $\|v(\mathbf{x})\|_2 \leq C_H d(\mathbf{x},\partial\Omega)^\beta$. It follows

$$\|v\|_{L^\infty(\{d(\mathbf{x},\partial\Omega)\leq\delta^\alpha\})} \leq C_H \delta^{\alpha\beta}.$$

The first term on the right hand side of (27) is, using that the $L^1(\mathbb{R}^d)$ norm of the Gaussian filter is equal to one,

$$\|\overline{v} - v\|_{L^\infty(\{d(\mathbf{x},\partial\Omega)\leq\delta^\alpha\})} = \operatorname*{ess\,sup}_{\mathbf{x}\in\{d(\mathbf{x},\partial\Omega)\leq\delta^\alpha\}} \left| \int_{\mathbb{R}^d} g_\delta(\mathbf{x}-\mathbf{y})(v(\mathbf{x}) - v(\mathbf{y}))\mathrm{d}\mathbf{y} \right|.$$

Since v vanishes outside Ω, it can be easily proven that

$$|v(\mathbf{x}) - v(\mathbf{y})| \leq C_H \|\mathbf{x} - \mathbf{y}\|^\beta_2$$

holds for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. We obtain, using the symmetry of the Gaussian filter,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} g_\delta(\mathbf{x}-\mathbf{y})(v(\mathbf{x})-v(\mathbf{y}))\mathrm{d}\mathbf{y}\right| \\
&\leq C_H \int_{\mathbb{R}^d} g_\delta(\mathbf{x}-\mathbf{y})\|\mathbf{x}-\mathbf{y}\|_2^\beta \mathrm{d}\mathbf{y} = C_H \int_{\mathbb{R}^d} g_\delta(\delta\mathbf{z})\delta^{\beta+d}\|\mathbf{z}\|_2^\beta \mathrm{d}\mathbf{z} \\
&= C C_H \delta^\beta \int_{\mathbb{R}^d} \exp(-6\|\mathbf{z}\|_2^2)\|\mathbf{z}\|_2^\beta \mathrm{d}\mathbf{z}.
\end{aligned}$$

The last integral is finite. Thus, we can conclude

$$\|\overline{v}-v\|_{L^\infty(\{d(x,\partial\Omega)\leq\delta^\alpha\})} \leq C C_H \delta^\beta.$$

Since δ^β decays faster for small δ than $\delta^{\beta\alpha}$, we obtain the estimate

$$\|\overline{v}\|^k_{L^\infty(\{d(\mathbf{x},\partial\Omega)\leq\delta^\alpha\})} \leq C_H^k \delta^{\beta\alpha k}.$$

Combining this estimate with (26) and using the estimate for C_H, we get

$$\int_{\mathbb{R}^d} |\overline{v}(\mathbf{x})B_\delta(\mathbf{x})|^k \mathrm{d}\mathbf{x} \leq C\delta^{1+(-d+\frac{(d-1)\alpha}{q}+\beta\alpha)k}\|v\|^k_{H^2(\Omega)}.$$

This dominates estimate (25) for small δ. Collecting terms, gives the final result. □

An easy consequence of Proposition 6.1 is the following

Corollary 6.1. *Let the assumptions of Proposition 6.1 be fulfilled. Then, the weak form of the commutation error is bounded:*

$$\left|\int_{\mathbb{R}^d} \overline{v}(\mathbf{x}) \int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})\mathrm{d}\mathbf{s}\right| \mathrm{d}\mathbf{x} \leq C\delta^{1-d+\frac{(d-1)\alpha}{q}+\beta\alpha}\|\psi\|_{L^p(\partial\Omega)}\|v\|_{H^2(\Omega)}. \quad (28)$$

Remark 6.1. Let $d = 2$ and $p < \infty$ arbitrary large. Then q is arbitrary close to one. Choosing α and β also arbitrary close to one leads to the following power of δ in (28)

$$1 + (-2 + (1-\epsilon_1) + (1-\epsilon_2)) = 1 - (\epsilon_1+\epsilon_2) = 1 - \epsilon_3$$

for arbitrary small $\epsilon_1, \epsilon_2, \epsilon_3 > 0$. In this case, the convergence is almost of first order.

The result of Proposition 6.1 does not provide an order of convergence for $d = 3$. Lemma 2.1 suggests choosing $p = 4$, i.e. $q = 4/3$. Then, the power of δ in (28) becomes $2(\alpha - 1)$, which is negative for $\alpha < 1$.

7. The boundedness of the kinetic energy for $\overline{\mathbf{u}}$ in some LES models

The space averaged Navier–Stokes equations

$$\begin{aligned}
\overline{\mathbf{u}}_t - \nu\Delta\overline{\mathbf{u}} + \nabla\cdot(\overline{\mathbf{u}\mathbf{u}^T}) + \nabla\overline{p} & \qquad (29)\\
= \overline{\mathbf{f}} + \int_{\partial\Omega} g(\mathbf{x}-\mathbf{s})\left[\nu\nabla\mathbf{u}(t,\mathbf{s})\mathbf{n}(\mathbf{s}) - p(t,\mathbf{s})\mathbf{n}(\mathbf{s})\right] d\mathbf{s} \text{ in } (0,T)\times\mathbb{R}^d &\\
\nabla\cdot\overline{\mathbf{u}} = 0 \qquad \text{in } (0,T)\times\mathbb{R}^d & \qquad (30)\\
\overline{\mathbf{u}}\,|_{t=0} = \overline{\mathbf{u}}_0 \qquad \text{in } \mathbb{R}^d & \qquad (31)
\end{aligned}$$

are not yet a closed system since the tensor $\overline{\mathbf{u}\mathbf{u}^T}$ is a priori not related to $\overline{\mathbf{u}}$ and $\overline{p}$. One central issue in LES is closure: modeling this tensor in terms of $\overline{\mathbf{u}}$ and $\overline{p}$.

In this section, we will apply the results of Section 6 to show that the kinetic energy of $\overline{\mathbf{u}}$ is bounded for a number of LES models including the previously omitted $A_\delta(\mathbb{S}(\mathbf{u},p))$ commutation error term if δ is sufficiently small.

7.1. The Smagorinsky model

We consider first one of the simplest LES model which goes back to Smagorinsky [24]. This model is obtained by

$$\overline{\mathbf{u}\mathbf{u}^T} \approx \overline{\mathbf{u}}\,\overline{\mathbf{u}}^T - C_\nu\delta^2\|\nabla\overline{\mathbf{u}}\|_F\nabla\overline{\mathbf{u}} + \text{terms absorbed into } \nabla\overline{p},$$

where $C_\nu > 0$ and $\|\nabla\overline{\mathbf{u}}\|_F = (\nabla\overline{\mathbf{u}} : \nabla\overline{\mathbf{u}})^{1/2}$ is the Frobenius norm of $\nabla\overline{\mathbf{u}}$.

The existence and uniqueness of a weak solution of the Smagorinsky model posed in a bounded domain, with homogeneous Dirichlet boundary conditions and without the boundary integral in (29) has been proven by Ladyzhenskaya [17, 18] and Du and Gunzburger [7].

The momentum equation of the Smagorinsky model has the form

$$\begin{aligned}
\overline{\mathbf{u}}_t - \nabla\cdot\left(\left(\nu + C_\nu\delta^2\|\nabla\overline{\mathbf{u}}\|_F\right)\nabla\overline{\mathbf{u}}\right) + \nabla\cdot(\overline{\mathbf{u}}\,\overline{\mathbf{u}}^T) + \nabla\overline{p}&\\
= \overline{\mathbf{f}} + \int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(t,\mathbf{s})\mathrm{d}\mathbf{s} \qquad \text{in } (0,T)\times\mathbb{R}^d & \qquad (32)
\end{aligned}$$

with $\psi(t,\mathbf{s}) = \nu\nabla\mathbf{u}(t,\mathbf{s})\mathbf{n}(\mathbf{s}) - p(t,\mathbf{s})\mathbf{n}(\mathbf{s})$. We assume, there exists a solution $(\mathbf{u},p)$ of (32), (30), (31) Multiplying 32 by $\overline{\mathbf{u}}$ and integrating on $\mathbb{R}^d$ give

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{\|\overline{\mathbf{u}}\|^2_{(L^2(\mathbb{R}^d))^d}}{2} - \int_{\mathbb{R}^d}\nabla\cdot\left(\left(\nu + C_\nu\delta^2\|\nabla\overline{\mathbf{u}}\|_F\right)\nabla\overline{\mathbf{u}}\right)\cdot\overline{\mathbf{u}}\mathrm{d}\mathbf{x}\\
&\quad + \int_{\mathbb{R}^d}\nabla\cdot(\overline{\mathbf{u}}\,\overline{\mathbf{u}}^T)\cdot\overline{\mathbf{u}}\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d}\nabla\overline{p}\cdot\overline{\mathbf{u}}\mathrm{d}\mathbf{x}\\
&= \int_{\mathbb{R}^d}\overline{\mathbf{f}}\cdot\overline{\mathbf{u}}\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d}\overline{\mathbf{u}}(t,\mathbf{x})\cdot\left(\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(t,\mathbf{s})\mathrm{d}\mathbf{s}\right)\mathrm{d}\mathbf{x}. \qquad (33)
\end{aligned}$$

Next, the terms on the left hand side are studied. Using the definition of $\overline{\mathbf{u}}$, changing variables, noting that $\mathbf{u}$ vanishes outside Ω and applying Fubini's theorem, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \nabla\overline{p}(\mathbf{x}) \cdot \overline{\mathbf{u}}(\mathbf{x})\mathrm{d}\mathbf{x} \\
&= \int_{\mathbb{R}^d} \mathbf{u}(\mathbf{y}) \cdot \left(\int_{\mathbb{R}^d} \nabla\overline{p}(\mathbf{y}+\mathbf{z}) g_\delta(\mathbf{z})\mathrm{d}\mathbf{z}\right)\mathrm{d}\mathbf{y} \\
&= \int_{\mathbb{R}^d} g_\delta(\mathbf{z}) \left(\int_{\Omega} \nabla\overline{p}(\mathbf{y}+\mathbf{z}) \cdot \mathbf{u}(\mathbf{y})\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{z} \\
&= \int_{\mathbb{R}^d} g_\delta(\mathbf{z}) \left(-\int_{\Omega} \overline{p}(\mathbf{y}+\mathbf{z})(\nabla\cdot\mathbf{u})(\mathbf{y})\mathrm{d}\mathbf{y} + \int_{\partial\Omega} \overline{p}(\mathbf{s}+\mathbf{z})\mathbf{u}(\mathbf{s})\cdot\mathbf{n}(\mathbf{s})\mathrm{d}\mathbf{s}\right)\mathrm{d}\mathbf{z} \\
&= 0,
\end{aligned}$$

since $\mathbf{u}$ is divergence free and vanishes on $\partial\Omega$. With an index calculation, using that $\overline{\mathbf{u}}$ is divergence free in $\mathbb{R}^d$, and applying the same arguments as for the pressure term, one obtains

$$\int_{\mathbb{R}^d} \nabla\cdot(\overline{\mathbf{u}}\,\overline{\mathbf{u}}^T)\cdot\overline{\mathbf{u}}\mathrm{d}\mathbf{x} = \frac{1}{2}\int_{\mathbb{R}^d} \nabla(\overline{\mathbf{u}}\cdot\overline{\mathbf{u}})\cdot\overline{\mathbf{u}}\mathrm{d}\mathbf{x} = 0.$$

By applying the Fubini's theorem in the same way as before, follows that

$$\begin{aligned}
&-\int_{\mathbb{R}^d} \nabla\cdot((\nu + C_\nu\delta^2\|\nabla\overline{\mathbf{u}}\|_F)\nabla\overline{\mathbf{u}})\cdot\overline{\mathbf{u}}\mathrm{d}\mathbf{x} \\
&= \int_{\mathbb{R}^d} ((\nu + C_\nu\delta^2\|\nabla\overline{\mathbf{u}}\|_F)\nabla\overline{\mathbf{u}}) : \nabla\overline{\mathbf{u}}\mathrm{d}\mathbf{x} \\
&\geq 0.
\end{aligned}$$

The first term on the right hand side of (33) is estimated by the Cauchy–Schwarz inequality and Young's inequality

$$\int_{\mathbb{R}^d} \overline{\mathbf{f}}\cdot\overline{\mathbf{u}}\mathrm{d}\mathbf{x} \leq \|\overline{\mathbf{f}}\|_{(L^2(\mathbb{R}^d))^d}\|\overline{\mathbf{u}}\|_{(L^2(\mathbb{R}^d))^d} \leq \frac{\|\overline{\mathbf{f}}\|^2_{(L^2(\mathbb{R}^d))^d}}{2} + \frac{\|\overline{\mathbf{u}}\|^2_{(L^2(\mathbb{R}^d))^d}}{2}.$$

We obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{\|\overline{\mathbf{u}}\|^2_{(L^2(\mathbb{R}^d))^d}}{2} \\
&\leq \frac{\|\overline{\mathbf{f}}\|^2_{(L^2(\mathbb{R}^d))^d}}{2} + \frac{\|\overline{\mathbf{u}}\|^2_{(L^2(\mathbb{R}^d))^d}}{2} + \int_{\mathbb{R}^d} \overline{\mathbf{u}}(t,\mathbf{x})\cdot\left(\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(t,\mathbf{s})\mathrm{d}\mathbf{s}\right)\mathrm{d}\mathbf{x}.
\end{aligned}$$

Assuming $\mathbf{f} \in L^2(0,T;(L^2(\mathbb{R}^d))^d)$, so that Gronwall's lemma can be applied, gives

$$\begin{aligned} &\frac{\|\overline{\mathbf{u}}(T)\|^2_{(L^2(\mathbb{R}^d))^d}}{2} \\ &\le \exp(T)\frac{\|\overline{\mathbf{u}}_0\|^2_{(L^2(\mathbb{R}^d))^d}}{2} + \int_0^T \exp(T-t)\frac{\|\overline{\mathbf{f}}(t)\|^2_{(L^2(\mathbb{R}^d))^d}}{2} dt \\ &\quad + \int_0^T \exp(T-t)\left(\int_{\mathbb{R}^d} \overline{\mathbf{u}}(t,\mathbf{x})\cdot\left(\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(t,\mathbf{s})\mathrm{d}\mathbf{s}\right)\mathrm{d}\mathbf{x}\right) dt. \end{aligned} \tag{34}$$

Thus, the kinetic energy of $\overline{\mathbf{u}}$ at time T is bounded by the kinetic energy of the initial data, by the right hand side and by a third term, which vanishes as $\delta \to 0$ by Proposition 6.1. For $d = 2$, it follows from Remark 6.1, (5) and Young's inequality

$$\begin{aligned} &\frac{\|\overline{\mathbf{u}}(T)\|^2_{(L^2(\mathbb{R}^2))^2}}{2} \\ &\le \exp(T)\frac{\|\overline{\mathbf{u}}_0\|^2_{(L^2(\mathbb{R}^2))^2}}{2} + \int_0^T \exp(T-t)\frac{\|\overline{\mathbf{f}}(t)\|^2_{(L^2(\mathbb{R}^2))^2}}{2} dt \\ &\quad + C\delta^{1-\epsilon_3}\int_0^T \exp(T-t)\left(\|\mathbf{u}(t)\|^2_{(H^2(\Omega))^2} + \|p(t)\|^2_{H^1(\Omega)}\right) dt \end{aligned} \tag{35}$$

with arbitrary $\epsilon_3 > 0$.

Remark 7.1. One obtains the same result for the deformation tensor formulation of the momentum equation and a Smagorinsky model of the form

$$\overline{\mathbf{u}\mathbf{u}^T} \approx \overline{\mathbf{u}}\,\overline{\mathbf{u}}^T - C_\nu\delta^2\|\mathbb{D}(\overline{\mathbf{u}})\|_F\mathbb{D}(\overline{\mathbf{u}}) + \text{terms absorbed into } \nabla\overline{p}.$$

7.2. The Taylor LES model

The second model which we will discuss is called variously gradient method of the Taylor LES model developed by Leonard [19] and Clark et al. [4]. In the mixed Taylor LES model, a Smagorinsky model for the turbulent fluctuations is included in the model of the non-linear convective term given by

$$\overline{\mathbf{u}\mathbf{u}^T} \approx \overline{\mathbf{u}}\,\overline{\mathbf{u}}^T - C_\nu\delta^2\|\nabla\overline{\mathbf{u}}\|_F\nabla\overline{\mathbf{u}} + \frac{\delta^2}{12}\nabla\overline{\mathbf{u}}\nabla\overline{\mathbf{u}}^T.$$

The existence and uniqueness of a weak solution of the Taylor LES model in a bounded domain, equipped with homogeneous Dirichlet boundary conditions and without the boundary integral in (29) has been proven for C_ν large enough by Coletti [5]. We study the energy balance of the Taylor LES model including the term $A_\delta(\mathbb{S}(\mathbf{u},p))$.

Inserting the Taylor LES model into (29), multiplying by $\overline{\mathbf{u}}$ and integrating by parts, the non-linear convective term and the pressure term vanish as in the Smagorinsky model. The bilinear viscous term is obviously non-negative. The non-linear viscous term is treated in connection with the third term in the Taylor LES model. Using the same arguments as in the Smagorinsky model, one finds

$$\int_{\mathbb{R}^d} \nabla \cdot \left(-C_\nu \delta^2 \|\nabla \overline{\mathbf{u}}\|_F \nabla \overline{\mathbf{u}} + \frac{\delta^2}{12} \nabla \overline{\mathbf{u}} \nabla \overline{\mathbf{u}}^T \right) \cdot \overline{\mathbf{u}} \mathrm{d}\mathbf{x}$$
$$= \int_{\mathbb{R}^d} C_\nu \delta^2 \|\nabla \overline{\mathbf{u}}\|_F \nabla \overline{\mathbf{u}} : \nabla \overline{\mathbf{u}} - \frac{\delta^2}{12} (\nabla \overline{\mathbf{u}} \nabla \overline{\mathbf{u}}^T) : \nabla \overline{\mathbf{u}} \mathrm{d}\mathbf{x}. \tag{36}$$

By norm equivalence in $\mathbb{R}^d \times \mathbb{R}^d$,

$$\left(\sum_{i=1}^d |a_{ij}|^3 \right)^{\frac{1}{3}} \leq \|A\|_F \leq C(d) \left(\sum_{i=1}^d |a_{ij}|^3 \right)^{\frac{1}{3}},$$

we get

$$\int_{\mathbb{R}^d} C_\nu \delta^2 \|\nabla \overline{\mathbf{u}}\|_F \nabla \overline{\mathbf{u}} : \nabla \overline{\mathbf{u}} \mathrm{d}\mathbf{x}$$
$$= \int_{\mathbb{R}^d} C_\nu \delta^2 \|\nabla \overline{\mathbf{u}}\|_F^3 \mathrm{d}\mathbf{x} \geq C_\nu \delta^2 \|\nabla \overline{\mathbf{u}}\|_{L^3(\mathbb{R}^d)}^3.$$

The second term in (36) is estimated in a similar way. Using twice the Cauchy–Schwarz inequality, one obtains

$$\int_{\mathbb{R}^d} \frac{\delta^2}{12} (\nabla \overline{\mathbf{u}} \nabla \overline{\mathbf{u}}^T) : \nabla \overline{\mathbf{u}} \mathrm{d}\mathbf{x} \leq \int_{\mathbb{R}^d} \frac{\delta^2}{12} \|\nabla \overline{\mathbf{u}}\|_F^3 \mathrm{d}\mathbf{x} \leq C(d) \frac{\delta^2}{12} \|\nabla \overline{\mathbf{u}}\|_{L^3(\mathbb{R}^d)}^3.$$

We get finally

$$-\int_{\mathbb{R}^d} \left(-C_\nu \delta^2 \|\nabla \overline{\mathbf{u}}\|_F \nabla \overline{\mathbf{u}} + \frac{\delta^2}{12} \nabla \overline{\mathbf{u}} \nabla \overline{\mathbf{u}}^T \right) : \nabla \overline{\mathbf{u}} \mathrm{d}\mathbf{x}$$
$$\geq C_\nu \delta^2 \|\nabla \overline{\mathbf{u}}\|_{L^3(\mathbb{R}^d)}^3 - C(d) \frac{\delta^2}{12} \|\nabla \overline{\mathbf{u}}\|_{L^3(\mathbb{R}^d)}^3 \geq 0$$

if C_ν is sufficiently large.

Now, we can continue as for the Smagorinsky model and obtain also the estimates (34) and (35) for the kinetic energy of $\overline{\mathbf{u}}$ if C_ν is chosen large enough and if δ is sufficiently small.

7.3. The rational LES model

The rational LES model was proposed in [11]. Including the Smagorinsky model for the effects of turbulent fluctuations, there are two variants of this model, one of them has the form

$$\overline{\mathbf{u}\mathbf{u}^T} \approx \overline{\mathbf{u}}\, \overline{\mathbf{u}}^T - C_\nu \delta^2 \|\nabla \overline{\mathbf{u}}\|_F \nabla \overline{\mathbf{u}} + \frac{\delta^2}{12} g_\delta * \left(\nabla \overline{\mathbf{u}} \nabla \overline{\mathbf{u}}^T \right).$$

The existence and uniqueness of generalized solutions of the rational LES model in the space periodic case for small data and small times has been proven by Berselli et al. [3].

Proceeding as in the Taylor LES model, the only difference is the term

$$\int_{\mathbb{R}^d} C_\nu \delta^2 \|\nabla \overline{\mathbf{u}}\|_F^3 - \frac{\delta^2}{12} g_\delta * \left(\nabla \overline{\mathbf{u}} \nabla \overline{\mathbf{u}}^T\right) : \nabla \overline{\mathbf{u}} \mathrm{d}\mathbf{x}.$$

The application of Fubini's theorem and the symmetry of the Gaussian filter yield

$$\int_{\mathbb{R}^d} g_\delta * (\nabla \overline{\mathbf{u}} \nabla \overline{\mathbf{u}}^T) : \nabla \overline{\mathbf{u}} \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^d} \nabla \overline{\mathbf{u}} \nabla \overline{\mathbf{u}}^T : (g_\delta * \nabla \overline{\mathbf{u}}) \mathrm{d}\mathbf{x}.$$

It follows with the same arguments as in the estimate for the Taylor LES model, using in addition Hölder's inequality for convolutions,

$$\begin{aligned}
\int_{\mathbb{R}^d} g_\delta * \left(\nabla \overline{\mathbf{u}} \nabla \overline{\mathbf{u}}^T\right) : \nabla \overline{\mathbf{u}} \mathrm{d}\mathbf{x} &\leq \int_{\mathbb{R}^d} \|\nabla \overline{\mathbf{u}}\|_F^2 \|g_\delta * \nabla \overline{\mathbf{u}}\|_F \mathrm{d}\mathbf{x} \\
&\leq \|\nabla \overline{\mathbf{u}}\|_{L^3(\mathbb{R}^d)}^2 \|g_\delta * \nabla \overline{\mathbf{u}}\|_{L^3(\mathbb{R}^d)} \\
&\leq \|\nabla \overline{\mathbf{u}}\|_{L^3(\mathbb{R}^d)}^3 \|g_\delta\|_{L^1(\mathbb{R}^d)} = \|\nabla \overline{\mathbf{u}}\|_{L^3(\mathbb{R}^d)}^3.
\end{aligned}$$

This gives the estimate

$$\begin{aligned}
&\int_{\mathbb{R}^d} C_\nu \delta^2 \|\nabla \overline{\mathbf{u}}\|_F^3 - \frac{\delta^2}{12} g_\delta * \left(\nabla \overline{\mathbf{u}} \nabla \overline{\mathbf{u}}^T\right) : \nabla \overline{\mathbf{u}} \mathrm{d}\mathbf{x} \\
&\geq \left(C_\nu \delta^2 - C_K^3 C(d) \frac{\delta^2}{12}\right) \|\nabla \overline{\mathbf{u}}\|_{L^3(\mathbb{R}^d)}^3 \geq 0
\end{aligned}$$

if C_ν is large enough.

That means, also for the rational LES model, the kinetic energy of $\overline{\mathbf{u}}$ can be estimated in form (34) and (35) if C_ν is chosen sufficiently large and δ sufficiently small.

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