On Divergence-Free Finite Element Methods for the Stokes Equations

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Abstract

The classical equations for the mathematical description of an incompressible flow via the velocity $\mathbf{u}$ and the pressure p are the Stokes equations. Based on its weak formulation, the finite element method provides a possibility to determine approximations of the solution $(\mathbf{u}, p)$ in finite element spaces.
On the one hand, the well-posedness of the discrete Stokes problem for a particular choice of finite element spaces requires the satisfaction of the discrete inf-sup condition on general meshes. On the other hand, for physical reasonability, it is desirable to construct methods which fulfill the so-called invariance property and automatically provide discrete velocity solutions that conserve mass weakly.
Unfortunately, the classical low order finite element pairs either are not inf-sup stable or do not guarantee these qualitative properties. Therefore this is a problem of current research and we will introduce two methods which lead to a well-posed problem and yield a weakly divergence-free velocity solution, respectively satisfy the invariance property.

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List of Abbreviations

$\exists$:	there exists
$\exists!$:	there exists a unique
$\forall$:	for all
∂:	boundary or partial derivative
$\mathrm{int}(\cdot)$:	interior of $(\cdot)$
$\ker(\cdot)$:	kernel of $(\cdot)$
$\mathrm{im}(\cdot)$:	image of $(\cdot)$
$\overline{\{\dots\}}$:	closure of $\{\dots\}$
$\dim\,(\cdot)$:	dimension of $(\cdot)$
$\mathrm{diam}\,(\cdot)$:	diameter of $(\cdot)$
$(\cdot)^T$:	transposed of $(\cdot)$
$\mathrm{conv}\,\{\dots\}$:	convex hull of $\{\dots\}$
$\mathrm{span}\{\dots\}$:	linear span of $\{\dots\}$

1 Introduction

The numerical simulation of fluid flows is nowadays a very important field in mathematics with diversified fields of application. It is used, e.g., for the development of stents in medicine, optimization of the trim for means of transport like airplanes, and for weather forecasts which are embedded in everyone's daily grind. Therefore, expensive, complex, and maybe impracticable experiments can be replaced by computer simulations.
The fundamental system for describing (incompressible) motion of Newtonian fluids arises from the law of conservation of mass, energy, and linear momentum. The resulting equations are called (incompressible) Navier–Stokes equations.
For a detailed derivation, it is referred to [John13].

The analytical solution or more precisely the proof of the existence of strong solutions in $\mathbb{R}^3$, which is on the one hand the velocity field of the fluid and on the other hand the pressure field of the fluid, is one of the seven Millennium Problems awarded with 1.000.000 dollars. This paper focuses on a special case of the (incompressible) Navier–Stokes equations, the Stokes equations. They occur when considering stationary flows at small fluid velocities. Solving them will support the process of understanding the numerical simulation of the Navier–Stokes equations by introducing generally useful tools.

After this introductory chapter, Chapter 2 provides a basis for the upcoming considerations. At the beginning there is a short repetition of, for the later chapters, important operator definitions. Shortly after we will in particular introduce the Lebesgue spaces and the Sobolev spaces, which are used afterwards. Additionally, theorems like integration by parts will be discussed.

In Chapter 3, the Stokes equations are presented and we will derive their weak formulation in order to search for weak solutions. The so-called inf-sup condition will turn out to be a necessary condition on the test spaces, which guarantees the well-posedness of the Stokes problem.
The main result will be the unique existence of a weak solution of the Stokes equations.

The topic of Chapter 4 is the discretization of the Stokes equations with the finite element method. We will give a short introduction into the finite element theory and apply it to the Stokes equations. For the finite element discretization of the Stokes problem a discrete version of the inf-sup condition will turn out to be the crucial factor for the existence of a unique solution. This discrete inf-sup condition can be interpreted as a compatibility condition between the finite element spaces. Apart from the discrete inf-sup condition there is another very important property one would like to be satisfied, the weak mass conservation. The problem is that one desires to get weakly divergence-free velocity approximations but discretely divergence-free vector fields are not necessarily weakly divergence-free and by construction, the general finite element velocity approximation is only assured to be discretely divergence-free. Some popular finite element spaces for the Stokes problem will be discussed concerning these two crucial features.

Chapter 5 deals with a modified method based on the Crouzeix–Raviart element. This method satisfies the invariance property and thus leads to a finite element error estimate for the velocity which is independent of the continuous pressure. It is realized by a projection of the test functions on the right-hand side of the finite element formulation into the lowest-order Raviart–Thomas space. Additionally this projection provides a possibility to get a weakly divergence-free finite element solution.

The purpose of Chapter 6 is the analysis of another example for a divergence-free method. Here the well-posedness of the finite element pair $\mathbf{P}_1/P_0$ is established by adding stabilizing extra terms. These lead to the violation of the mass conservation which is actually present for this finite element pair. Therefore, a post-processing which re-establishes the mass conservation by adding a vector field located in the lowest order Raviart–Thomas field is presented.

Finally, this thesis is completed by numerical studies and a summary.

2 Basics and Preliminaries

The aim of this chapter is to introduce operators we will use, relations between them, some important function spaces, theorems, and inequalities.

For the sake of formality, instead of $\frac{\partial u}{\partial x_i}$, we will also adopt the common notation u_{x_i} for partial derivatives and a bold faced letter will indicate that we are dealing with a Cartesian product, e.g., a vector-valued function. This is assumed throughout this paper also for function spaces.
In addition, $\mathbf{\Omega} \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is assumed to be a bounded domain.

2.1 Operators

Most of the operator definitions are valid for arbitrary dimensions $n \in \mathbb{N}$. An exception is the rotation or curl which has different definitions for different dimensions of the vector field. We will mention it for two- and three-dimensional vector fields, since these are the cases we need.

Definition 2.1.1 *Let $v : \mathbf{\Omega} \to \mathbb{R}$ be a scalar function, $\mathbf{u} : \mathbf{\Omega} \to \mathbb{R}^n$, $\mathbf{u} = (u_1, u_2, \dots, u_n)^T$ a vector-valued function, and $\tilde{\mathbf{u}} : \mathbf{\Omega} \to \mathbb{R}^n$ another vector-valued function, all three on $\mathbf{\Omega} \subset \mathbb{R}^n$, $n \in \mathbb{N}$. We define the following operators:*

1. **Laplace operator:**

$$\Delta v := \sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i^2},$$

$$\Delta \mathbf{u} \in \mathbb{R}^n, \qquad (\Delta \mathbf{u})_i := \sum_{j=1}^{n} \frac{\partial^2 u_i}{\partial x_j^2}, \qquad i = 1, \dots, n,$$

2. **nabla operator:**

$$\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n} \right)^T,$$

3. **gradient:**

$$\nabla v := \left(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \cdots, \frac{\partial v}{\partial x_n} \right)^T,$$

$$\nabla \mathbf{u} \in \mathbb{R}^{n\times n}, \quad (\nabla \mathbf{u})_{ij} := \frac{\partial u_i}{\partial x_j}, \quad i,j = 1,\dots,n,$$

4. **divergence for vector-valued functions:**

$$\nabla \cdot \mathbf{u} := \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i},$$

5. **rotation/curl:**

$$\begin{aligned}
\nabla \times v &:= \begin{pmatrix} -v_{x_2} \\ v_{x_1} \end{pmatrix}, && \textit{for } n = 2,\\
\nabla \times \mathbf{u} &:= (u_2)_{x_1} - (u_1)_{x_2}, && \textit{for } n = 2,\\
\nabla \times \mathbf{u} &:= \begin{pmatrix} (u_3)_{x_2} - (u_2)_{x_3} \\ (u_1)_{x_3} - (u_3)_{x_1} \\ (u_2)_{x_1} - (u_1)_{x_2} \end{pmatrix}, && \textit{for } n = 3,
\end{aligned}$$

6. **tensor-product:**

$$\nabla \mathbf{u} : \nabla \tilde{\mathbf{u}} := \sum_{i=1}^{n} \nabla u_i \cdot \nabla \tilde{u}_i = \sum_{i,j=1}^{n} \frac{\partial u_i}{\partial x_j} \frac{\partial \tilde{u}_i}{\partial x_j}.$$

Lemma 2.1.1 *For v and $\mathbf{u}$ as previously stated and both sufficiently smooth the following identities hold:*

$$\begin{aligned}
&(i) && \nabla \cdot \nabla v = \Delta v \in \mathbb{R}, && \\
& && \nabla \cdot \nabla \mathbf{u} = \Delta \mathbf{u} \in \mathbb{R}^n, && \\
&(ii) && \nabla \cdot (v\mathbf{u}) = \nabla v \cdot \mathbf{u} + v \nabla \cdot \mathbf{u}, && \\
&(iii) && \nabla \times (\nabla v) = 0, && \textit{for } n = 2,3,\\
&(iv) && \nabla \times (\nabla \times \mathbf{u}(\mathbf{x})) = -\Delta \mathbf{u}(\mathbf{x}) + \nabla(\nabla \cdot \mathbf{u}(\mathbf{x})), && \textit{for } n = 3,\\
&(v) && \nabla \cdot (\nabla \times v(\mathbf{x})) = 0, && \textit{for } n = 2,\\
& && \nabla \cdot (\nabla \times \mathbf{u}(\mathbf{x})) = 0, && \textit{for } n = 3.
\end{aligned}$$

Proof: The statements are proved by direct calculations. In (iii), (iv), and (v) one has to apply the symmetry of second derivatives. □

Remark 2.1.1

The term smooth has to be interpreted as weakly differentiable in the sense of Definition 2.2.3 and sufficiently smooth should indicate that the used functions are smooth enough for the applied calculations and considerations.

2.2 Some Function Spaces and Norms

This section provides definitions and elementary properties of some basic function spaces.

2.2.1 The Lebesgue Spaces

Definition 2.2.1 (Lebesgue space) *The Lebesgue space*

$$L^p(\mathbf{\Omega}) := \left\{ q : \mathbf{\Omega} \to \mathbb{R} \text{ } \textit{measurable} \text{ } : \int_{\mathbf{\Omega}} |q(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} < \infty \right\}, \qquad p \in [1, \infty),$$

with the norm

$$\|q\|_{L^p(\mathbf{\Omega})} := \left(\int_{\mathbf{\Omega}} |q(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{p}}$$

is the space of all functions which are Lebesgue measurable and have finite norm $\| \cdot \|_{L^p(\mathbf{\Omega})}$.
To complete the definition:

$$L^\infty(\mathbf{\Omega}) := \left\{ q : \mathbf{\Omega} \to \mathbb{R} \text{ } \textit{measurable} \text{ } : \operatorname*{ess\,sup}_{\mathbf{x} \in \mathbf{\Omega}} |q(\mathbf{x})| < \infty \right\}$$

with

$$\|q\|_{L^\infty(\mathbf{\Omega})} := \operatorname*{ess\,sup}_{\mathbf{x} \in \mathbf{\Omega}} |q(\mathbf{x})|.$$

Additionally we define two special spaces which will be needed later:

$$L^2_0(\mathbf{\Omega}) := \left\{ q \in L^2(\mathbf{\Omega}) : \int_{\mathbf{\Omega}} q(\mathbf{x}) \, \mathrm{d}\mathbf{x} = 0 \right\}$$

and

$$\begin{aligned} L^1_{loc}(\mathbf{\Omega}) : &= \left\{ q : \mathbf{\Omega} \to \mathbb{R} \text{ } \textit{measurable} \text{ } : \int_{\mathbf{\Omega}'} |q(\mathbf{x})| \, \mathrm{d}\mathbf{x} < \infty, \ \forall \mathbf{\Omega}' \subset_{compact} \mathbf{\Omega} \right\} \\ &= \left\{ q \in L^1(\mathbf{\Omega}') : \mathbf{\Omega}' \subset_{compact} \mathbf{\Omega} \right\}, \end{aligned}$$

the space of locally integrable functions.

Remark 2.2.1

1. The spaces $L^p(\mathbf{\Omega})$ are Banach spaces, see [AdaFou05], pp. 29. and they actually consist of equivalence classes.

2. $L^2(\mathbf{\Omega})$ becomes a Hilbert space with the inner product

$$(q,g)_{L^2(\mathbf{\Omega})} := \int_{\mathbf{\Omega}} q(\mathbf{x})g(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

and induced norm

$$\|q\|_{L^2(\mathbf{\Omega})} = (q,q)^{\frac{1}{2}}_{L^2(\mathbf{\Omega})}.$$

3. $L_0^2(\mathbf{\Omega})$ is a closed subspace of $L^2(\mathbf{\Omega})$ and hence a Hilbert space, too.
4. Conventionally, the dual pairing $\langle\cdot,\cdot\rangle$ of elements in L^2 is equivalent to the L^2-scalar product $(\cdot,\cdot)_{L^2(\mathbf{\Omega})}$.

2.2.2 The Smooth Spaces

Definition 2.2.2 ($C^k(\mathbf{\Omega})$) *Define the space of k times continuously differentiable functions by*

$$\begin{aligned}
C^k(\mathbf{\Omega}) &:= \left\{v:\mathbf{\Omega}\to\mathbb{R} : D^{\boldsymbol{\alpha}}v \in C^0(\mathbf{\Omega}),\ \forall|\boldsymbol{\alpha}|\le k\right\}, \qquad \boldsymbol{\alpha}\in\mathbb{N}_0^n,\ k\in\mathbb{N}_0,\\
C^\infty(\mathbf{\Omega}) &:= \bigcap_{k=0}^{\infty} C^k(\mathbf{\Omega}),\\
C^0(\mathbf{\Omega}) &:= \{v:\mathbf{\Omega}\to\mathbb{R} : v \text{ is continuous}\},
\end{aligned}$$

and their analogons with the additional property that the closure of the support of the functions is a compact subset of $\mathbf{\Omega}$ by

$$\begin{aligned}
C_0^k(\mathbf{\Omega}) &:= \left\{v\in C^k(\mathbf{\Omega}) : supp(v)\subset_{compact}\mathbf{\Omega}\right\},\\
C_0^\infty(\mathbf{\Omega}) &:= \{v\in C^\infty(\mathbf{\Omega}) : supp(v)\subset_{compact}\mathbf{\Omega}\},
\end{aligned}$$

with

$$supp(v) := \overline{\{\mathbf{x}\in\mathbf{\Omega} : v(\mathbf{x})\neq 0\}}$$

being the support of the function $v:\mathbf{\Omega}\to\mathbb{R}$.

2.2.3 The Sobolev Spaces

For the use of the finite element theory the Sobolev spaces play a key role. They are defined via some kind of relaxed derivatives, called weak derivatives, which will be the starting point of this section.

Definition 2.2.3 (Weak derivative) *Let $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_n)$, $\alpha_i\in\mathbb{N}_0$, be a multi-index with $|\boldsymbol{\alpha}|=\alpha_1+\cdots+\alpha_n$ and $f(\mathbf{x})$, $D^{\boldsymbol{\alpha}}f(\mathbf{x})\in L^1_{loc}(\mathbf{\Omega})$. We call*

$D^{\boldsymbol{\alpha}} f(\mathbf{x})$ weak derivative of $f(\mathbf{x})$ w.r.t. $\boldsymbol{\alpha}$ if it holds for all $g \in C_0^\infty(\mathbf{\Omega})$:

$$\int_{\mathbf{\Omega}} f(\mathbf{x}) D^{\boldsymbol{\alpha}} g(\mathbf{x}) \, \mathrm{d}\mathbf{x} = (-1)^{|\boldsymbol{\alpha}|} \int_{\mathbf{\Omega}} D^{\boldsymbol{\alpha}} f(\mathbf{x}) g(\mathbf{x}) \, \mathrm{d}\mathbf{x},$$

where

$$D^{\boldsymbol{\alpha}} f(\mathbf{x}) = \frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Remark 2.2.2

1. In order to simplify the notation, the classical and the weak derivatives are both denoted by D. From the context it will be clear, which one is meant.
2. If $f(\mathbf{x})$ has a classical derivative, then it coincides with the corresponding weak derivative.
3. Since the Lebesgue integral is not affected by function values on null sets, the notion of a weak derivative works for functions which are not classically differentiable on a set of Lebesgue measure zero.
   Note that a function can have several weak derivatives, but up to a null set, they are equal. So the weak derivative is unique up to sets of Lebesgue measure zero.

In an analogue manner we define the weak divergence.

Definition 2.2.4 (Weak divergence) *A vector field $\mathbf{w} \in \mathbf{L}^2(\mathbf{\Omega})$ is said to have a weak divergence (a divergence in $\mathbf{L}^2(\mathbf{\Omega})$) if there is a function $s \in L^2(\mathbf{\Omega})$ such that*

$$-\int_{\mathbf{\Omega}} \mathbf{w} \cdot \nabla \phi \, \mathrm{d}\mathbf{x} = \int_{\mathbf{\Omega}} s \phi \, \mathrm{d}\mathbf{x}, \quad \forall \phi \in C_0^\infty(\mathbf{\Omega}).$$

We then write $s = \nabla \cdot \mathbf{w}$.

Definition 2.2.5 (Sobolev space) *The Sobolev space*

$$W^{k,p}(\mathbf{\Omega}) := \{u \in L^p(\mathbf{\Omega}) : D^{\boldsymbol{\alpha}} u \in L^p(\mathbf{\Omega}), \ \forall |\boldsymbol{\alpha}| \leq k\}, \quad p \in [1, \infty], \, k \in \mathbb{N}_0,$$

is the space of all L^p-functions whose weak derivatives of order $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| \leq k$ are in $L^p(\mathbf{\Omega})$, too.
Additionally define the closure of $C_0^\infty(\mathbf{\Omega})$ in $W^{k,p}(\mathbf{\Omega})$ by

$$W_0^{k,p}(\mathbf{\Omega}) := \overline{C_0^\infty(\mathbf{\Omega})}^{\|\cdot\|_{W^{k,p}(\mathbf{\Omega})}}.$$

Remark 2.2.3

1. Equipped with the norm

$$\|u\|_{W^{k,p}(\boldsymbol{\Omega})} := \begin{cases} \left(\sum_{|\boldsymbol{\alpha}|\le k} \int_{\boldsymbol{\Omega}} |D^{\boldsymbol{\alpha}} u|^p \,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}}, & \text{if } p < \infty, \\ \max\limits_{|\boldsymbol{\alpha}|\le k} \left(\operatorname*{ess\,sup}\limits_{\mathbf{x}\in\boldsymbol{\Omega}} |D^{\boldsymbol{\alpha}} u| \right), & \text{if } p = \infty, \end{cases}$$

   Sobolev spaces are Banach spaces, see [Ev10], pp. 262.
2. The Sobolev spaces $H^k(\boldsymbol{\Omega}) := W^{k,2}(\boldsymbol{\Omega})$ are Hilbert spaces equipped with the inner product

$$(u,v)_{H^k(\boldsymbol{\Omega})} := \sum_{|\boldsymbol{\alpha}|\le k} (D^{\boldsymbol{\alpha}} u, D^{\boldsymbol{\alpha}} v)_{L^2(\boldsymbol{\Omega})} = \sum_{|\boldsymbol{\alpha}|\le k} \int_{\boldsymbol{\Omega}} D^{\boldsymbol{\alpha}} u D^{\boldsymbol{\alpha}} v \,\mathrm{d}\mathbf{x}$$

   and norm

$$\|u\|_{H^k(\boldsymbol{\Omega})} = (u,u)^{\frac{1}{2}}_{H^k(\boldsymbol{\Omega})}.$$

   A seminorm is defined on $H^k(\boldsymbol{\Omega})$ by

$$|u|_{H^k(\boldsymbol{\Omega})} := \left(\sum_{|\boldsymbol{\alpha}|=k} \|D^{\boldsymbol{\alpha}} u\|^2_{L^2(\boldsymbol{\Omega})} \right)^{\frac{1}{2}} = \left(\sum_{|\boldsymbol{\alpha}|=k} \int_{\boldsymbol{\Omega}} |D^{\boldsymbol{\alpha}} u|^2 \,\mathrm{d}\mathbf{x} \right)^{\frac{1}{2}}.$$

3. It is $|u|_{H^1(\boldsymbol{\Omega})} = \|\nabla u\|_{\mathbf{L}^2(\boldsymbol{\Omega})}$.
4. We have the identity $H^0(\boldsymbol{\Omega}) = L^2(\boldsymbol{\Omega})$.
5. It holds $W_0^{k,p}(\boldsymbol{\Omega}) \subset W^{k,p}(\boldsymbol{\Omega})$ and $u \in W_0^{k,p}(\boldsymbol{\Omega}) \iff \exists$ a sequence $u_k \in C_0^\infty(\boldsymbol{\Omega})$ with $\|u - u_k\|_{W^{k,p}(\boldsymbol{\Omega})} \xrightarrow{k\to\infty} 0$.
6. Since one special Sobolev space will play an important role within this thesis, we will briefly name it:

$$\begin{aligned} \mathbf{H}_0^1(\boldsymbol{\Omega}) &= \underbrace{H_0^1(\boldsymbol{\Omega}) \times H_0^1(\boldsymbol{\Omega}) \times \cdots \times H_0^1(\boldsymbol{\Omega})}_{\text{n times}} \\ &= \left\{ \mathbf{u} \in \mathbf{H}^1(\boldsymbol{\Omega}) : \mathbf{u}|_{\partial\boldsymbol{\Omega}} = \mathbf{0} \right\} \\ &= \left\{ \mathbf{u} \in \mathbf{L}^2(\boldsymbol{\Omega}) : \nabla\mathbf{u} \in \mathbf{L}^2(\boldsymbol{\Omega}), \mathbf{u}|_{\partial\boldsymbol{\Omega}} = \mathbf{0} \right\}. \end{aligned}$$

   Note that $H_0^1(\boldsymbol{\Omega})$ is a Hilbert space, too, since

$$(u,v) := \int_{\boldsymbol{\Omega}} \nabla u \cdot \nabla v \,\mathrm{d}\mathbf{x}$$

   is indeed an inner product and $|\cdot|_{H^1(\boldsymbol{\Omega})}$ is a norm on $H_0^1(\boldsymbol{\Omega})$. By the Poincaré–Friedrichs inequality (Theorem 2.3.2) $|u|_{H^1(\boldsymbol{\Omega})}$ and $\|u\|_{H^1(\boldsymbol{\Omega})}$ are equivalent in $H_0^1(\boldsymbol{\Omega})$, see for example [AdaFou05], pp. 184. So for $H_0^1(\boldsymbol{\Omega})$-functions, $|\cdot|_{H^1(\boldsymbol{\Omega})}$ may be used instead of $\|\cdot\|_{H^1(\boldsymbol{\Omega})}$. From now on one should be aware of the identity $\|u\|_{H_0^1(\boldsymbol{\Omega})} = \|\nabla u\|_{\mathbf{L}^2(\boldsymbol{\Omega})}$.

Remark 2.2.4

Many properties of Sobolev spaces require $\partial\boldsymbol{\Omega}$ to be Lipschitz continuous, so we will assume that this is the case. For that reason, in the following, we can be sure to have a (unit) outer normal vector almost everywhere at the boundary.

Definition 2.2.6 (Dual space) *A bounded linear operator $t : X \longrightarrow \mathbb{R}$ is called a bounded linear functional on the space X.*

$$X' := \{\textit{bounded, linear functionals on } X\}$$

is called the dual space of X.

For further reading it is also necessary to be aware of the definition of Sobolev spaces with negative exponents.
Denote by q the conjugate exponent corresponding to p such that

$$q = \begin{cases} \infty, & \text{if } p = 1, \\ \frac{p}{p-1}, & \text{if } p \in (1, \infty), \\ 1, & \text{if } p = \infty. \end{cases}$$

As short notation we write $1 = \frac{1}{p} + \frac{1}{q}$.

Definition 2.2.7 ($W^{-k,q}(\boldsymbol{\Omega})$) *Let $k \in \mathbb{N}_0$, $p \in [1, \infty]$ and $1 = \frac{1}{p} + \frac{1}{q}$. We define*

$$W^{-k,q}(\boldsymbol{\Omega}) := \Big\{\phi \in (C_0^\infty(\boldsymbol{\Omega}))' : \|\phi\|_{W^{-k,q}(\boldsymbol{\Omega})} < \infty\Big\}$$

with the norm

$$\|\phi\|_{W^{-k,q}(\boldsymbol{\Omega})} := \sup_{u \in W_0^{k,p}(\boldsymbol{\Omega})\setminus\{0\}} \frac{\langle \phi, u\rangle}{\|u\|_{W_0^{k,p}(\boldsymbol{\Omega})}}.$$

Remark 2.2.5

The Sobolev space $W^{-k,q}(\boldsymbol{\Omega})$ is the dual space of $W_0^{k,p}(\boldsymbol{\Omega})$, i.e., $W^{-k,q}(\boldsymbol{\Omega}) = (W_0^{k,p}(\boldsymbol{\Omega}))'$. Hence

$$W^{-k,q}(\boldsymbol{\Omega}) = \Big\{f : \psi \mapsto f(\psi) := \langle f, \psi\rangle : \psi \in W_0^{k,q}(\boldsymbol{\Omega}), \\ f \text{ bounded and linear}\Big\}.$$

In particular, it holds $H^{-1}(\boldsymbol{\Omega}) = (H_0^1(\boldsymbol{\Omega}))'$.

Throughout this thesis, for spaces of vector-valued functions we use the intuitive extensions by considering the Cartesian product, as noted in the beginning of this chapter.
For example it is

$$\mathbf{L}^2(\mathbf{\Omega}) = [L^2(\mathbf{\Omega})]^n = \Big\{\mathbf{v} : \mathbf{\Omega} \to \mathbb{R}^n : v_i \in L^2(\mathbf{\Omega}),\ \forall i = 1, \dots, n\Big\}$$

and

$$\|\mathbf{v}\|_{\mathbf{L}^2(\mathbf{\Omega})} = \left(\sum_{i=1}^{n} \|v_i\|^2_{L^2(\mathbf{\Omega})}\right)^{\frac{1}{2}}.$$

2.3 Some Inequalities and Theorems

In numerous proofs we will make use of the inequalities mentioned in this section.

Theorem 2.3.1 (Hölder inequality and Cauchy–Schwarz inequality) *Let $p,\ q > 1$ with $1 = \frac{1}{p} + \frac{1}{q}$, $u \in L^p(\mathbf{\Omega})$ and $v \in L^q(\mathbf{\Omega})$. Then, $uv \in L^1(\mathbf{\Omega})$ and the generalized Cauchy–Schwarz inequality or Hölder inequality holds:*

$$\|uv\|_{L^1(\mathbf{\Omega})} \leq \|u\|_{L^p(\mathbf{\Omega})}\|v\|_{L^q(\mathbf{\Omega})}.$$

A particularly important case corresponds to $p = q = 2$ and results in the Cauchy–Schwarz inequality:

$$\|uv\|_{L^1(\mathbf{\Omega})} \leq \|u\|_{L^2(\mathbf{\Omega})}\|v\|_{L^2(\mathbf{\Omega})}.$$

Proof: See [John13] or [Fors11], pp. 129. □

Theorem 2.3.2 (The classical Poincaré–Friedrichs inequality) *Let $\mathbf{\Omega} \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\mathbf{\Omega}$. Then*

$$\forall v \in H^1_0(\mathbf{\Omega}) : \|v\|_{L^2(\mathbf{\Omega})} \leq C\|\nabla v\|_{\mathbf{L}^2(\mathbf{\Omega})},$$

with a constant $C = C(\mathrm{diam}(\mathbf{\Omega})) > 0$.

Proof: The proof can be found in [GiRa86], page 3. □

The below stated tools will be applied especially in proofs during the next chapter when talking about the so-called weak formulation.

Theorem 2.3.3 *Let $\mathbf{\Omega} \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with Lipschitz boundary $\partial\mathbf{\Omega}$ and v, w sufficiently smooth.*
Moreover, denote by $\mathbf{n}$ the unit outer normal vector on $\partial\mathbf{\Omega}$.
Then it holds

1. **Integration by parts (Ibp):**
   Let $v \in W^{1,p}(\mathbf{\Omega})$, $w \in W^{1,q}(\mathbf{\Omega})$, $p \in (1,\infty)$, and $1 = \frac{1}{p} + \frac{1}{q}$, then it holds
   $$\int_{\mathbf{\Omega}} \partial_i v(\mathbf{x}) w(\mathbf{x}) \,\mathrm{d}\mathbf{x} = \int_{\partial\mathbf{\Omega}} v(\mathbf{s}) w(\mathbf{s}) n_i(\mathbf{s}) \,\mathrm{d}\mathbf{s} - \int_{\mathbf{\Omega}} v(\mathbf{x}) \partial_i w(\mathbf{x}) \,\mathrm{d}\mathbf{x}.$$
   This statement generalizes for vector fields $\mathbf{v}$, $\mathbf{w}$ to:
   $$\int_{\mathbf{\Omega}} \nabla \cdot \mathbf{v}(\mathbf{x}) \mathbf{w}(\mathbf{x}) \,\mathrm{d}\mathbf{x} = \int_{\partial\mathbf{\Omega}} \mathbf{v}(\mathbf{s}) \left(\mathbf{w}(\mathbf{s}) \cdot \mathbf{n}(\mathbf{s})\right) \,\mathrm{d}\mathbf{s} - \int_{\mathbf{\Omega}} \mathbf{v}(\mathbf{x}) \nabla \cdot \mathbf{w}(\mathbf{x}) \,\mathrm{d}\mathbf{x}.$$
2. **Green's first formula:**
   For all $v \in H^2(\mathbf{\Omega})$ and $w \in H^1(\mathbf{\Omega})$ it is:
   $$\begin{aligned}\int_{\mathbf{\Omega}} \nabla v(\mathbf{x}) \cdot \nabla w(\mathbf{x}) \,\mathrm{d}\mathbf{x} &= \int_{\partial\mathbf{\Omega}} \frac{\partial v}{\partial n}(\mathbf{s}) \cdot w(\mathbf{s}) \,\mathrm{d}\mathbf{s} - \int_{\mathbf{\Omega}} \Delta v(\mathbf{x}) \cdot w(\mathbf{x}) \,\mathrm{d}\mathbf{x} \\ &= \int_{\partial\mathbf{\Omega}} \left(\nabla v(\mathbf{s}) \cdot \mathbf{n}(\mathbf{s})\right) w(\mathbf{s}) \,\mathrm{d}\mathbf{s} - \int_{\mathbf{\Omega}} \Delta v(\mathbf{x}) \cdot w(\mathbf{x}) \,\mathrm{d}\mathbf{x},\end{aligned}$$
   where
   $$\int_{\mathbf{\Omega}} \nabla v(\mathbf{x}) \cdot \nabla w(\mathbf{x}) \,\mathrm{d}\mathbf{x} = \int_{\mathbf{\Omega}} \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} \,\mathrm{d}\mathbf{x}.$$
3. **Gaussian theorem:**
   For all $v \in W^{1,1}(\mathbf{\Omega})$ it is:
   $$\int_{\mathbf{\Omega}} \partial_i v(\mathbf{x}) \,\mathrm{d}\mathbf{x} = \int_{\partial\mathbf{\Omega}} v(\mathbf{s}) n_i(\mathbf{s}) \,\mathrm{d}\mathbf{s}.$$
   Generalizing this formula to vector fields $\mathbf{v} \in \mathbf{W}^{1,1}(\mathbf{\Omega})$ yields :
   $$\int_{\mathbf{\Omega}} \nabla \cdot \mathbf{v}(\mathbf{x}) \,\mathrm{d}\mathbf{x} = \int_{\partial\mathbf{\Omega}} \mathbf{v}(\mathbf{s}) \cdot \mathbf{n}(\mathbf{s}) \,\mathrm{d}\mathbf{s}.$$

Proof: For the proof it is recommended [John13]. □

Lemma 2.3.1 *Let $\mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega})$ and $\mathbf{\Omega} \subset \mathbb{R}^3$. Then*
$$\|\nabla \mathbf{v}\|^2_{\mathbf{L}^2(\mathbf{\Omega})} = \|\nabla \cdot \mathbf{v}\|^2_{L^2(\mathbf{\Omega})} + \|\nabla \times \mathbf{v}\|^2_{\mathbf{L}^2(\mathbf{\Omega})}.$$
In particular, it holds
$$\|\nabla \mathbf{v}\|_{\mathbf{L}^2(\mathbf{\Omega})} \geq \|\nabla \cdot \mathbf{v}\|_{L^2(\mathbf{\Omega})}.$$

Proof: Let $\mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega})$. By Lemma (2.1.1$(iv)$) it holds

$$-\Delta \mathbf{v}(\mathbf{x}) = -\nabla(\nabla \cdot \mathbf{v}(\mathbf{x})) + \nabla \times (\nabla \times \mathbf{v}(\mathbf{x})).$$

Multiplication with a test function $\mathbf{w} \in \mathbf{H}_0^1(\mathbf{\Omega})$, integration using integration by parts and Green's first formula results in:

$$\int_{\mathbf{\Omega}} \nabla \mathbf{v} : \nabla \mathbf{w} \, \mathrm{d}\mathbf{x} = \int_{\mathbf{\Omega}} (\nabla \cdot \mathbf{v})(\nabla \cdot \mathbf{w}) \, \mathrm{d}\mathbf{x} + \int_{\mathbf{\Omega}} (\nabla \times \mathbf{v}) \cdot (\nabla \times \mathbf{w}) \, \mathrm{d}\mathbf{x}.$$

Setting $\mathbf{v} = \mathbf{w}$ finishes the proof. □

We finish this chapter with a number of facts we will get back to later on.

Theorem 2.3.4 (Symmetry of second derivatives) *Let* $\mathbf{\Omega} \subset \mathbb{R}^n$ *and* $f \in C^2(\mathbf{\Omega})$. *Then it holds*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}), \quad \forall i, j \in \{1, \ldots, n\}.$$

Proof: See for example [Kab97], pp. 112. □

Weak derivatives also have the property that they can be interchanged, see [Ev10], pp. 261.

Theorem 2.3.5 (Symmetry of weak derivatives) *Let* $\mathbf{\Omega} \subset \mathbb{R}^n$, $u \in W^{k,p}(\mathbf{\Omega})$ *and* $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ *multi-indices with* $|\boldsymbol{\alpha}| + |\boldsymbol{\beta}| \leq k$. *Then it holds*

$$D^{\boldsymbol{\alpha}} \left(D^{\boldsymbol{\beta}} u \right) = D^{\boldsymbol{\beta}} \left(D^{\boldsymbol{\alpha}} u \right).$$

Proof: Since $u \in W^{k,p}(\mathbf{\Omega})$ the weak derivatives up to order k exist and $g \in C_0^\infty(\mathbf{\Omega})$ implies that one can test by $D^{\boldsymbol{\alpha}} g$ respectively $D^{\boldsymbol{\beta}} g$:

$$\begin{aligned} \int_{\mathbf{\Omega}} D^{\boldsymbol{\alpha}} u D^{\boldsymbol{\beta}} g \, \mathrm{d}\mathbf{x} &= (-1)^{|\boldsymbol{\alpha}|} \int_{\mathbf{\Omega}} u \left(D^{\boldsymbol{\alpha}+\boldsymbol{\beta}} g \right) \mathrm{d}\mathbf{x} \\ &= (-1)^{|\boldsymbol{\alpha}|} (-1)^{|\boldsymbol{\alpha}+\boldsymbol{\beta}|} \int_{\mathbf{\Omega}} \left(D^{\boldsymbol{\alpha}+\boldsymbol{\beta}} u \right) g \, \mathrm{d}\mathbf{x} \\ &= (-1)^{|\boldsymbol{\beta}|} \int_{\mathbf{\Omega}} \left(D^{\boldsymbol{\alpha}+\boldsymbol{\beta}} u \right) g \, \mathrm{d}\mathbf{x}. \end{aligned}$$

By the definition of the weak derivative this means that $D^{\boldsymbol{\beta}}\left(D^{\boldsymbol{\alpha}}u\right) = D^{\boldsymbol{\alpha}+\boldsymbol{\beta}}u$. Exchanging the roles of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ one obtains

$$\begin{aligned}\int_{\boldsymbol{\Omega}} D^{\boldsymbol{\beta}}uD^{\boldsymbol{\alpha}}g\,\mathrm{d}\mathbf{x} &= (-1)^{|\boldsymbol{\beta}|}\int_{\boldsymbol{\Omega}} u\left(D^{\boldsymbol{\alpha}+\boldsymbol{\beta}}g\right)\,\mathrm{d}\mathbf{x}\\ &= (-1)^{|\boldsymbol{\beta}|}(-1)^{|\boldsymbol{\alpha}+\boldsymbol{\beta}|}\int_{\boldsymbol{\Omega}}\left(D^{\boldsymbol{\alpha}+\boldsymbol{\beta}}u\right)g\,\mathrm{d}\mathbf{x}\\ &= (-1)^{|\boldsymbol{\alpha}|}\int_{\boldsymbol{\Omega}}\left(D^{\boldsymbol{\alpha}+\boldsymbol{\beta}}u\right)g\,\mathrm{d}\mathbf{x},\end{aligned}$$

i.e., $D^{\boldsymbol{\alpha}}\left(D^{\boldsymbol{\beta}}u\right) = D^{\boldsymbol{\alpha}+\boldsymbol{\beta}}u$. □

Theorem 2.3.6 (Rank–nullity theorem) *Let $f : V \to W$ be a linear map between vector spaces and V be finite-dimensional. Then it is*

$$\dim(V) = \dim(\mathrm{im}(f)) + \dim(\ker(f)).$$

Proof: This statement is proven in [Fi03], pp. 117. □

Definition 2.3.1 (Rayleigh quotient) *Given $A \in \mathbb{R}^{n\times n}$ and $\mathbf{x} \in \mathbb{C}^n\setminus\{\mathbf{0}\}$, the term*

$$R_A(\mathbf{x}) := \frac{\mathbf{x}^\star A\mathbf{x}}{\mathbf{x}^\star\mathbf{x}}$$

is called the Rayleigh quotient of A to $\mathbf{x}$.

Remark 2.3.1

The transposed of the complex conjugated is here denoted by a star, i.e., $\mathbf{x}^\star := (\overline{\mathbf{x}}^T)$. For $\mathbf{x} \in \mathbb{R}^n$ it holds $\mathbf{x}^T = (\overline{\mathbf{x}}^T)$.

Lemma 2.3.2 *For $A \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{C}$ an eigenvalue of A with corresponding eigenvector $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ it is*

$$R_A(\mathbf{b}) = \lambda.$$

Proof: The vector $\mathbf{b}$ is an eigenvector to the eigenvalue λ if and only if $A\mathbf{b} = \lambda\mathbf{b}$ and hence

$$R_A(\mathbf{b}) := \frac{\mathbf{b}^\star A\mathbf{b}}{\mathbf{b}^\star\mathbf{b}} = \frac{\mathbf{b}^\star\lambda\mathbf{b}}{\mathbf{b}^\star\mathbf{b}} = \lambda\frac{\mathbf{b}^\star\mathbf{b}}{\mathbf{b}^\star\mathbf{b}} = \lambda.$$

□

Lemma 2.3.3 *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. Then it holds*

$$\lambda_1 \le R_A(\mathbf{x}) \le \lambda_n.$$

Proof: Take the eigenvectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$ corresponding to the eigenvalues

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n. \tag{2.1}$$

Since A is a symmetric matrix, $\mathbf{b}_1, \ldots, \mathbf{b}_n$ form an orthogonal basis of $\mathbb{R}^n$. Assume that the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are normalized, i.e., they form an orthonormal basis. This means that

$$\mathbf{b}_i^T\mathbf{b}_j = \delta_{ij} := \begin{cases} 1, & \text{if } i = j, \\ 0, & \textit{else}, \end{cases} \qquad \forall i, j = 1, \ldots, n. \tag{2.2}$$

So we can represent any $\mathbf{x} \in \mathbb{R}^n$ as a linear combination of the eigenvectors:

$$\mathbf{x} = \sum_{i=1}^n c_i\mathbf{b}_i, \qquad c_i \in \mathbb{R}. \tag{2.3}$$

Multiplication with A yields

$$A\mathbf{x} = \sum_{i=1}^n c_i A\mathbf{b}_i = \sum_{i=1}^n c_i\lambda_i\mathbf{b}_i. \tag{2.4}$$

After plugging in (2.3) and (2.4) into $R_A(\mathbf{x})$ one obtains

$$R_A(\mathbf{x}) = \frac{\mathbf{x}^TA\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \overset{(2.3)}{\underset{(2.4)}{=}} \frac{\left(\sum\limits_{i=1}^n c_i\mathbf{b}_i\right)^T\left(\sum\limits_{i=1}^n c_i\lambda_i\mathbf{b}_i\right)}{\left(\sum\limits_{i=1}^n c_i\mathbf{b}_i\right)^T\left(\sum\limits_{i=1}^n c_i\mathbf{b}_i\right)} = \frac{\sum\limits_{i,j=1}^n c_ic_j\lambda_j(\mathbf{b}_i)^T\mathbf{b}_j}{\sum\limits_{i,j=1}^n c_ic_j(\mathbf{b}_i)^T\mathbf{b}_j}$$

$$\overset{(2.2)}{=} \frac{\sum\limits_{i=1}^n c_i^2\lambda_i}{\sum\limits_{i=1}^n c_i^2}.$$

Using (2.1) we conclude

$$R_A(\mathbf{x}) = \frac{\sum\limits_{i=1}^n c_i^2\lambda_i}{\sum\limits_{i=1}^n c_i^2} \overset{(2.1)}{\ge} \frac{\sum\limits_{i=1}^n c_i^2\lambda_1}{\sum\limits_{i=1}^n c_i^2} = \lambda_1\frac{\sum\limits_{i=1}^n c_i^2}{\sum\limits_{i=1}^n c_i^2} = \lambda_1$$

and

$$R_A(\mathbf{x}) = \frac{\sum\limits_{i=1}^n c_i^2\lambda_i}{\sum\limits_{i=1}^n c_i^2} \overset{(2.1)}{\le} \frac{\sum\limits_{i=1}^n c_i^2\lambda_n}{\sum\limits_{i=1}^n c_i^2} = \lambda_n\frac{\sum\limits_{i=1}^n c_i^2}{\sum\limits_{i=1}^n c_i^2} = \lambda_n.$$

□

3 The Stokes Equations

The incompressible Navier–Stokes equations are nowadays the classical tool for describing fluid flows. They model the n-dimensional motion of viscous, incompressible (Newtonian) fluids namely incompressible flows, for $n = 2, 3$, subject to an external force. A flow is called incompressible if the density of the fluid is constant along trajectories of a fluid element for constant temperature and changing pressure.

The dimensionless, incompressible Navier–Stokes equations can be formulated as follows:

$$\underbrace{\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}}_{\substack{\text{total acceleration}\\ \text{of a particle}\\ \text{in the fluid}}} \underbrace{-\frac{1}{Re}\Delta \mathbf{u}}_{\substack{\text{friction between}\\ \text{the particles}\\ \text{of the fluid}}} + \nabla p = \mathbf{f} \qquad \text{in } \mathbf{\Omega} \times (0, T), \tag{3.1a}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \mathbf{\Omega} \times (0, T), \tag{3.1b}$$

where

- $\mathbf{\Omega} \subset \mathbb{R}^n$, $n \geq 2$ – a bounded, non-empty, polyhedral domain,
- $\partial\mathbf{\Omega}$ – Lipschitz boundary,
- $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{\Omega}$ – space variable,
- $t \in [0, T) \subset \mathbb{R}$ – time variable,
- $\mathbf{u} : \mathbf{\Omega} \times [0, T) \to \mathbb{R}^n$,
  $\mathbf{u}(\mathbf{x}, t) = (u_1(\mathbf{x}, t), \ldots, u_n(\mathbf{x}, t))$ – velocity of the fluid at $(\mathbf{x}, t)$,
- $p : \mathbf{\Omega} \times (0, T) \to \mathbb{R}$,
  $p(\mathbf{x}, t)$ – pressure at $(\mathbf{x}, t)$,
- $\mathbf{f} : \mathbf{\Omega} \times [0, T) \to \mathbb{R}^n$,
  $\mathbf{f}(\mathbf{x}, t) = (f_1(\mathbf{x}, t), \ldots, f_n(\mathbf{x}, t))$ – external body force,
- $Re > 0$ – Reynolds number of the fluid, constant in $(\mathbf{x}, t)$,
- $\frac{1}{Re}$ – constant kinematic viscosity of the fluid.

Here the first equation (3.1a) arises from Newton's law of balanced forces, $\mathbf{F} = m \cdot \mathbf{a}$, and it is referred to as the momentum equation.
The second equation (3.1b) models the incompressibility, i.e., the constant

density and is therefore called incompressibility constraint or continuity equation. It is derived from the law of conservation of mass.

There are several ways to derive these equations and for a detailed derivation from continuum mechanics, [John12] is recommended to the interested reader.
This is also the setting we are studying in the whole thesis. As $\mathbf{\Omega}$ is a domain, it is always connected and open, and for our studies it will suffice to assume a Lipschitz boundary $\partial\mathbf{\Omega}$.

Considering only very slow fluid flows, i.e., Re is very small, e.g., for honey, the non-linear convective term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ can be omitted, since in comparison to the viscous term $-\frac{1}{Re}\Delta\mathbf{u}$ it is negligibly small. The resulting system of linear partial differential equations is called the nonstationary Stokes system. If $\mathbf{u}$, p, and $\mathbf{f}$ are independent of t, then particularly $\mathbf{u}_t = \mathbf{0}$ and we obtain the stationary Stokes system. Assuming both special cases mentioned before, we end up with a simplification of (3.1) which is called the Stokes equations:

Find $\mathbf{u} : \mathbf{\Omega} \to \mathbb{R}^n$ and $p : \mathbf{\Omega} \to \mathbb{R}$, such that for a given force field $\mathbf{f} : \mathbf{\Omega} \to \mathbb{R}^n$ and kinematic viscosity $\frac{1}{Re} > 0$, $\mathbf{u}$ and p fulfill

$$\begin{aligned} -\frac{1}{Re}\Delta\mathbf{u} + \nabla p &= \mathbf{f} \qquad \text{in } \mathbf{\Omega}, \\ \nabla \cdot \mathbf{u} &= 0 \qquad \text{in } \mathbf{\Omega}. \end{aligned}$$

Multiplying the first equation by Re, i.e., scaling by the Reynolds number, we get for a modified pressure p and right-hand side $\mathbf{f}$, for the sake of simplicity without new notation,

$$\begin{aligned} -\Delta\mathbf{u} + \nabla p &= \mathbf{f} \qquad \text{in } \mathbf{\Omega}, \\ \nabla \cdot \mathbf{u} &= 0 \qquad \text{in } \mathbf{\Omega}. \end{aligned} \tag{3.2}$$

In order to receive a well-posed problem, (3.2) has to be equipped with suitable boundary conditions.
We choose the homogeneous Dirichlet boundary conditions introduced below.

Definition 3.0.2 (Dirichlet / No-slip boundary condition) *The system* (3.1*a*) *and* (3.1*b*) *is said to be equipped with Dirichlet boundary conditions, if*

$$\mathbf{u} = \mathbf{g} \qquad \text{on } \partial\mathbf{\Omega}, \qquad \text{i.e., the velocity } \mathbf{u} \text{ is prescribed at the boundary.}$$

If $\mathbf{g} \equiv \mathbf{0}$, the boundary condition is called homogeneous Dirichlet boundary condition or no-slip boundary condition.

Boundary conditions are only meaningful for the Stokes system if they satisfy a certain compatibility constraint:

Lemma 3.0.4 *Boundary conditions for the problem* (3.2) *have to be chosen such that*

$$\int_{\partial\mathbf{\Omega}} \mathbf{g} \cdot \mathbf{n} \, \mathrm{ds} = 0, \qquad \textit{for } \mathbf{n} \textit{ the (unit) outer normal on } \partial\mathbf{\Omega}. \tag{3.3}$$

Proof: Let

$$\mathbf{u} = \mathbf{g} \qquad \text{on } \partial\mathbf{\Omega}.$$

Then the incompressibility constraint in the Stokes problem (3.2) and the Gaussian theorem imply

$$0 = \int_{\mathbf{\Omega}} \nabla \cdot \mathbf{u} \, \mathrm{d}\mathbf{x} = \int_{\partial\mathbf{\Omega}} \mathbf{u} \cdot \mathbf{n} \, \mathrm{ds} = \int_{\partial\mathbf{\Omega}} \mathbf{g} \cdot \mathbf{n} \, \mathrm{ds}.$$

$\square$

For simplicity we restrict our analysis to homogeneous Dirichlet boundary conditions for $\mathbf{u}$. The compatibility constraint (3.3) is therefore fulfilled. Thus, the problem we want to concentrate on is the

Strong formulation of the Stokes problem:

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \mathbf{\Omega}, \tag{3.4a}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \mathbf{\Omega}, \tag{3.4b}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \partial\mathbf{\Omega}. \tag{3.4c}$$

Apparently, the solution of the strong formulation of the Stokes problem (3.4) has to fulfill $\mathbf{u} \in \mathbf{C}^2(\mathbf{\Omega}) \cap \mathbf{C}(\bar{\mathbf{\Omega}})$ and $p \in C^1(\mathbf{\Omega})$.

Definition 3.0.3 (Classical solution) *A pair* $(\mathbf{u}, p) \in \left(\mathbf{C}^2(\mathbf{\Omega}) \cap \mathbf{C}(\bar{\mathbf{\Omega}})\right) \times C^1(\mathbf{\Omega})$ *is called classical solution of the Stokes problem if it fulfills the equations* (3.4) *for a given force* $\mathbf{f} \in \mathbf{C}(\mathbf{\Omega})$.

In the system (3.4) only the gradient of p appears and there are no requirements imposed on the behavior of p itself on the boundary or in $\mathbf{\Omega}$. Therefore, in general the pressure p, if it exists, is not unique. If $(\mathbf{u}, p)$ is a solution of the Stokes problem (3.4), then, for any constant $c \in \mathbb{R}$, $(\mathbf{u}, p + c)$ is a solution, too. The reason behind this is that $\nabla p = \nabla(p + c)$.
As a remedy, one has to insist on an additional condition for the pressure p in order to be able to find a unique solution. One possible way to fix the additive constant is to search for a pressure p with vanishing integral mean value over the domain $\mathbf{\Omega}$:

$$\int_{\mathbf{\Omega}} p \, \mathrm{d}\mathbf{x} = 0.$$

3.1 The Weak Formulation

The weak or variational formulation of the Stokes problem is the basis for using the mixed finite element method.
Many naturally arising, interesting applications, e.g., from physics, have solutions which violate the smoothness requirements for the classical solution. For example, it is also necessary to be able to find the solutions of problems with discontinuous right-hand side.

Example:
This is illustrated for a much more simple partial differential equation. Given the following Poisson equation with homogeneous Dirichlet boundary condition on $\mathbf{\Omega} = (-1, 1)^2$:

$$\begin{aligned} -\Delta u &= \operatorname{sgn}\left(\frac{1}{2} - |\mathbf{x}|\right), && \text{in } \mathbf{\Omega}, \\ u &= 0, && \text{on } \partial\mathbf{\Omega}. \end{aligned}$$

This problem does not have a classical solution $u \in C^2(\mathbf{\Omega}) \cap C(\bar{\mathbf{\Omega}})$ because otherwise this would imply $-\Delta u = \operatorname{sgn}(\frac{1}{2} - |\mathbf{x}|)$ to be continuous which is not the case.

The advantage of using the weak formulation is that less smooth solutions can be considered. For that purpose we introduced in Chapter 2 the concept of weak derivatives and Sobolev spaces which are essential for the weak formulation of partial differential equations.

The usual way to get the weak formulation of a partial differential equation is to multiply (dot product) the equations by so-called test functions, integrate them on $\mathbf{\Omega}$, and afterwards apply integration by parts. The last step enables the transfer of derivatives to the test functions.
To that end we should be aware of some theorems from Section 2.3.

Let $\mathbf{v} \in \mathbf{C}_0^\infty(\mathbf{\Omega})$ and $q \in C^\infty_{\int=0}(\mathbf{\Omega}) := C^\infty(\mathbf{\Omega}) \cap \{\phi : \int\limits_{\mathbf{\Omega}} \phi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0\}$ be test functions. Multiplying the first equation (3.4a) with $\mathbf{v}$ and the second equation (3.4b) with q, (3.4) reformulates to

$$\begin{aligned} -\Delta\mathbf{u}\cdot\mathbf{v} + \nabla p\cdot\mathbf{v} &= \mathbf{f}\cdot\mathbf{v}, && \forall \mathbf{v} \in \mathbf{C}_0^\infty(\mathbf{\Omega}),\\ (\nabla\cdot\mathbf{u})q &= 0, && \forall q \in C^\infty_{\int=0}(\mathbf{\Omega}). \end{aligned}$$

By integrating on $\mathbf{\Omega}$ we get

$$\begin{aligned} \int_{\mathbf{\Omega}} -\Delta\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_{\mathbf{\Omega}} \nabla p\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} &= \int_{\mathbf{\Omega}} \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}, && \forall \mathbf{v} \in \mathbf{C}_0^\infty(\mathbf{\Omega}),\\ \int_{\mathbf{\Omega}} (\nabla\cdot\mathbf{u})q\,\mathrm{d}\mathbf{x} &= 0, && \forall q \in C^\infty_{\int=0}(\mathbf{\Omega}). \end{aligned}$$

Using integration by parts and Green's first formula in combination with the compact support of $\mathbf{v}$, i.e., the vanishing boundary integrals, yields

$$\begin{aligned} \int_{\mathbf{\Omega}} \nabla\mathbf{u} : \nabla\mathbf{v}\,\mathrm{d}\mathbf{x} - \int_{\mathbf{\Omega}} (\nabla\cdot\mathbf{v})p\,\mathrm{d}\mathbf{x} &= \int_{\mathbf{\Omega}} \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}, && \forall \mathbf{v} \in \mathbf{C}_0^\infty(\mathbf{\Omega}),\\ \int_{\mathbf{\Omega}} (\nabla\cdot\mathbf{u})q\,\mathrm{d}\mathbf{x} &= 0, && \forall q \in C^\infty_{\int=0}(\mathbf{\Omega}). \end{aligned} \tag{3.5}$$

Now we define the function spaces such that the arising integrals in (3.5) are well-defined. An appropriate ansatz space for the pressure p is a subset of $L^2(\mathbf{\Omega})$. Furthermore, for the uniqueness we enforce $p \in L_0^2(\mathbf{\Omega})$.
For $\mathbf{u}$ one has to require $\nabla\mathbf{u} \in \mathbf{L}^2(\mathbf{\Omega})$ and the homogeneous Dirichlet boundary condition $\mathbf{u}|_{\partial\mathbf{\Omega}} = \mathbf{0}$. Hence the above formulation makes sense if $\mathbf{u} \in \mathbf{H}_0^1(\mathbf{\Omega})$.
Under these conditions, one has to think about the regularity assumptions for the test functions leading to well-defined integrals.
The space $C_0^\infty(\mathbf{\Omega})$ is dense in $H_0^1(\mathbf{\Omega})$ and the space $C^\infty_{\int=0}(\mathbf{\Omega})$ is dense in $L_0^2(\mathbf{\Omega})$. Hence, it is allowed to test the first equation by $H_0^1(\mathbf{\Omega})$ functions and the second equation by $L_0^2(\mathbf{\Omega})$ functions.

At this point we have all basics to write down the

Weak formulation of the Stokes problem:

For a given $\mathbf{f} \in \mathbf{H}^{-1}(\mathbf{\Omega})$ find $(\mathbf{u}, p) \in \mathbf{H}_0^1(\mathbf{\Omega}) \times L_0^2(\mathbf{\Omega})$ such that

$$\begin{aligned} \int_{\mathbf{\Omega}} \nabla\mathbf{u} : \nabla\mathbf{v} \, \mathrm{d}\mathbf{x} - \int_{\mathbf{\Omega}} (\nabla \cdot \mathbf{v}) p \, \mathrm{d}\mathbf{x} &= \int_{\mathbf{\Omega}} \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x}, \qquad && \forall \mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega}), \\ \int_{\mathbf{\Omega}} (\nabla \cdot \mathbf{u}) q \, \mathrm{d}\mathbf{x} &= 0, && \forall q \in L_0^2(\mathbf{\Omega}). \end{aligned} \tag{3.6}$$

Definition 3.1.1 (Weak solution) *Let* $\mathbf{f} \in \mathbf{H}^{-1}(\mathbf{\Omega})$.
A pair $(\mathbf{u}, p) \in \mathbf{H}_0^1(\mathbf{\Omega}) \times L_0^2(\mathbf{\Omega})$ *is called weak solution of the Stokes problem if it fulfills the system* (3.6)*.*

Remark 3.1.1

A classical solution of the Stokes problem (3.4) is of course a solution of the weak Stokes problem (3.6) by construction and if the weak solution is smooth enough, then it is a classical solution. To realize this we prove the following lemma.

Lemma 3.1.1 *For* $\mathbf{f} \in \mathbf{C}(\mathbf{\Omega})$ *a weak solution* $(\mathbf{u}, p) \in \mathbf{H}_0^1(\mathbf{\Omega}) \times L_0^2(\mathbf{\Omega})$ *with* $\mathbf{u} \in \left(\mathbf{C}^2(\mathbf{\Omega}) \cap \mathbf{C}(\bar{\mathbf{\Omega}})\right)$ *and* $p \in C^1(\mathbf{\Omega})$ *is a classical solution.*

Proof: Let $(\mathbf{u}, p) \in \mathbf{H}_0^1(\mathbf{\Omega}) \times L_0^2(\mathbf{\Omega})$ be a weak solution, i.e.,

$$\int_{\mathbf{\Omega}} \nabla\mathbf{u} : \nabla\mathbf{v} \, \mathrm{d}\mathbf{x} - \int_{\mathbf{\Omega}} (\nabla \cdot \mathbf{v}) p \, \mathrm{d}\mathbf{x} = \int_{\mathbf{\Omega}} \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x}, \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega}), \tag{3.7a}$$

$$\int_{\mathbf{\Omega}} (\nabla \cdot \mathbf{u}) q \, \mathrm{d}\mathbf{x} = 0, \qquad \forall q \in L_0^2(\mathbf{\Omega}). \tag{3.7b}$$

Furthermore, let $\mathbf{u} \in \left(\mathbf{C}^2(\mathbf{\Omega}) \cap \mathbf{C}(\bar{\mathbf{\Omega}})\right)$ and $p \in C^1(\mathbf{\Omega})$.
We start by proving the incompressibility condition (3.4b).
Choose $q := \nabla \cdot \mathbf{u} \in L^2(\mathbf{\Omega})$. Then there exists a constant c such that $q - c = \nabla \cdot \mathbf{u} - c \in L_0^2(\mathbf{\Omega})$. Using (3.7$b$) one obtains

$$\begin{aligned} \int_{\mathbf{\Omega}} (\nabla \cdot \mathbf{u}) q \, \mathrm{d}\mathbf{x} = 0, \ \forall q \in L_0^2(\mathbf{\Omega}) \xRightarrow{q := \nabla\cdot\mathbf{u} - c} & \int_{\mathbf{\Omega}} (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{u} - c) \, \mathrm{d}\mathbf{x} = 0 \\ \Longleftrightarrow & \int_{\mathbf{\Omega}} (\nabla \cdot \mathbf{u})^2 \, \mathrm{d}\mathbf{x} = c \int_{\mathbf{\Omega}} (\nabla \cdot \mathbf{u}) \, \mathrm{d}\mathbf{x} \\ \Longleftrightarrow & \|\nabla \cdot \mathbf{u}\|_{\mathbf{L}^2(\mathbf{\Omega})}^2 = c \int_{\mathbf{\Omega}} (\nabla \cdot \mathbf{u}) \, \mathrm{d}\mathbf{x}. \end{aligned}$$

The Gaussian theorem and $\mathbf{u}|_{\partial\mathbf{\Omega}} = \mathbf{0}$ lead to

$$\int\limits_{\mathbf{\Omega}} \nabla \cdot \mathbf{u} \, \mathrm{d}\mathbf{x} = \int\limits_{\partial\mathbf{\Omega}} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}\mathbf{s} = 0.$$

$$\Longrightarrow \|\nabla \cdot \mathbf{u}\|^2_{\mathbf{L}^2(\mathbf{\Omega})} = 0 \Longrightarrow \nabla \cdot \mathbf{u} = 0$$

almost everywhere in $\mathbf{\Omega}$. The assumption $\mathbf{u} \in C^2(\mathbf{\Omega}) \subset C^1(\mathbf{\Omega})$ implies incompressibility for all $\mathbf{x} \in \mathbf{\Omega}$.

For the weak solution $\mathbf{u}$ it holds (3.7a):

$$\int_{\mathbf{\Omega}} \nabla\mathbf{u} : \nabla\mathbf{v} \, \mathrm{d}\mathbf{x} = \int_{\mathbf{\Omega}} \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} + \int_{\mathbf{\Omega}} (\nabla \cdot \mathbf{v}) p \, \mathrm{d}\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\mathbf{\Omega}).$$

Therefore using integration by parts reversed and the continuity of $\Delta\mathbf{u} + \mathbf{f} - \nabla p$, $(\mathbf{u}, p)$ is a classical solution of

$$-\Delta\mathbf{u} = \mathbf{f} - \nabla p$$

with homogeneous Dirichlet boundary conditions which is equivalent to (3.4a). $\square$

3.2 Existence and Uniqueness of Weak Solutions

After reducing the regularity expectations for a solution of the strong Stokes problem via the idea of weak derivatives, we are now interested in the question under which conditions one can guarantee the existence of a unique weak solution.

The proof of the existence of a unique weak solution $(\mathbf{u}, p)$ is split into two steps. The first step is to restrict ourselves to the weakly solenoidal functions in $\mathbf{H}^1_0(\mathbf{\Omega})$ as the test space, which leads to the decoupling of the velocity $\mathbf{u}$ from the pressure p and enables a separate treatment of $\mathbf{u}$. The second step is then to prove the existence of a unique corresponding pressure.

3.2.1 A Reformulation of the Weak Stokes Problem

We will start by discussing some important properties of the arising integrals in the weak formulation.

Notation: Define the following two bilinear forms and the linear form:

- $a(\cdot,\cdot) : \mathbf{H}_0^1(\boldsymbol{\Omega}) \times \mathbf{H}_0^1(\boldsymbol{\Omega}) \to \mathbb{R}$,

$$a(\mathbf{u},\mathbf{v}) := \int_{\boldsymbol{\Omega}} \nabla\mathbf{u} : \nabla\mathbf{v}\,\mathrm{d}\mathbf{x}, \qquad \forall \mathbf{u},\mathbf{v} \in \mathbf{H}_0^1(\boldsymbol{\Omega})$$

  with

$$\|a\| := \sup_{\mathbf{u},\mathbf{v}\in\mathbf{H}_0^1(\boldsymbol{\Omega})\setminus\{\mathbf{0}\}} \frac{a(\mathbf{u},\mathbf{v})}{\|\mathbf{u}\|_{\mathbf{H}_0^1(\boldsymbol{\Omega})}\|\mathbf{v}\|_{\mathbf{H}_0^1(\boldsymbol{\Omega})}},$$

- $b(\cdot,\cdot) : \mathbf{H}_0^1(\boldsymbol{\Omega}) \times L_0^2(\boldsymbol{\Omega}) \to \mathbb{R}$,

$$b(\mathbf{v},p) := -\int_{\boldsymbol{\Omega}} (\nabla\cdot\mathbf{v})p\,\mathrm{d}\mathbf{x}, \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\boldsymbol{\Omega}), p \in L_0^2(\boldsymbol{\Omega})$$

  with

$$\|b\| := \sup_{\substack{\mathbf{v}\in\mathbf{H}_0^1(\boldsymbol{\Omega})\setminus\{\mathbf{0}\}\\ p\in L_0^2(\boldsymbol{\Omega})\setminus\{0\}}} \frac{b(\mathbf{v},p)}{\|\mathbf{v}\|_{\mathbf{H}_0^1(\boldsymbol{\Omega})}\|p\|_{L_0^2(\boldsymbol{\Omega})}},$$

- $f : \mathbf{H}_0^1(\boldsymbol{\Omega}) \to \mathbb{R}$,

$$\begin{aligned} f(\mathbf{v}) &:= \langle \mathbf{f},\mathbf{v}\rangle, && \forall \mathbf{f} \in \mathbf{H}^{-1}(\boldsymbol{\Omega}),\ \mathbf{v} \in \mathbf{H}_0^1(\boldsymbol{\Omega}),\\ f(\mathbf{v}) &:= \int_{\boldsymbol{\Omega}} \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}, && \forall \mathbf{f} \in \mathbf{L}^2(\boldsymbol{\Omega}),\ \mathbf{v} \in \mathbf{H}_0^1(\boldsymbol{\Omega}). \end{aligned}$$

Definition 3.2.1 *Let W be a Banach space (more specific a Hilbert space) and $c(\cdot,\cdot) : W \times W \to \mathbb{R}$ a map. Then we call $c(\cdot,\cdot)$*

1. **bounded/continuous** *if*

$$\exists M > 0 : \ |c(u,v)| \le M\|u\|_W\|v\|_W, \qquad \forall u,v \in W,$$

2. **coercive/W-elliptic** *if*

$$\exists m > 0 : \ c(u,u) \ge m\|u\|_W^2, \qquad \forall u \in W,$$

3. **symmetric** *if*

$$c(u,v) = c(v,u), \qquad \forall u,v \in W,$$

4. **positive definite** *if*

$$c(u,u) > 0, \qquad \forall u \in W \setminus \{0\},$$

5. **bilinear form** *if*

$$c(\alpha u + \beta v, w) = \alpha c(u,w) + \beta c(v,w)$$

   and

$$c(u, \alpha v + \beta w) = \alpha c(u,v) + \beta c(u,w), \qquad \forall \alpha,\beta \in \mathbb{R}, u,v,w \in W.$$

Lemma 3.2.1 *It holds*

1. *$a(\cdot,\cdot)$ is a symmetric, bounded, positive definite, and coercive bilinear form,*
2. *$b(\cdot,\cdot)$ is a bounded bilinear form,*
3. *$f(\cdot)$ is a bounded linear functional.*

Proof: 1. $a(\cdot,\cdot)$:

Symmetry: This is trivially true by the symmetry of the tensor product:

$$
\begin{aligned}
a(\mathbf{u},\mathbf{v}) &= \int_{\mathbf{\Omega}} \nabla\mathbf{u} : \nabla\mathbf{v}\,\mathrm{d}\mathbf{x} = \int_{\mathbf{\Omega}} \sum_{i,j=1}^{n} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,\mathrm{d}\mathbf{x} \\
&= \int_{\mathbf{\Omega}} \sum_{i,j=1}^{n} \frac{\partial v_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\,\mathrm{d}\mathbf{x} = \int_{\mathbf{\Omega}} \nabla\mathbf{v} : \nabla\mathbf{u}\,\mathrm{d}\mathbf{x} = a(\mathbf{v},\mathbf{u}),
\end{aligned}
$$

$\forall \mathbf{u},\mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega})$.

Boundedness:

$$
\begin{aligned}
|a(\mathbf{u},\mathbf{v})| &= \left|\int_{\mathbf{\Omega}} \nabla\mathbf{u} : \nabla\mathbf{v}\,\mathrm{d}\mathbf{x}\right| \le \int_{\mathbf{\Omega}} |\nabla\mathbf{u} : \nabla\mathbf{v}|\;\mathrm{d}\mathbf{x} = \|\nabla\mathbf{u} : \nabla\mathbf{v}\|_{L^1(\mathbf{\Omega})} \\
&\overbrace{\le}^{\text{Cauchy–Schwarz}} \quad \|\nabla\mathbf{u}\|_{\mathbf{L}^2(\mathbf{\Omega})}\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\mathbf{\Omega})} = \|\mathbf{u}\|_{\mathbf{H}_0^1(\mathbf{\Omega})}\|\mathbf{v}\|_{\mathbf{H}_0^1(\mathbf{\Omega})},
\end{aligned}
$$

$\forall \mathbf{u},\mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega})$.
So choosing $M = 1$ finishes the proof.

Positive Definiteness:

$$
a(\mathbf{u},\mathbf{u}) = \int_{\mathbf{\Omega}} \nabla\mathbf{u} : \nabla\mathbf{u}\,\mathrm{d}\mathbf{x} = \|\nabla\mathbf{u}\|^2_{L^2(\mathbf{\Omega})} > 0, \quad \forall \mathbf{u} \in \mathbf{H}_0^1(\mathbf{\Omega}) \setminus \{0\}.
$$

Coercivity:

$$
a(\mathbf{u},\mathbf{u}) = \int_{\mathbf{\Omega}} \nabla\mathbf{u} : \nabla\mathbf{u}\,\mathrm{d}\mathbf{x} = \|\nabla\mathbf{u}\|^2_{\mathbf{L}^2(\mathbf{\Omega})} = \|\mathbf{u}\|^2_{\mathbf{H}_0^1(\mathbf{\Omega})},
$$

$\forall \mathbf{u} \in \mathbf{H}_0^1(\mathbf{\Omega})$.
The constant $m = 1$ leads to coercivity.

Bilinearity: We already proved that $a(\cdot,\cdot)$ is symmetric, so for bilinearity it suffices to proof the linearity in one argument:

$$
\begin{aligned}
a(\alpha(\mathbf{u}+\mathbf{v}),\mathbf{w}) &= \int_{\mathbf{\Omega}} \nabla(\alpha(\mathbf{u}+\mathbf{v})) : \nabla \mathbf{w}\,\mathrm{d}\mathbf{x} \\
&= \int_{\mathbf{\Omega}} \sum_{i,j=1}^{n} \frac{\partial(\alpha(u_i+v_i))}{\partial x_j}\frac{\partial w_i}{\partial x_j}\,\mathrm{d}\mathbf{x} \\
&= \alpha \int_{\mathbf{\Omega}} \sum_{i,j=1}^{n} \frac{\partial u_i + \partial v_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}\,\mathrm{d}\mathbf{x} \\
&= \alpha \int_{\mathbf{\Omega}} \sum_{i,j=1}^{n} \frac{\partial u_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}\,\mathrm{d}\mathbf{x} + \alpha \int_{\mathbf{\Omega}} \sum_{i,j=1}^{n} \frac{\partial v_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}\,\mathrm{d}\mathbf{x} \\
&= \alpha a(\mathbf{u},\mathbf{w}) + \alpha a(\mathbf{v},\mathbf{w}),\ \ \forall \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{H}_0^1(\mathbf{\Omega}), \alpha \in \mathbb{R}.
\end{aligned}
$$

2. $b(\cdot,\cdot)$:
   Boundedness:

$$
\begin{aligned}
|b(\mathbf{u},q)| &= \left| - \int_{\mathbf{\Omega}} (\nabla\cdot\mathbf{u})q\,\mathrm{d}\mathbf{x}\right| = \left|\int_{\mathbf{\Omega}} (\nabla\cdot\mathbf{u})q\,\mathrm{d}\mathbf{x}\right| \leq \int_{\mathbf{\Omega}} |(\nabla\cdot\mathbf{u})q|\,\mathrm{d}\mathbf{x} \\
&= \|(\nabla\cdot\mathbf{u})q\|_{L^1(\mathbf{\Omega})} \overbrace{\leq}^{\text{Cauchy–Schwarz}} \|\nabla\cdot\mathbf{u}\|_{L^2(\mathbf{\Omega})}\|q\|_{L^2(\mathbf{\Omega})} \\
&\leq \|\nabla\mathbf{u}\|_{\mathbf{L}^2(\mathbf{\Omega})}\|q\|_{L^2(\mathbf{\Omega})} = \|\mathbf{u}\|_{H_0^1(\mathbf{\Omega})}\|q\|_{L^2(\mathbf{\Omega})},
\end{aligned}
$$

   $\forall \mathbf{u} \in \mathbf{H}_0^1(\mathbf{\Omega}), q \in L_0^2(\mathbf{\Omega})$.
   The last inequality is proven in Lemma 2.3.1. Hence $M = 1$ satisfies the condition.

   Bilinearity: The bilinearity is proven similarly as for $a(\cdot,\cdot)$ but here one has to check the properties for both arguments, the first argument for $\mathbf{H}_0^1(\mathbf{\Omega})$ and the second for $L_0^2(\mathbf{\Omega})$, by reason of absent symmetry.

3. $f(\cdot)$:
   Boundedness:
   To show: $\exists M > 0 : \ |f(\mathbf{v})| \overset{!}{\leq} M\|\mathbf{v}\|_{\mathbf{H}_0^1(\mathbf{\Omega})} \quad , \forall \mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega})$.
   For $\mathbf{f} \in \mathbf{L}^2(\mathbf{\Omega})$ it is

$$
\begin{aligned}
|f(\mathbf{v})| &= \left|\int_{\mathbf{\Omega}} \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}\right| \leq \int_{\mathbf{\Omega}} |\mathbf{f}\cdot\mathbf{v}|\,\mathrm{d}\mathbf{x} = \|\mathbf{f}\cdot\mathbf{v}\|_{L^1(\mathbf{\Omega})} \\
&\overbrace{\leq}^{\text{Cauchy–Schwarz}} \|\mathbf{f}\|_{\mathbf{L}^2(\mathbf{\Omega})}\|\mathbf{v}\|_{\mathbf{L}^2(\mathbf{\Omega})} \overbrace{\leq}^{\text{cl. Poinc.–Fr.}} \|\mathbf{f}\|_{\mathbf{L}^2(\mathbf{\Omega})} C\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\mathbf{\Omega})} \\
&= C\|\mathbf{f}\|_{\mathbf{L}^2(\mathbf{\Omega})}\|\mathbf{v}\|_{\mathbf{H}_0^1(\mathbf{\Omega})}, \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega}).
\end{aligned}
$$

   Setting $M = C\|\mathbf{f}\|_{\mathbf{L}^2(\mathbf{\Omega})}$, for $C > 0$ from the classical Poincaré-Friedrichs inequality, is a possible choice.

A way to proof the continuity of $f(\cdot)$ for $\mathbf{f} \in \mathbf{H}^{-1}(\boldsymbol{\Omega})$ is obtained by using the definition of the H^{-1}-norm as presented below:

$$\|\mathbf{f}\|_{\mathbf{H}^{-1}(\boldsymbol{\Omega})} = \sup_{\mathbf{v}\in\mathbf{H}_0^1(\boldsymbol{\Omega})\setminus\{\mathbf{0}\}} \frac{\langle \mathbf{f}, \mathbf{v}\rangle}{\|\mathbf{v}\|_{\mathbf{H}_0^1(\boldsymbol{\Omega})}} \geq \frac{\langle \mathbf{f}, \mathbf{v}\rangle}{\|\mathbf{v}\|_{\mathbf{H}_0^1(\boldsymbol{\Omega})}}, \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\boldsymbol{\Omega})$$
$$\Longrightarrow |f(\mathbf{v})| = |\langle \mathbf{f}, \mathbf{v}\rangle| \leq \|\mathbf{f}\|_{\mathbf{H}^{-1}(\boldsymbol{\Omega})}\|\mathbf{v}\|_{\mathbf{H}_0^1(\boldsymbol{\Omega})}, \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\boldsymbol{\Omega}),$$

with $M = \|\mathbf{f}\|_{\mathbf{H}^{-1}(\boldsymbol{\Omega})}$.

Linearity: For $\mathbf{f} \in \mathbf{L}^2(\boldsymbol{\Omega})$ we obtain

$$\begin{aligned}
f(\alpha\mathbf{u} + \beta\mathbf{v}) &= \int_{\boldsymbol{\Omega}} \mathbf{f} \cdot (\alpha\mathbf{u} + \beta\mathbf{v})\, \mathrm{d}\mathbf{x} \\
&= \int_{\boldsymbol{\Omega}} (\alpha\mathbf{f} \cdot \mathbf{u} + \beta\mathbf{f} \cdot \mathbf{v})\, \mathrm{d}\mathbf{x} \\
&= \alpha \int_{\boldsymbol{\Omega}} \mathbf{f} \cdot \mathbf{u}\, \mathrm{d}\mathbf{x} + \beta \int_{\boldsymbol{\Omega}} \mathbf{f} \cdot \mathbf{v}\, \mathrm{d}\mathbf{x} \\
&= \alpha f(\mathbf{u}) + \beta f(\mathbf{v}), \qquad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}_0^1(\boldsymbol{\Omega}), \alpha, \beta \in \mathbb{R}.
\end{aligned}$$

If $\mathbf{f} \in \mathbf{H}^{-1}(\boldsymbol{\Omega})$ it follows

$$\begin{aligned}
f(\alpha\mathbf{u} + \beta\mathbf{v}) = \langle \mathbf{f}, \alpha\mathbf{u} + \beta\mathbf{v}\rangle &= \langle \mathbf{f}, \alpha\mathbf{u}\rangle + \langle \mathbf{f}, \beta\mathbf{v}\rangle \\
&= \alpha\langle \mathbf{f}, \mathbf{u}\rangle + \beta\langle \mathbf{f}, \mathbf{v}\rangle = \alpha f(\mathbf{u}) + \beta f(\mathbf{v}),
\end{aligned}$$
$$\forall \mathbf{u}, \mathbf{v} \in \mathbf{H}_0^1(\boldsymbol{\Omega}), \alpha, \beta \in \mathbb{R}.$$

$\square$

Remark 3.2.1

For normed spaces V and W a linear operator $L : V \to W$ is continuous if and only if it is bounded. Hence, the boundedness of a bilinear form is equivalent to its continuity. This statement is proved in [Wer06], pp. 305.

With the bilinear forms $a(\cdot, \cdot)$, $b(\cdot, \cdot)$, and the linear functional $f(\cdot)$ we can transform (3.6) into the following setting:

Reformulation of the Weak Stokes problem:

For given $\mathbf{f} \in \mathbf{H}^{-1}(\boldsymbol{\Omega})$, find $(\mathbf{u}, p) \in \mathbf{H}_0^1(\boldsymbol{\Omega}) \times L_0^2(\boldsymbol{\Omega})$ such that

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = f(\mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\boldsymbol{\Omega}), \tag{3.8a}$$
$$b(\mathbf{u}, q) = 0, \qquad \forall q \in L_0^2(\boldsymbol{\Omega}). \tag{3.8b}$$

3.2.2 The Saddle Point Approach and the Inf-Sup Condition

With the bounded bilinear maps $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ one can associate linear operators. Thus, (3.8) can be rewritten as the following abstract system which is equivalent to the weak formulation.

Definition 3.2.2 (Saddle point problem) *For given* $\mathbf{f} \in \mathbf{H}^{-1}(\mathbf{\Omega})$, *find* $(\mathbf{u}, p) \in \mathbf{H}_0^1(\mathbf{\Omega}) \times L_0^2(\mathbf{\Omega})$ *such that*

$$\begin{aligned} A\mathbf{u} + B'p &= f, \\ B\mathbf{u} &= 0 \end{aligned} \tag{3.9}$$

with the linear operators A, B, and B' defined by

- $A : \mathbf{H}_0^1(\mathbf{\Omega}) \to (\mathbf{H}_0^1(\mathbf{\Omega}))'$,
  $\langle A\mathbf{u}, \mathbf{v}\rangle_{(\mathbf{H}_0^1(\mathbf{\Omega}))', \mathbf{H}_0^1(\mathbf{\Omega})} := a(\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega})$,

- $B : \mathbf{H}_0^1(\mathbf{\Omega}) \to (L_0^2(\mathbf{\Omega}))'$,
  $\langle B\mathbf{v}, q\rangle_{(L_0^2(\mathbf{\Omega}))', L_0^2(\mathbf{\Omega})} := b(\mathbf{v}, q), \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega}), q \in L_0^2(\mathbf{\Omega})$,

- $B' : L_0^2(\mathbf{\Omega}) \to (\mathbf{H}_0^1(\mathbf{\Omega}))'$,
  $\langle B'q, \mathbf{v}\rangle_{(\mathbf{H}_0^1(\mathbf{\Omega}))', \mathbf{H}_0^1(\mathbf{\Omega})} = \langle B\mathbf{v}, q\rangle_{(L_0^2(\mathbf{\Omega}))', L_0^2(\mathbf{\Omega})} = b(\mathbf{v}, q)$,
  $\forall \mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega}), q \in L_0^2(\mathbf{\Omega})$.

This is an operator form of a so-called linear saddle point problem on the Hilbert spaces $\mathbf{H}_0^1(\mathbf{\Omega})$ and $L_0^2(\mathbf{\Omega})$.

Lemma 3.2.2 *It holds*

$$\|A\|_{\mathcal{L}(\mathbf{H}_0^1(\mathbf{\Omega}), \mathbf{H}^{-1}(\mathbf{\Omega}))} = \|a\| \text{ and } \|B\|_{\mathcal{L}(\mathbf{H}_0^1(\mathbf{\Omega}), (L_0^2(\mathbf{\Omega}))')} = \|b\|.$$

Proof: The norm of a linear operator $L : V \to W$ is defined by

$$\|L\| := \sup_{x \in V \setminus \{0\}} \frac{\|L(x)\|_W}{\|x\|_V}.$$

It follows that

$$\begin{aligned} \|A\|_{\mathcal{L}(\mathbf{H}_0^1(\mathbf{\Omega}), \mathbf{H}^{-1}(\mathbf{\Omega}))} &= \sup_{\mathbf{u} \in \mathbf{H}_0^1(\mathbf{\Omega}) \setminus \{\mathbf{0}\}} \frac{\|A\mathbf{u}\|_{\mathbf{H}^{-1}(\mathbf{\Omega})}}{\|\mathbf{u}\|_{\mathbf{H}_0^1(\mathbf{\Omega})}} \\ &= \sup_{\mathbf{u}, \mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega}) \setminus \{\mathbf{0}\}} \frac{\langle A\mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|_{\mathbf{H}_0^1(\mathbf{\Omega})} \|\mathbf{v}\|_{\mathbf{H}_0^1(\mathbf{\Omega})}} = \|a\|. \end{aligned}$$

The proof for B is done similarly. $\square$

Therefore, the boundedness of $a(\cdot,\cdot)$ respectively $b(\cdot,\cdot)$ implies the boundedness of A respectively B.

Next, we will discuss the problem of solving such a linear saddle point problem.

Definition 3.2.3 (Well-posedness) *Problem* (3.9) *is called well-posed, which means unique solvability for all* $\mathbf{f} \in \mathbf{H}^{-1}(\mathbf{\Omega})$*, if the linear operator (solution operator)*

$$\mathcal{I} : \mathbf{H}_0^1(\mathbf{\Omega}) \times L_0^2(\mathbf{\Omega}) \to (\mathbf{H}_0^1(\mathbf{\Omega}))' \times (L_0^2(\mathbf{\Omega}))',$$

$$\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} \mapsto \begin{pmatrix} f \\ 0 \end{pmatrix}$$

defined by

$$\mathcal{I} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} := \begin{pmatrix} A\mathbf{u} + B'p \\ B\mathbf{u} \end{pmatrix}$$

is an isomorphism.

Note that $\mathcal{I}$ is a continuous, linear operator because A and B are continuous, linear operators.

In general, this is not a condition which is easy to verify, so we have to derive criteria for $\mathcal{I}$ to be an isomorphism. Considering finite-dimensional spaces, (3.9) is equivalent to a system of the form

$$\underbrace{\begin{pmatrix} A_h & B_h^T \\ B_h & \mathbf{0} \end{pmatrix}}_{=:M_h} \cdot \begin{pmatrix} \mathbf{u}_h \\ \mathbf{p}_h \end{pmatrix} = \begin{pmatrix} \mathbf{f}_h \\ \mathbf{0} \end{pmatrix}.$$

From linear algebra it is known that a linear operator on finite-dimensional (vector-) spaces is an isomorphism if and only if the corresponding matrix, here M_h, is invertible. Hence, to compute $\mathbf{u}_h$ and $\mathbf{p}_h$ one would have to solve

$$\begin{pmatrix} \mathbf{u}_h \\ \mathbf{p}_h \end{pmatrix} = \begin{pmatrix} A_h & B_h^T \\ B_h & \mathbf{0} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \mathbf{f}_h \\ \mathbf{0} \end{pmatrix}.$$

This idea will be discussed more detailed when talking about the finite element method in Chapter 4. Unfortunately, at the moment we cannot restrict to the finite-dimensional case and hence have to attack the problem with some other techniques.

A criterion for the existence and uniqueness of a weak solution for elliptic boundary value problems is given by the popular:

Lemma 3.2.3 (Lemma of Lax–Milgram) *Let $a(\cdot,\cdot) : W \times W \to \mathbb{R}$ be a bounded and coercive bilinear form and W a Hilbert space. Then for each $f \in W'$ there exists a unique $u \in W$ such that*

$$a(u, v) = f(v), \quad \forall v \in W.$$

Proof: See literature, e.g., [John13], Chapter 4. □

This lemma can be used to analyze the Stokes equations by taking into account weakly divergence-free velocity functions only.

Definition 3.2.4 (Weakly divergence-free) *A function $\mathbf{u} \in \mathbf{H}_0^1(\boldsymbol{\Omega})$ is called weakly divergence-free if*

$$b(\mathbf{u}, q) = 0, \quad \forall q \in L_0^2(\boldsymbol{\Omega}).$$

Furthermore, define the space of weakly divergence-free functions (divergence vanishes almost everywhere) by

$$\begin{aligned}
\mathbf{V}^{\text{div}} :&= \left\{\mathbf{u} \in \mathbf{H}_0^1(\boldsymbol{\Omega}) : b(\mathbf{u}, q) = 0,\ \forall q \in L_0^2(\boldsymbol{\Omega})\right\} \\
&= \left\{\mathbf{u} \in \mathbf{H}_0^1(\boldsymbol{\Omega}) : B\mathbf{u} = 0\right\} = \ker(B) \\
&= \left\{\mathbf{u} \in \mathbf{H}_0^1(\boldsymbol{\Omega}) : \nabla \cdot \mathbf{u} = 0 \ \textit{almost everywhere in } \boldsymbol{\Omega}\right\}.
\end{aligned}$$

Lets denote by

$$\mathbf{V}^{\text{div},\perp} := \left\{\mathbf{v} \in \mathbf{H}_0^1(\boldsymbol{\Omega}) : a(\mathbf{v}, \mathbf{w}) = 0,\ \forall \mathbf{w} \in \mathbf{V}^{\text{div}}\right\}$$

the orthogonal complement of $\mathbf{V}^{\text{div}}$ in $\mathbf{H}_0^1(\boldsymbol{\Omega})$.

Remark 3.2.2

Note that any function which is divergence-free in the classical sense is automatically weakly divergence-free, i.e., belongs to $\mathbf{V}^{\text{div}}$.

Lemma 3.2.4 $\mathbf{V}^{\text{div}}$ *is a linear, closed subspace of the Hilbert space* $\mathbf{H}_0^1(\boldsymbol{\Omega})$.

Proof: First we show that $\mathbf{V}^{\text{div}}$ is a linear subspace of $\mathbf{H}_0^1(\boldsymbol{\Omega})$.

- By the definition of the space $\mathbf{V}^{\text{div}}$ it is $\mathbf{V}^{\text{div}} \subset \mathbf{H}_0^1(\boldsymbol{\Omega})$,
- $\mathbf{V}^{\text{div}} \neq \emptyset$, since $\mathbf{0} \in \mathbf{V}^{\text{div}}$,
- Consider any $\mathbf{v}, \mathbf{w} \in \mathbf{V}^{\text{div}}$ and any $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned} b(\alpha\mathbf{v} + \beta\mathbf{w}, q) &= -\int_{\boldsymbol{\Omega}} \nabla \cdot (\alpha\mathbf{v} + \beta\mathbf{w}) q \, \mathrm{d}\mathbf{x} \\ &= -\alpha \int_{\boldsymbol{\Omega}} (\nabla \cdot \mathbf{v}) \, q \, \mathrm{d}\mathbf{x} - \beta \int_{\boldsymbol{\Omega}} (\nabla \cdot \mathbf{w}) \, q \, \mathrm{d}\mathbf{x} \\ &= \alpha b(\mathbf{v}, q) + \beta b(\mathbf{w}, q) = 0 \\ &\Longrightarrow \alpha\mathbf{v} + \beta\mathbf{w} \in \mathbf{V}^{\text{div}}. \end{aligned}$$

Now only the closedness remains to be proved. Let $\mathbf{v} \in \mathbf{H}_0^1(\boldsymbol{\Omega})$ has the property that there exists a sequence $\mathbf{v}_n \in \mathbf{V}^{\text{div}}$, $n = 1, 2, \ldots$, with

$$\|\mathbf{v} - \mathbf{v}_n\|_{\mathbf{H}_0^1(\boldsymbol{\Omega})} \xrightarrow{n \to \infty} 0.$$

We have to show that $\mathbf{v} \in \mathbf{V}^{\text{div}}$. We have seen that $b(\cdot, \cdot)$ is continuous, so for any fixed $q \in L_0^2(\boldsymbol{\Omega})$ it holds

$$b(\mathbf{v}, q) = b(\lim_{n \to \infty} \mathbf{v}_n, q) = \lim_{n \to \infty} \underbrace{b(\mathbf{v}_n, q)}_{=0} = 0$$

$$\begin{aligned} &\Longrightarrow b(\mathbf{v}, q) = 0, \quad \forall q \in L_0^2(\boldsymbol{\Omega}) \\ &\Longrightarrow \mathbf{v} \in \mathbf{V}^{\text{div}}. \end{aligned}$$

□

Remark 3.2.3

A linear, closed subspace of a Hilbert space is a Hilbert space itself. Hence, by Lemma 3.2.4, $\mathbf{V}^{\text{div}}$ is a Hilbert space.

Integrating the weak incompressibility into the solution space for $\mathbf{u}$, i.e., searching $\mathbf{u} \in \mathbf{V}^{\text{div}}$ results instead of (3.8) in:
For $\mathbf{f} \in \mathbf{H}^{-1}(\boldsymbol{\Omega})$ given, find $\mathbf{u} \in \mathbf{V}^{\text{div}}$ such that

$$a(\mathbf{u}, \mathbf{v}) = f(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}^{\text{div}}. \tag{3.10}$$

Obviously, with this procedure we eliminated the pressure p.

The two main questions we have to answer are:

1. Is the problem (3.10) uniquely solvable?
2. If $\mathbf{u}$ solves (3.10), does there exist a unique pressure $p \in L_0^2(\boldsymbol{\Omega})$ such that $(\mathbf{u}, p)$ solves (3.8)?

The first question is easily answerable.
With the inner product and norm of $\mathbf{H}_0^1(\boldsymbol{\Omega})$, $\mathbf{V}^{\text{div}}$ becomes a Hilbert space. For that reason, the bounded bilinear form $a(\cdot,\cdot)$ on $\mathbf{H}_0^1(\boldsymbol{\Omega}) \times \mathbf{H}_0^1(\boldsymbol{\Omega})$ and the bounded linear functional $f(\cdot)$ on $\mathbf{H}_0^1(\boldsymbol{\Omega})$ remain bounded when restricted to $\mathbf{V}^{\text{div}}$.

Corollary 3.2.1 *The bilinear form $a(\cdot,\cdot)$ is coercive on $\mathbf{V}^{\text{div}} \times \mathbf{V}^{\text{div}}$, i.e.,*

$$\exists m > 0: \ \ a(\mathbf{v},\mathbf{v}) \geq m \|\mathbf{v}\|^2_{\mathbf{H}_0^1(\boldsymbol{\Omega})}, \qquad \forall \mathbf{v} \in \mathbf{V}^{\text{div}}$$

and $m = 1$.

Proof: By the proof of the coercivity of $a(\cdot,\cdot)$ on $\mathbf{H}_0^1(\boldsymbol{\Omega}) \times \mathbf{H}_0^1(\boldsymbol{\Omega})$ it follows immediately

$$a(\mathbf{v},\mathbf{v}) = \|\mathbf{v}\|^2_{\mathbf{H}_0^1(\boldsymbol{\Omega})}, \qquad \forall \mathbf{v} \in \mathbf{V}^{\text{div}} \subset \mathbf{H}_0^1(\boldsymbol{\Omega}).$$

□

Fulfilling these requirements, the theorem of Lax–Milgram guarantees a unique solution $\mathbf{u} \in \mathbf{V}^{\text{div}}$ of (3.10) which automatically, by the definition of the space $\mathbf{V}^{\text{div}}$, satisfies the second equation in (3.8).

It is still not clarified if there exists a unique pressure $p \in L_0^2(\boldsymbol{\Omega})$, solving for the $\mathbf{u} \in \mathbf{V}^{\text{div}}$ which was produced above

$$\begin{aligned} & a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = f(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\boldsymbol{\Omega}) \\ \iff \qquad & b(\mathbf{v},p) = f(\mathbf{v}) - a(\mathbf{u},\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\boldsymbol{\Omega}). \end{aligned}$$

For the solution of this subproblem we cannot use the theorem of Lax–Milgram, since the bilinear form $b(\cdot,\cdot)$ operates on two different spaces. However, as an idea we try to modify it appropriately.
We know that
(i) $b(\cdot,\cdot)$ is a bounded bilinear functional on $\mathbf{H}_0^1(\boldsymbol{\Omega}) \times L_0^2(\boldsymbol{\Omega})$
and
(ii) for $\mathbf{u} \in \mathbf{V}^{\text{div}}$ fixed, $\mathbf{v} \mapsto f(\mathbf{v}) - a(\mathbf{u},\mathbf{v}) =: \mathfrak{L}(\mathbf{v})$ is a bounded linear

functional on $\mathbf{H}_0^1(\mathbf{\Omega})$ because $a(\cdot,\cdot)$ and $f(\cdot)$ are bounded.
So now, for our "modified Lax–Milgram" approach, the question is how to interpret coercivity in the case of a bilinear functional on the Cartesian product of two different Hilbert spaces, here on $\mathbf{H}_0^1(\mathbf{\Omega}) \times L_0^2(\mathbf{\Omega})$.

For that reason we take a closer look at the coercivity condition:

$$\exists m > 0: \ \ c(\mathbf{v},\mathbf{v}) \geq m\|\mathbf{v}\|^2_{\mathbf{H}_0^1(\mathbf{\Omega})}, \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega})$$
$$\Longrightarrow \exists m > 0: \ \ m\|\mathbf{v}\|_{\mathbf{H}_0^1(\mathbf{\Omega})} \leq \frac{c(\mathbf{v},\mathbf{v})}{\|\mathbf{v}\|_{\mathbf{H}_0^1(\mathbf{\Omega})}} \leq \sup_{\mathbf{w}\in\mathbf{H}_0^1(\mathbf{\Omega})\setminus\{\mathbf{0}\}} \frac{c(\mathbf{w},\mathbf{v})}{\|\mathbf{w}\|_{\mathbf{H}_0^1(\mathbf{\Omega})}},$$

$\forall \mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega}) \setminus \{\mathbf{0}\}$.
This inequality applied to $b(\cdot,\cdot)$ is somehow equivalent to

$$\exists \beta > 0: \ \ \beta\|q\|_{L_0^2(\mathbf{\Omega})} \leq \sup_{\mathbf{w}\in\mathbf{H}_0^1(\mathbf{\Omega})\setminus\{\mathbf{0}\}} \frac{b(\mathbf{w},q)}{\|\mathbf{w}\|_{\mathbf{H}_0^1(\mathbf{\Omega})}}, \qquad \forall q \in L_0^2(\mathbf{\Omega})$$
$$\Longrightarrow \quad \exists \beta > 0: \ \beta \leq \inf_{q\in L_0^2(\mathbf{\Omega})\setminus\{0\}} \ \sup_{\mathbf{w}\in\mathbf{H}_0^1(\mathbf{\Omega})\setminus\{\mathbf{0}\}} \frac{b(\mathbf{w},q)}{\|\mathbf{w}\|_{\mathbf{H}_0^1(\mathbf{\Omega})}\|q\|_{L_0^2(\mathbf{\Omega})}}$$

and is referred to as the inf-sup condition.

The inf-sup condition or BB condition, named after Babuška ([Bab71]) and Brezzi ([Br74]), is a powerful criterion as we will see in the next theorem. It provides the possibility to guarantee under certain conditions a unique solution of a linear saddle point problem, so in particular of the weak Stokes problem.

Definition 3.2.5 (Inf-Sup condition) *A bilinear form $c(\cdot,\cdot): V \times W \to \mathbb{R}$ on the Hilbert spaces V, W fulfills the inf-sup condition*

$$\Longleftrightarrow \qquad \exists \beta > 0: \ \ \beta \leq \inf_{q\in W\setminus\{0\}} \ \sup_{v\in V\setminus\{0\}} \frac{c(v,q)}{\|v\|_V\|q\|_W}. \tag{3.11}$$

Remark 3.2.4

Assume we are given the solution $(\mathbf{u},p) \in \mathbf{H}_0^1(\mathbf{\Omega}) \times L_0^2(\mathbf{\Omega})$ of (3.8). Then $\mathbf{u} \in \mathbf{H}_0^1(\mathbf{\Omega})$ fulfills (3.8b): $b(\mathbf{u},q) = 0, \ \forall q \in L_0^2$, and the space $\mathbf{V}^{\text{div}}$ is a subspace of $\mathbf{H}_0^1(\mathbf{\Omega})$. Thus, $\mathbf{u} \in \mathbf{V}^{\text{div}}$. Equation (3.8$a$) is valid for all $\mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega})$ so it is in particular satisfied for all $\mathbf{v} \in \mathbf{V}^{\text{div}} \subset \mathbf{H}_0^1(\mathbf{\Omega})$.
Therefore $\mathbf{u} \in \mathbf{V}^{\text{div}}$ and it solves (3.10).
The answer to the second question is given by the following theorems.

Definition 3.2.6 *The set*

$$\mathbf{V}^o := \left\{ \mathbf{g} \in \mathbf{H}^{-1}(\mathbf{\Omega}) : \langle \mathbf{g}, \mathbf{v} \rangle = 0, \quad \forall \mathbf{v} \in \mathbf{V}^{\text{div}} \right\}$$

is the polar set of $\mathbf{V}^{\text{div}}$.

Theorem 3.2.1 *The following properties are equivalent:*

1. *there exists a constant* $\beta > 0$ *with*

$$\inf_{q \in L_0^2(\mathbf{\Omega}) \setminus \{0\}} \sup_{\mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega}) \setminus \{\mathbf{0}\}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbf{H}_0^1(\mathbf{\Omega})} \|q\|_{L_0^2(\mathbf{\Omega})}} \geq \beta,$$

2. *the operator* B' *is an isomorphism from* $L_0^2(\mathbf{\Omega})$ *onto* $\mathbf{V}^o$ *and*

$$\|B'q\|_{\mathbf{H}^{-1}(\mathbf{\Omega})} \geq \beta \|q\|_{L_0^2(\mathbf{\Omega})}, \quad \forall q \in L_0^2(\mathbf{\Omega}),$$

3. *the operator* B *is an isomorphism from* $\mathbf{V}^{\text{div},\perp}$ *onto* $(L_0^2(\mathbf{\Omega}))'$ *and*

$$\|B\mathbf{v}\|_{(L_0^2(\mathbf{\Omega}))'} \geq \beta \|\mathbf{v}\|_{\mathbf{H}_0^1(\mathbf{\Omega})}, \quad \forall \mathbf{v} \in \mathbf{V}^{\perp}.$$

Proof: See [GiRa86], page 58/59. □

Definition 3.2.7 *Define a linear continuous operator* $\boldsymbol{\pi} : \mathbf{H}^{-1}(\mathbf{\Omega}) \to (\mathbf{V}^{\text{div}})'$ *by*

$$\langle \boldsymbol{\pi}\mathbf{f}, \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle, \quad \forall \mathbf{f} \in \mathbf{H}^{-1}(\mathbf{\Omega}), \mathbf{v} \in \mathbf{V}^{\text{div}}.$$

Remark 3.2.5

The functional $\boldsymbol{\pi}\mathbf{f}$ is the restriction of the functional $\mathbf{f}$ from $\mathbf{H}_0^1(\mathbf{\Omega})$ onto $\mathbf{V}^{\text{div}}$. Note that $\mathbf{V}^{\text{div}} \subset \mathbf{H}_0^1(\mathbf{\Omega})$ and is therefore equipped with the same norm. The functional is bounded since

$$\begin{aligned} \|\boldsymbol{\pi}\mathbf{f}\|_{(\mathbf{V}^{\text{div}})'} &= \sup_{\mathbf{v} \in \mathbf{V}^{\text{div}} \setminus \{\mathbf{0}\}} \frac{\langle \boldsymbol{\pi}\mathbf{f}, \mathbf{v} \rangle}{\|\mathbf{v}\|_{\mathbf{V}^{\text{div}}}} = \sup_{\mathbf{v} \in \mathbf{V}^{\text{div}} \setminus \{\mathbf{0}\}} \frac{\langle \mathbf{f}, \mathbf{v} \rangle}{\|\mathbf{v}\|_{\mathbf{V}^{\text{div}}}} \\ &\leq \sup_{\mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega}) \setminus \{\mathbf{0}\}} \frac{\langle \mathbf{f}, \mathbf{v} \rangle}{\|\mathbf{v}\|_{\mathbf{V}^{\text{div}}}} = \|\mathbf{f}\|_{\mathbf{H}^{-1}(\mathbf{\Omega})}. \end{aligned}$$

Theorem 3.2.2 *Problem* (3.8) *is well-posed, i.e., for every* $\mathbf{f} \in \mathbf{H}^{-1}(\mathbf{\Omega})$ *there exists a unique pair* $(\mathbf{u}, p) \in \mathbf{H}_0^1(\mathbf{\Omega}) \times L_0^2(\mathbf{\Omega})$ *solving problem* (3.8)

$$\Longleftrightarrow$$

1. *the operator* $\boldsymbol{\pi} \circ A$ *is an isomorphism from* $\mathbf{V}^{\mathrm{div}}$ *onto* $(\mathbf{V}^{\mathrm{div}})'$ *and*

2. *the bilinear form* $b(\cdot,\cdot)$ *fulfills the inf-sup condition* (3.11).

Proof: For a proof see [Br74], Theorem 1.1, or [GiRa86], pp. 59. □

Corollary 3.2.2 *Let the bilinear form* $a(\cdot,\cdot)$ *be coercive in* $\mathbf{V}^{\mathrm{div}}$ ($\mathbf{V}^{\mathrm{div}}$*-elliptic*). *Then* (3.9) *is well-posed* $\Longleftrightarrow$ *the bilinear form* $b(\cdot,\cdot)$ *satisfies the inf-sup condition.*

Proof: The idea of the proof is to show that the coercivity of $a(\cdot,\cdot)$ implies the first condition in Theorem 3.2.2. Then, the result follows immediately. For a detailed version see [GiRa86], page 61. □

Remark 3.2.6

Corollary 3.2.2 is in the literature often referred to as Brezzi's splitting theorem, due to its appearance in [Br74].

We draw the conclusion, that the problems (3.8) and (3.10) are equivalent if the inf-sup condition is satisfied and if $a(\cdot,\cdot)$ is coercive in $\mathbf{V}^{\mathrm{div}}$.

We will see that the inf-sup condition for $\mathbf{H}_0^1(\mathbf{\Omega})$ and $L_0^2(\mathbf{\Omega})$ is a consequence of the following lemma.

Lemma 3.2.5 *Let* $q \in L_0^2(\mathbf{\Omega})$. *Then*

$$\exists!\mathbf{v} \in \mathbf{V}^{\mathrm{div},\perp} : \ \nabla \cdot \mathbf{v} = q \quad \textit{and} \quad \|\mathbf{v}\|_{\mathbf{H}_0^1(\mathbf{\Omega})} \le C\|q\|_{L_0^2(\mathbf{\Omega})},$$

for a constant $C > 0$ *independent of* $\mathbf{v}$ *and* q.

Proof: See [John14], pp. 40. □

Theorem 3.2.3 *Let* $\mathbf{\Omega}$ *be a bounded domain with Lipschitz boundary and* $\mathbf{f} \in \mathbf{H}^{-1}(\mathbf{\Omega})$. *Then the weak Stokes problem* (3.8) *has a unique solution* $(\mathbf{u}, p) \in \mathbf{H}_0^1(\mathbf{\Omega}) \times L_0^2(\mathbf{\Omega})$.

Proof: By Corollary 3.2.2 two conditions have to be shown:

1. The bilinear form $a(\cdot,\cdot)$ is V^{div}-elliptic.
2. The bilinear form $b(\cdot,\cdot)$ fulfills the inf-sup condition:

$$\exists \beta > 0: \quad \beta \leq \inf_{q \in L_0^2(\mathbf{\Omega})\setminus\{0\}} \sup_{\mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega})\setminus\{\mathbf{0}\}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbf{H}_0^1(\mathbf{\Omega})} \|q\|_{L_0^2(\mathbf{\Omega})}}.$$

The coercivity of $a(\cdot,\cdot)$ was proved in Lemma 3.2.1. For the second property let $q \in L_0^2(\mathbf{\Omega})$. Then by Lemma 3.2.5

$$\Longrightarrow \exists! \mathbf{v} \in \mathbf{V}^{\text{div},\perp}: \quad \nabla \cdot \mathbf{v} = q \quad \text{and} \quad \|\mathbf{v}\|_{\mathbf{H}_0^1(\mathbf{\Omega})} \leq C \|q\|_{L_0^2(\mathbf{\Omega})}.$$

Using this, we get

$$\begin{aligned}
\sup_{\mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega})\setminus\{\mathbf{0}\}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbf{H}_0^1(\mathbf{\Omega})}} &= \sup_{\mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega})\setminus\{\mathbf{0}\}} \frac{(\nabla \cdot \mathbf{v}, q)_{L_0^2(\mathbf{\Omega})}}{\|\mathbf{v}\|_{\mathbf{H}_0^1(\mathbf{\Omega})}} \\
&\geq \frac{(\nabla \cdot \mathbf{v}, q)_{L_0^2(\mathbf{\Omega})}}{\|\mathbf{v}\|_{\mathbf{H}_0^1(\mathbf{\Omega})}} = \frac{(q, q)_{L_0^2(\mathbf{\Omega})}}{\|\mathbf{v}\|_{\mathbf{H}_0^1(\mathbf{\Omega})}}, \qquad \text{for } \nabla \cdot \mathbf{v} = q \\
&= \frac{\|q\|_{L_0^2(\mathbf{\Omega})}^2}{\|\mathbf{v}\|_{\mathbf{H}_0^1(\mathbf{\Omega})}} \geq \frac{1}{C} \|q\|_{L_0^2(\mathbf{\Omega})}.
\end{aligned}$$

Since we can choose $q \in L_0^2(\mathbf{\Omega})$ arbitrarily, it follows

$$\inf_{q \in L_0^2(\mathbf{\Omega})\setminus\{0\}} \sup_{\mathbf{v} \in \mathbf{H}_0^1(\mathbf{\Omega})\setminus\{\mathbf{0}\}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbf{H}_0^1(\mathbf{\Omega})} \|q\|_{L_0^2(\mathbf{\Omega})}} \geq \frac{1}{C} =: \beta.$$

□

With this theorem, we have shown that for a solution $\mathbf{u} \in \mathbf{V}^{\text{div}}$ of (3.10) there exists a unique $p \in L_0^2(\mathbf{\Omega})$ such that $(\mathbf{u}, p)$ solves the weak Stokes problem (3.6).

3.2.3 An Alternative Formulation of the Stokes Problem for the Pressure

In this section, another representation of the Stokes problem in the orthogonal complement of $\mathbf{V}^{\text{div}}$ is derived. It will be analyzed in more detail in Chapter 5.

Lemma 3.2.6 (Orthogonal Decomposition) *Let H be a Hilbert space with inner product $(\cdot,\cdot)_H$ and $S \subset H$ a closed linear subspace. Then $S^\perp$, the orthogonal complement of S w.r.t. $(\cdot,\cdot)_H$, is a closed, linear subspace of H, too. Moreover, every $h \in H$ can be uniquely decomposed into $h = s + s^\perp$, where $s \in S$ and $s^\perp \in S^\perp$. That means*

$$H = S \oplus S^\perp.$$

The element s is called the orthogonal projection of h upon S.

Proof: For the proof see [Yo71], pp. 82. □

Remark 3.2.7

We can apply the previous lemma to our setting with $H = \mathbf{H}_0^1(\mathbf{\Omega})$, $S = \mathbf{V}^{\text{div}}$, and $S^\perp = \mathbf{V}^{\text{div},\perp}$ leading to

$$\mathbf{H}_0^1(\mathbf{\Omega}) = \mathbf{V}^{\text{div}} \oplus \mathbf{V}^{\text{div},\perp}.$$

If we want to determine the pressure p corresponding to the velocity solution $\mathbf{u}$ of (3.8), it suffices to test (3.8a) for $\mathbf{v} \in \mathbf{V}^{\text{div},\perp}$. This is the case since for $\mathbf{v} \in \mathbf{V}^{\text{div}}$ the pressure p does not appear anymore and by the previous lemma we know that

$$\mathbf{H}_0^1(\mathbf{\Omega}) = \mathbf{V}^{\text{div}} \oplus \mathbf{V}^{\text{div},\perp}.$$

Therefore we get

$$\underbrace{a(\underbrace{\mathbf{u}}_{\in \mathbf{V}^{\text{div}}}, \underbrace{\mathbf{v}}_{\in \mathbf{V}^{\text{div},\perp}})}_{=0} + b(\mathbf{v}, p) = f(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}^{\text{div},\perp}.$$

The resulting problem is:

Let $\mathbf{u} \in \mathbf{V}^{\text{div}}$ be the velocity solution of (3.8), then for given $\mathbf{f} \in \mathbf{H}^{-1}(\mathbf{\Omega})$ find $p \in L_0^2(\mathbf{\Omega})$ such that

$$b(\mathbf{v}, p) = f(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}^{\text{div},\perp}. \tag{3.12}$$

This problem will be analyzed in Section 5.2 in more detail.

4 Low Order Finite Element Discretizations

The existence of a unique weak solution of the Stokes problem was proven in Chapter 3. However, the aim is to find a satisfactory approximation of the solution $(\mathbf{u}, p)$. To that end, one has to use a discretization method, since the computers are not able to deal with infinite dimensions. The finite element method is a common technique to discretize systems of partial differential equations in space, in order to find an approximate solution.

As we have seen for the continuous setting, for the unique solvability in particular the inf-sup condition has to be satisfied. This is also the case for the discretization. Not every combination of velocity and pressure finite element spaces leads to unique solvability. For the analysis in this chapter $n \in \{2, 3\}$ is assumed.

4.1 (Mixed) Finite Element Methods

When using the finite element method for the discretization of the Stokes problem, one has to replace the infinite-dimensional test spaces, $\mathbf{V} := \mathbf{H}_0^1(\mathbf{\Omega})$ and $Q := L_0^2(\mathbf{\Omega})$, by finite-dimensional spaces. The use of different test spaces for the different variables when searching for a solution yields the term mixed finite element method.
Since we choose two finite element spaces, a velocity finite element space and a pressure finite element space, the finite element method for the Stokes equations is naturally arising to be mixed.

Let $\mathbf{\Omega}$ be a polyhedral domain. The first step is to decompose the domain $\overline{\mathbf{\Omega}} \subset \mathbb{R}^n$, $n = 2, 3$, into polyhedrons. The result is called triangulation $\mathcal{T}_h$, with h denoting the grid-size defined by $h := \max\limits_{\mathbf{T} \in \mathcal{T}_h} \operatorname{diam}(\mathbf{T})$. For $\mathbf{\Omega} \subset \mathbb{R}^2$ one typically uses triangles or rectangles and their three-dimensional equivalents, tetrahedrons and hexahedrons, for $n = 3$. These polyhedrons are called mesh cells and their union is the grid or mesh.

We assume $\{\mathcal{T}_h\}$, $h > 0$, to be a family of regular triangulations of $\overline{\mathbf{\Omega}}$. Thereby, the term regular triangulation involves for $\mathbf{T}, \mathbf{T}_1, \mathbf{T}_2 \in \mathcal{T}_h$ the following properties:

1. $\mathbf{T}$ is closed, $\partial\mathbf{T}$ is Lipschitz continuous and $\mathrm{int}(\mathbf{T}) \neq \emptyset$.
2. For $\mathbf{T}_1 \neq \mathbf{T}_2$ it is $\mathrm{int}(\mathbf{T}_1) \cap \mathrm{int}(\mathbf{T}_2) = \emptyset$.
3. $\mathbf{T}_1 \cap \mathbf{T}_2 = \mathbf{R}$ with $\mathbf{R} \in \{$m-faces: $m = -1, \ldots, n-1\}$, where a (-1)-face is defined as the empty set.
4. $\overline{\mathbf{\Omega}} = \bigcup\limits_{\mathbf{T} \in \mathcal{T}_h} \mathbf{T}$.
5. Regularity: There is a constant $C > 0$ independent of $h > 0$ such that $\frac{h_{\mathbf{T}}}{\rho_{\mathbf{T}}} \leq C$, $\forall \mathbf{T} \in \mathcal{T}_h$, where $h_{\mathbf{T}}$ is the diameter of $\mathbf{T}$, i.e., the diameter of the smallest circumscribed sphere and $\rho_{\mathbf{T}}$ denotes the diameter of the largest inscribed sphere of $\mathbf{T}$.

In the second step, one has to choose the finite element spaces. They will serve as test spaces for the solution. The idea is to replace $\mathbf{V}$ and Q by finite-dimensional spaces. Subsequently, we will denote the finite-dimensional velocity space by $\mathbf{V}_h$ and the finite-dimensional pressure space by Q_h, where h determines the grid-size again.

Finite element methods can be classified according to the relation between their finite element spaces and the primary infinite-dimensional trial and test spaces:

Definition 4.1.1 (Conforming / Nonconforming) *A finite element method for the Stokes system is called conforming if for the corresponding finite element spaces it holds:* $\mathbf{V}_h \subset \mathbf{H}_0^1(\mathbf{\Omega})$ *and* $Q_h \subset L_0^2(\mathbf{\Omega})$*, otherwise nonconforming.*

The velocity and pressure fields are usually approximated by elementwise polynomial functions. For simplicity, the theory is developed for a so-called reference mesh cell. A more detailed introduction into finite element theory can be found in [John13].

Definition 4.1.2 (Affine independence) *A set of* $n+1$ *points* $\mathbf{x}_0, \ldots, \mathbf{x}_n \in \mathbb{R}^n$ *is called affinely independent, if* $\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0$ *are linearly independent.*

Definition 4.1.3 (n-simplex) *A n-dimensional simplex, short n-simplex, is the convex hull of* $n+1$ *affinely independent points in* $\mathbb{R}^n$*.*

Definition 4.1.4 (Reference cell) *The reference cell* $\hat{\mathbf{T}} \subset \mathbb{R}^n$ *is an* n-*simplex formed by taking the convex hull of the points* $\hat{\mathbf{x}}_0 = \mathbf{0}$, $\hat{\mathbf{x}}_i := \mathbf{e}_i$, $i = 1, \ldots, n$.

Remark 4.1.1

1. The vectors $\mathbf{e}_i$ are the cartesian unit vectors in $\mathbb{R}^n$, i.e.,

$$\mathbf{e}_i := \begin{array}{ccccccc} & & & i & & & \\ (0 & \cdots & 0 & 1 & 0 & \cdots & 0)^T . \end{array}$$

2. Assume $\hat{\mathbf{T}} = \operatorname{conv}(\hat{\mathbf{x}}_0, \ldots, \hat{\mathbf{x}}_n)$ and $\mathbf{T} = \operatorname{conv}(\mathbf{r}_0, \ldots, \mathbf{r}_n)$. Using an invertible affine map $F_{\mathbf{T}} : \hat{\mathbf{T}} \to \mathbf{T}$, $F_{\mathbf{T}}(\hat{\mathbf{x}}_i) = \mathbf{r}_i$, $i = 0, \ldots, n$, one can define finite element spaces on $\mathbf{T} \in \mathcal{T}_h$, see Figure 4.1.

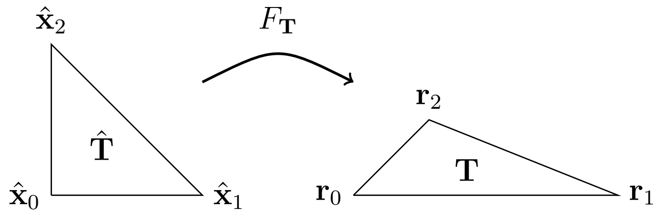


Figure 4.1: Invertible, affine map between the reference triangle and another triangle in 2D.

Definition 4.1.5 (Polynomial spaces on T) *Let* $\mathcal{T}_h$ *be a triangulation of the domain* $\mathbf{\Omega}$. *The space of polynomials of degree smaller or equal to* k *on an element* $\mathbf{T} \in \mathcal{T}_h$ *is defined by*

$$\begin{aligned} P_k(\mathbf{T}) :&= \operatorname{span}\{\mathbf{x}^{\boldsymbol{\alpha}} : |\boldsymbol{\alpha}| \leq k\} \\ &= \left\{ p : \mathbf{T} \to \mathbb{R} : p(\mathbf{x}) = \sum_{|\boldsymbol{\alpha}| \leq k} a_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}, \ a_{\boldsymbol{\alpha}} \in \mathbb{R} \right\} \end{aligned}$$

with

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n), \ \mathbf{x}^{\boldsymbol{\alpha}} := \prod_{i=1}^{n} x_i^{\alpha_i} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \text{ and } k \in \mathbb{N}_0.$$

Remark 4.1.2

The space of all linear polynomials on a mesh cell $\mathbf{T}$ is

$$P_1(\mathbf{T}) := \left\{ a_0 + \sum_{i=1}^{n} a_i x_i : \mathbf{x} = (x_1, \ldots x_n) \in \mathbf{T}, \ a_i \in \mathbb{R} \right\}.$$

For $n = 2$ one obtains $P_1(\mathbf{T}) = \operatorname{span}\{1, x_1, x_2\}$. The space of constant polynomials on $\mathbf{T}$ is $P_0(\mathbf{T}) = \operatorname{span}\{1\}$.

Lemma 4.1.1 *The dimension of $P_k(\mathbf{T})$ is*

$$dim\,(P_k(\mathbf{T})) = \binom{n+k}{n}.$$

Proof: We proof this statement by giving a bijection:

The monomials $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ with $|\boldsymbol{\alpha}| \leq k$, $k \in \mathbb{N}_0$, form a basis of $P_k(\mathbf{T})$. So we have to compute the cardinality of

$$\mathbf{M}(n,k) := \{\boldsymbol{\alpha} \in \mathbb{N}_0^n : |\boldsymbol{\alpha}| \leq k\}.$$

Define a map

$$\mathbf{f} : \mathbf{M}(n,k) \to \left\{\mathbf{r} \in \{0,1\}^{n+k} : |\mathbf{r}| = n\right\},$$

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) \longmapsto (\underbrace{0,\ldots,0}_{\alpha_1}, 1, \underbrace{0\ldots,0}_{\alpha_2}, 1, \ldots, 1, \underbrace{0\ldots0}_{\alpha_n}, 1, \underbrace{0,\ldots,0}_{k-|\boldsymbol{\alpha}|}).$$

So for any monomial with $|\boldsymbol{\alpha}| \leq k$ there exists exactly one $(n + |\boldsymbol{\alpha}| + k - |\boldsymbol{\alpha}|) = (n+k)$-tuple with entries in $\{0,1\}$ with exactly $\#1 = |\mathbf{r}| = n$ and vice versa. Obviously this is by construction a bijection, hence

$$dim\,(P_k(\mathbf{T})) = |\mathbf{M}(n,k)| = \left|\left\{\mathbf{r} \in \{0,1\}^{n+k} : |\mathbf{r}| = n\right\}\right| = \binom{n+k}{n}.$$

□

For the definition of finite elements one has to specify linear functionals defined on $P_k(\mathbf{T})$. Finite elements whose linear functionals evaluate the polynomials on certain points in $\mathbf{T}$ are called Lagrangian finite elements.

Definition 4.1.6 (Unisolvence) *Let $\Phi_{\mathbf{T},1}, \ldots, \Phi_{\mathbf{T},\dim(P_k(\mathbf{T}))} : C(\mathbf{T}) \to \mathbb{R}$ be linearly independent, linear, and continuous functionals. The space $P_k(\mathbf{T})$ is unisolvent with respect to the functionals, i.e.,*

$$\forall \mathbf{a} \in \mathbb{R}^{\dim(P_k(\mathbf{T}))}\ \exists! p \in P_k(\mathbf{T}) : \Phi_{\mathbf{T},i}(p) = a_i, \quad i = 1, \ldots, \dim(P_k(\mathbf{T})).$$

Remark 4.1.3

1. If we choose a basis of $\mathbb{R}^{\dim(P_k(\mathbf{T}))}$, lets say $\mathbf{a} = \mathbf{e}_i$, the unisolvence implies that there exist $\dim(P_k(\mathbf{T}))$ elements of $P_k(\mathbf{T})$, $p_1, \ldots, p_{\dim(P_k(\mathbf{T}))}$, with
$$\Phi_{\mathbf{T},i}(p_j) = \delta_{ij}, \quad i,j = 1, \ldots, \dim(P_k(\mathbf{T})).$$

The elements $p_1, \dots, p_{\dim(P_k(\mathbf{T}))}$ form a basis, the so-called local basis of $P_k(\mathbf{T})$. Unisolvence means that any $p \in P_k(\mathbf{T})$ is uniquely determined by its values under the functionals, the degrees of freedom. For example, a linear polynomial on a triangle is uniquely determined by $\dim(P_k(\mathbf{T})) = \binom{n+k}{n} = \binom{2+1}{2} = 3$ linearly independent points in the triangle, e.g., the vertices.

2. One can define finite elements on quadrilaterals and their n-dimensional analogues and do a similar analysis for spaces of n-linear polynomials

$$Q_k(\mathbf{T}) := \operatorname{span}\{\mathbf{x}^{\boldsymbol{\alpha}} : 0 \le \alpha_i \le k, i = 1, \dots, n\}.$$

   For $k = 1$ and $n = 2$ this gives $Q_1(\mathbf{T}) := \operatorname{span}\{1, x_1, x_2, x_1x_2\}$.
   The reference cell is then the unit cube, i.e., $\hat{\mathbf{K}} = [0,1]^n$ or $\hat{\mathbf{K}} = [-1,1]^n$.
3. Global definitions result from the cellwise restrictions.

Notation:
In the next sections, especially in Chapter 5 and 6 we will use the following notation.

Let $\{\mathcal{T}_h\}$, $h > 0$, be a family of regular triangulations of $\overline{\mathbf{\Omega}}$ and $\mathbf{\Omega} \subset \mathbb{R}^n$, $n = 2, 3$, a polyhedral domain with Lipschitz boundary. Then denote by

- $\mathbf{x_T}$ – the barycenter of the simplex $\mathbf{T} \in \mathcal{T}_h$,
- $\mathcal{F}_h^\star$ – the set of all $(n-1)$-dim. simplex faces in $\mathcal{T}_h$,
- $\mathcal{F}_h$ – the set of all $(n-1)$-dim. interior simplex faces in $\mathcal{T}_h$,
- $\mathbf{x_F}$ – the barycenter of the face $\mathbf{F} \in \mathcal{F}_h^\star$,
- $\mathbf{n_F}$ – (unit) face normal on $\mathbf{F} \in \mathcal{F}_h^\star$
  with arbitrary but fixed orientation for $\mathbf{F} \in \mathcal{F}_h$
  and outwards (w.r.t. $\mathbf{\Omega}$) pointing orientation for $\mathbf{F} \in \mathcal{F}_h^\star \setminus \mathcal{F}_h$,
- $\mathcal{F}_\mathbf{T}$ – the set of faces of the simplex $\mathbf{T} \in \mathcal{T}_h$,
- $\mathbf{n_{T,F}}$ – (unit) outer normal of the simplex $\mathbf{T} \in \mathcal{T}_h$ at the face $\mathbf{F}$,
- $\mathbf{T}_1|\mathbf{T}_2 := \mathbf{F}$ – face between $\mathbf{T}_1, \mathbf{T}_2 \in \mathcal{T}_h$ for an interior face $\mathbf{F} \in \mathcal{F}_h$ belonging to $\mathbf{T}_1$ and $\mathbf{T}_2$, $\mathbf{T}_1 \neq \mathbf{T}_2$.

Definition 4.1.7 (Face jump) *For $\mathbf{T}_1, \mathbf{T}_2 \in \mathcal{T}_h$ and $\phi \in C(\mathbf{T}_1) \cap C(\mathbf{T}_2)$ we define the face jump for all $\mathbf{x} \in \mathbf{F} = \mathbf{T}_1|\mathbf{T}_2 \in \mathcal{F}_h$ by*

$$[\![\phi]\!]_\mathbf{F}(\mathbf{x}) := \left(\lim_{\mathbf{y}\to\mathbf{x}, \mathbf{y}\in\mathbf{T}_1} \phi(\mathbf{y})\mathbf{n}_{\mathbf{T}_1,\mathbf{F}} + \lim_{\mathbf{y}\to\mathbf{x}, \mathbf{y}\in\mathbf{T}_2} \phi(\mathbf{y})\mathbf{n}_{\mathbf{T}_2,\mathbf{F}} \right) \cdot \mathbf{n_F}.$$

Remark 4.1.4

Consider an elementwise constant function $q \in P_0(\mathbf{T}_1) \cap P_0(\mathbf{T}_2)$ with $\mathbf{F} = \mathbf{T}_1|\mathbf{T}_2$. Then for the face jump one obtains

$$\begin{aligned}[\![q]\!]_{\mathbf{F}}(\mathbf{x}) &= (q|_{\mathbf{T}_1}\mathbf{n}_{\mathbf{T}_1,\mathbf{F}} + q|_{\mathbf{T}_2}\mathbf{n}_{\mathbf{T}_2,\mathbf{F}}) \cdot \mathbf{n}_{\mathbf{F}} \\ &= q|_{\mathbf{T}_1}\mathbf{n}_{\mathbf{T}_1,\mathbf{F}} \cdot \mathbf{n}_{\mathbf{F}} + q|_{\mathbf{T}_2}\mathbf{n}_{\mathbf{T}_2,\mathbf{F}} \cdot \mathbf{n}_{\mathbf{F}}.\end{aligned}$$

For the scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ with angle α we have the characterization

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\alpha).$$

Therefore we conclude

$$\mathbf{n}_{\mathbf{T}_i,\mathbf{F}} \cdot \mathbf{n}_{\mathbf{F}} = \begin{cases} 1, & \text{if } \mathbf{n}_{\mathbf{T}_i,\mathbf{F}} = \mathbf{n}_{\mathbf{F}} \\ -1, & \text{if } \mathbf{n}_{\mathbf{T}_i,\mathbf{F}} = -\mathbf{n}_{\mathbf{F}} \end{cases}, \qquad \text{for } i = 1, 2.$$

Hence depending on how the face normal $\mathbf{n}_{\mathbf{F}}$ was chosen, we obtain

$$[\![q]\!]_{\mathbf{F}} = q|_{\mathbf{T}_1} - q|_{\mathbf{T}_2}$$

or

$$[\![q]\!]_{\mathbf{F}} = q|_{\mathbf{T}_2} - q|_{\mathbf{T}_1}.$$

4.2 Application to the Stokes Equations

Finite element formulation of the Stokes problem:

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) &= f_h(\mathbf{v}_h), \qquad && \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b_h(\mathbf{u}_h, q_h) &= 0, && \forall q_h \in Q_h, \end{aligned} \tag{4.1}$$

with

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) &:= \sum_{\mathbf{T}\in\mathcal{T}_h} \int_{\mathbf{T}} \nabla \mathbf{u}_h : \nabla \mathbf{v}_h \, \mathrm{d}\mathbf{x}, \\ b_h(\mathbf{v}_h, p_h) &:= -\sum_{\mathbf{T}\in\mathcal{T}_h} \int_{\mathbf{T}} (\nabla \cdot \mathbf{v}_h) p_h \, \mathrm{d}\mathbf{x}, \\ f_h(\mathbf{v}_h) &:= \sum_{\mathbf{T}\in\mathcal{T}_h} \int_{\mathbf{T}} \mathbf{f} \cdot \mathbf{v}_h \, \mathrm{d}\mathbf{x}. \end{aligned}$$

The usual notation for a pair of finite element spaces is $\mathbf{V}_h/Q_h$.

The aim of this section is to explain how a finite element approximation of the Stokes equations can be computed. An important issue consists in choosing appropriate finite element spaces.

In what follows, we will figure out how to choose the "right" finite element spaces for the Stokes equations.

Solving the finite element discretization of the Stokes problem:
$\mathbf{V}_h$ and Q_h are by definition finite-dimensional spaces, hence they both have finite bases. Let $\dim(\mathbf{V}_h) = J$ and $\dim(Q_h) = K$. By choosing a basis $\{\boldsymbol{\phi}_j\}_{j=1}^J$ of $\mathbf{V}_h$ and a basis $\{\psi_k\}_{k=1}^K$ of Q_h, we may write

$$\mathbf{u}_h = \sum_{j=1}^{J} \alpha_j \boldsymbol{\phi}_j, \qquad \forall \mathbf{u}_h \in \mathbf{V}_h, \tag{4.2}$$

and

$$p_h = \sum_{k=1}^{K} \beta_k \psi_k, \qquad \forall p_h \in Q_h, \tag{4.3}$$

for some constants $\alpha_j, \beta_k \in \mathbb{R}$.

Since the first equation in (4.1) holds for all $\mathbf{v}_h \in \mathbf{V}_h$ it suffices to test it for all basis elements of $\mathbf{V}_h$. For the same reason the analogous statement holds for the second equation for Q_h.

Inserting the basis representations of $\mathbf{u}_h$, (4.2), and p_h, (4.3), into the equations (4.1) and using the bilinearity of $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ we end up with

$$\begin{aligned} \sum_{j=1}^{J} a_h(\boldsymbol{\phi}_j, \boldsymbol{\phi}_l)\alpha_j + \sum_{k=1}^{K} b_h(\boldsymbol{\phi}_l, \psi_k)\beta_k &= f_h(\boldsymbol{\phi}_l), \qquad & l = 1, \dots, J, \\ \sum_{j=1}^{J} b_h(\boldsymbol{\phi}_j, \psi_i)\alpha_j &= 0, & i = 1, \dots, K. \end{aligned} \tag{4.4}$$

This is a quadratic linear system of equations where the number of unknowns $(\alpha_1, \dots, \alpha_J, \beta_1, \dots, \beta_K)$ equals the number of equations (J+K).

Linear operators on finite-dimensional spaces can be represented by matrices, provided that bases where chosen. Thats why the problem (4.4) can be expressed in matrix form as

$$\underbrace{\begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix}}_{=:M_h} \cdot \begin{pmatrix} \mathbf{u}_h \\ \mathbf{p}_h \end{pmatrix} = \begin{pmatrix} \mathbf{f}_h \\ \mathbf{0} \end{pmatrix} \tag{4.5}$$

with

- $(A_h)_{l,j} := a_h(\boldsymbol{\phi}_j, \boldsymbol{\phi}_l), \quad A_h \in \mathbb{R}^{J\times J}$,
- $(B_h)_{i,j} := b_h(\boldsymbol{\phi}_j, \psi_i), \quad B_h \in \mathbb{R}^{K\times J}$,
- $(u_h)_j := \alpha_j, \quad \mathbf{u}_h \in \mathbb{R}^J$,
- $(p_h)_k := \beta_k, \quad \mathbf{p}_h \in \mathbb{R}^K$,
- $(f_h)_l := f_h(\boldsymbol{\phi}_l), \quad \mathbf{f}_h \in \mathbb{R}^J$.

Remark 4.2.1

In what follows, we will deal with conforming finite element discretizations unless otherwise stated. For conforming finite element methods the discrete bilinear forms and the linear form given in (4.1) agree with their continuous counterparts from Section 3.2.1 when restricting $\mathbf{V}$ to $\mathbf{V}_h$ and Q to Q_h. Defining $\|(\cdot)\|_{V_h} := (a_h(\cdot,\cdot))^{\frac{1}{2}}$, we get for conforming finite element methods $\|(\cdot)\|_{V_h} = \|\nabla(\cdot)\|_{L^2}$.

The linear system of equations (4.5) is uniquely solvable if the corresponding coefficient matrix M_h has full rank which is equivalent to its invertibility.

Theorem 4.2.1 *Let $\{\boldsymbol{\phi}_j\}_{j=1}^J$ be the basis of the finite-dimensional space $\mathbf{V}_h \subset \mathbf{V}$. Then the matrix $A_h \in \mathbb{R}^{J\times J}$ defined by $(A_h)_{lj} := a_h(\boldsymbol{\phi}_j, \boldsymbol{\phi}_l)$, $j, l = 1, \ldots, J$, is*
(i) symmetric and positive definite,
(ii) invertible.

Proof: Represent $\mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h$ by

$$\mathbf{u}_h = \sum_{i=1}^{J} u_i \boldsymbol{\phi}_i \qquad \text{and} \qquad \mathbf{v}_h = \sum_{j=1}^{J} v_j \boldsymbol{\phi}_j.$$

Denote the vectors that consist of the coefficients by $\mathbf{x} := (u_1, \ldots, u_J)^T$ and $\mathbf{y} := (v_1, \ldots, v_J)^T$, then it holds

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) &= a_h(\sum_{i=1}^{J} u_i \boldsymbol{\phi}_i, \sum_{j=1}^{J} v_j \boldsymbol{\phi}_j) = \sum_{i,j=1}^{J} u_i v_j a_h(\boldsymbol{\phi}_i, \boldsymbol{\phi}_j) \\ &= \sum_{i,j=1}^{J} u_i v_j a_{ji} = \mathbf{y}^T A_h \mathbf{x} = \langle A_h \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}^J}. \end{aligned} \tag{4.6}$$

(i)

$$
\begin{aligned}
A_h \text{ is symmetric } &\iff A_h = A_h^T \\
&\iff \langle A_h \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}^J} = \langle \mathbf{x}, A_h \mathbf{y} \rangle_{\mathbb{R}^J} \\
&\overset{(4.6)}{\iff} a_h(\mathbf{u}_h, \mathbf{v}_h) = a_h(\mathbf{v}_h, \mathbf{u}_h), \qquad \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h \subset \mathbf{V}.
\end{aligned}
$$

The latter is true by Lemma 3.2.1.

$$
\begin{aligned}
A_h \text{ is positive definite } &\iff \mathbf{y}^T A_h \mathbf{y} > 0, \quad \forall \mathbf{y} \in \mathbb{R}^J \setminus \{\mathbf{0}\} \\
&\iff \langle A_h \mathbf{y}, \mathbf{y} \rangle_{\mathbb{R}^J} > 0, \quad \forall \mathbf{y} \in \mathbb{R}^J \setminus \{\mathbf{0}\} \\
&\iff a_h(\mathbf{v}_h, \mathbf{v}_h) > 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h \setminus \{\mathbf{0}\}.
\end{aligned}
$$

This was already shown in Lemma 3.2.1.
(ii) By part (i), A_h is positive definite, i.e., $\mathbf{x}^T A_h \mathbf{x} > 0, \forall \mathbf{x} \in \mathbb{R}^J \setminus \{\mathbf{0}\}$. This implies

$$
\ker(A_h) := \{\mathbf{x} \in \mathbb{R}^J : A_h \mathbf{x} = 0\} = \{\mathbf{0}\}.
$$

The rank-nullity theorem yields

$$
\begin{aligned}
\dim(\mathbb{R}^J) &= \dim(\operatorname{im}(A_h)) + \dim(\ker(A_h)) \\
&= \dim(\operatorname{im}(A_h)) + \underbrace{\dim(\{0\})}_{=0} = \dim(\operatorname{im}(A_h)).
\end{aligned}
$$

Hence, A_h has full rank and is invertible. □

The regularity of A_h is equivalent to the fact that A_h has full (row) rank. It remains to show that B_h has full row rank, i.e., $rank_{\text{row}}(B_h) = K$. Therefore, $K \leq J$ is necessary, since otherwise there were more conditions than variables and thus linear dependent rows in the system matrix M_h.

Theorem 4.2.2 *Let $B \in \mathbb{R}^{K \times J}$, $K \leq J$, and denote by $\|\cdot\|_2$ the Euclidean norm. Then*

$$
rank_{row}(B) = K \iff \inf_{\mathbf{q} \in \mathbb{R}^K \setminus \{\mathbf{0}\}} \sup_{\mathbf{v} \in \mathbb{R}^J \setminus \{\mathbf{0}\}} \frac{\mathbf{q}^T B \mathbf{v}}{\|\mathbf{q}\|_2 \|\mathbf{v}\|_2} \geq \beta > 0.
$$

Proof: „$\Longleftarrow$“: This direction is proven by contradiction. To that end, let

$$
\inf_{\mathbf{q} \in \mathbb{R}^K \setminus \{\mathbf{0}\}} \sup_{\mathbf{v} \in \mathbb{R}^J \setminus \{\mathbf{0}\}} \frac{\mathbf{q}^T B \mathbf{v}}{\|\mathbf{q}\|_2 \|\mathbf{v}\|_2} \geq \beta > 0 \tag{4.7}
$$

and we assume

$$rank_{\text{row}}(B) \neq K$$

which is in fact equivalent to

$$rank_{\text{row}}(B) < K.$$

Then,

$$\begin{aligned}
rank_{\text{row}}(B) < K \quad &\Leftrightarrow \quad \ker(B^T) \text{ is nontrivial} \\
&\Leftrightarrow \quad \exists \mathbf{q} \in \mathbb{R}^K \setminus \{\mathbf{0}\} : \ \ \mathbf{q}^T B = B^T \mathbf{q} = \mathbf{0} \\
&\Rightarrow \quad \exists \mathbf{q} \in \mathbb{R}^K \setminus \{\mathbf{0}\} : \ \ \mathbf{q}^T B \mathbf{v} = 0, \qquad \forall \mathbf{v} \in \mathbb{R}^J \\
&\Rightarrow \quad \exists \mathbf{q} \in \mathbb{R}^K \setminus \{\mathbf{0}\} : \quad \sup_{\mathbf{v} \in \mathbb{R}^J \setminus \{\mathbf{0}\}} \frac{\mathbf{q}^T B \mathbf{v}}{\|\mathbf{v}\|_2} = 0 \\
&\Rightarrow \quad \inf_{\mathbf{q} \in \mathbb{R}^K \setminus \{\mathbf{0}\}} \sup_{\mathbf{v} \in \mathbb{R}^J \setminus \{\mathbf{0}\}} \frac{\mathbf{q}^T B \mathbf{v}}{\|\mathbf{q}\|_2 \|\mathbf{v}\|_2} \leq 0
\end{aligned}$$

$$\lightning$$

$$\Longrightarrow rank_{\text{row}}(B) = K.$$

„$\Longrightarrow$ “: Let

$$rank_{\text{row}}(B) = K.$$

Then,

$$\begin{aligned}
rank_{\text{row}}(B) = K \quad &\Rightarrow \quad \ker(B^T) = \{\mathbf{0}\} \\
&\Leftrightarrow \quad \forall \mathbf{q} \in \mathbb{R}^K \setminus \{\mathbf{0}\} : \ \ \mathbf{q}^T B = B^T \mathbf{q} \neq \mathbf{0}.
\end{aligned}$$

Choosing $\mathbf{v} = B^T \mathbf{q}$ leads to

$$\begin{aligned}
\inf_{\mathbf{q} \in \mathbb{R}^K \setminus \{\mathbf{0}\}} \sup_{\mathbf{v} \in \mathbb{R}^J \setminus \{\mathbf{0}\}} \frac{\mathbf{q}^T B \mathbf{v}}{\|\mathbf{q}\|_2 \|\mathbf{v}\|_2} &\geq \inf_{\mathbf{q} \in \mathbb{R}^K \setminus \{\mathbf{0}\}} \frac{\mathbf{q}^T B B^T \mathbf{q}}{\|\mathbf{q}\|_2 \|B^T \mathbf{q}\|_2} \\
&= \inf_{\mathbf{q} \in \mathbb{R}^K \setminus \{\mathbf{0}\}} \frac{\|B^T \mathbf{q}\|_2^2}{\|\mathbf{q}\|_2 \|B^T \mathbf{q}\|_2} = \inf_{\mathbf{q} \in \mathbb{R}^K \setminus \{\mathbf{0}\}} \frac{\|B^T \mathbf{q}\|_2}{\|\mathbf{q}\|_2}.
\end{aligned}$$

The term

$$\frac{\|B^T \mathbf{q}\|_2^2}{\|\mathbf{q}\|_2^2} = \frac{\mathbf{q}^T B B^T \mathbf{q}}{\mathbf{q}^T \mathbf{q}}$$

is known as the Rayleigh quotient and by Lemma 2.3.3 it is bounded by the minimum and maximum eigenvalue of BB^T:

$$\lambda_{\min}(BB^T) \leq \frac{\mathbf{q}^T B B^T \mathbf{q}}{\mathbf{q}^T \mathbf{q}} \leq \lambda_{\max}(BB^T), \qquad \forall \mathbf{q} \in \mathbb{R}^K \setminus \{\mathbf{0}\}$$

$$\Rightarrow \inf_{\mathbf{q}\in\mathbb{R}^K\setminus\{\mathbf{0}\}} \frac{\|B^T\mathbf{q}\|_2^2}{\|\mathbf{q}\|_2^2} = \lambda_{\min}(BB^T).$$

It is $BB^T = (B^T)^T B^T = (BB^T)^T$, so BB^T is symmetric and it is positive semidefinite since $\mathbf{x}^T BB^T\mathbf{x} = \left(B^T\mathbf{x}\right)^2 \geq 0$. Therefore all of its eigenvalues are positive or zero. By assumption, B has full rank, which is equivalent to $\ker(B^T) = \{\mathbf{0}\}$, hence

$$\left(B^T\mathbf{x}\right)^2 = 0 \Leftrightarrow B^T\mathbf{x} = \mathbf{0} \Leftrightarrow \mathbf{x} = \mathbf{0}$$

and therefore zero is not an eigenvalue of BB^T. Hence $\lambda_{\min}(BB^T) > 0$. Altogether yields

$$\inf_{\mathbf{q}\in\mathbb{R}^K\setminus\{\mathbf{0}\}} \sup_{\mathbf{v}\in\mathbb{R}^J\setminus\{\mathbf{0}\}} \frac{\mathbf{q}^T B\mathbf{v}}{\|\mathbf{q}\|_2\|\mathbf{v}\|_2} \geq \left(\lambda_{\min}(BB^{T)}\right)^{\frac{1}{2}} > 0.$$

□

Remark 4.2.2

The reader might have noticed the similarity of the condition (4.7) to the inf-sup condition (3.11). We have seen that M_h is invertible if A_h is non-singular and (4.7) is fulfilled. Obviously, the finite element spaces cannot be chosen arbitrarily. For B_h to be invertible, there is a necessary condition stating $K \leq J$. This gives the relation

$$\dim(Q_h) \leq \dim(\mathbf{V}_h). \tag{4.8}$$

Similar as for the continuous setting, one can reduce (4.1) to a problem on the space of discretely divergence-free functions.

Definition 4.2.1 (Discretely divergence-free) *A function $\mathbf{v}_h \in \mathbf{V}_h$ is called discretely divergence-free if*

$$b_h(\mathbf{v}_h, q_h) = 0, \quad \forall q_h \in Q_h.$$

Furthermore, define the space of discretely divergence-free functions by

$$\mathbf{V}_h^{\mathrm{div}} := \left\{\mathbf{v}_h \in \mathbf{V}_h : b_h(\mathbf{v}_h, q_h) = 0,\ \forall q_h \in Q_h\right\}.$$

The finite element approximation of the velocity is the solution $\mathbf{u}_h \in \mathbf{V}_h^{\mathrm{div}}$ of

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = f_h(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathrm{div}}.$$

Definition 4.2.2 (Discrete Inf-Sup condition) *The spaces* $\mathbf{V}_h$ *and* Q_h *satisfy the discrete inf-sup condition if*

$$\exists \beta^\star > 0 : \inf_{q_h \in Q_h \setminus \{0\}} \sup_{\mathbf{u}_h \in \mathbf{V}_h \setminus \{\mathbf{0}\}} \frac{b_h(\mathbf{u}_h, q_h)}{\|q_h\|_Q \|\mathbf{u}_h\|_{\mathbf{V}}} \geq \beta^\star. \tag{4.9}$$

Now, requirements analog to the continuous case are derived for the guarantee of the unique solvability.

Theorem 4.2.3 *Let* $a_h(\cdot,\cdot)^{\frac{1}{2}}$ *define a norm in* $\mathbf{V}_h$ *and let the discrete inf-sup condition* (4.9) *be fulfilled by* $\mathbf{V}_h$ *and* Q_h*. Then the problem* (4.1) *has a unique solution* $(\mathbf{u}_h, p_h)$*.*

Proof: If the bilinear form $a_h(\cdot,\cdot)^{\frac{1}{2}}$ defines a norm in $\mathbf{V}_h$ one obtains

$$a_h(\mathbf{u}_h, \mathbf{u}_h) \geq 1 \cdot a_h(\mathbf{u}_h, \mathbf{u}_h) = 1 \cdot \left(a_h(\mathbf{u}_h, \mathbf{u}_h)^{\frac{1}{2}}\right)^2 = 1 \cdot \|\mathbf{u}_h\|^2_{\mathbf{V}_h}, \ \forall \mathbf{u}_h \in \mathbf{V}_h.$$

Thus, $a_h(\cdot,\cdot)$ is $\mathbf{V}_h$-elliptic and since $\mathbf{V}_h^{\text{div}}$ is a subspace of $\mathbf{V}_h$ it is also $\mathbf{V}_h^{\text{div}}$-elliptic. The rest of the proof follows from Corollary 3.2.2. □

Definition 4.2.3 (Interior / Exterior method) *The finite element method* (4.1) *based on the spaces* $\mathbf{V}_h$ *and* Q_h *is called interior if discretely divergence-free vector fields are weakly divergence-free, i.e.,*

$$\mathbf{V}_h^{\text{div}} \subset \mathbf{V}^{\text{div}}.$$

If $\mathbf{V}_h^{\text{div}} \not\subseteq \mathbf{V}^{\text{div}}$ *the method is called exterior.*

Remark 4.2.3

It is important to notice that discretely divergence-free functions do not have to be (weakly) divergence-free. Even if $\mathbf{V}_h \subset \mathbf{V}$ and $Q_h \subsetneq Q$ it is in general $\mathbf{V}_h^{\text{div}} \not\subset \mathbf{V}^{\text{div}}$. This is the case because functions in $\mathbf{V}^{\text{div}}$ fulfill more conditions than functions in $\mathbf{V}_h^{\text{div}}$. Assume $\mathbf{V}_h \subset \mathbf{V}$, then the vanishing of the bilinear form $b(\mathbf{v}_h, q_h) = 0, \ \forall q_h \in Q_h$, does not imply $b_h(\mathbf{v}_h, q) = 0, \ \forall q \in Q$, for $Q_h \subsetneq Q$.

The physical interpretation is that using a finite element method for approximating solutions of partial differential equations, describing an incompressible flow, might result in the violation of mass conservation, which is modeled by the incompressibility constraint (3.1*b*).

Thereby, the strength of the deviation depends on the choice of the finite element spaces. Indeed, there are finite element spaces leading to mass conservation, but the discrete inf-sup condition is then satisfied for special meshes only.

4.3 The Choice of the Finite Element Spaces

The theoretical analysis has shown that the finite element spaces have to be chosen carefully in order to get satisfactory approximations. Attention should be paid to two aspects:

- The discrete inf-sup condition, i.e., the existence of a unique solution.
- The conservation of mass, i.e., the physical reasonability.

This section is devoted to the examination of these aspects, especially by examples.

The more important aspect is of course the inf-sup stability, since the incompressibility only provides an information about the quality of the solution. Nevertheless, we start by investigating the incompressibility.

4.3.1 The Weak Mass Conservation

Apart from the inf-sup condition, in some applications mass conservation is of particular importance. So one would like to have a discrete solution $\mathbf{u}_h$, which is not just discretely divergence-free, but does conserve mass in a weak sense. Therefore $\mathbf{u}_h$ is required to be weakly divergence-free, $\mathbf{u}_h \in \mathbf{V}^{\text{div}}$. This could be realized by using so-called interior methods, i.e., methods with $\mathbf{V}_h^{\text{div}} \subset \mathbf{V}^{\text{div}}$ because then $\mathbf{u}_h \in \mathbf{V}_h^{\text{div}} \Longrightarrow \mathbf{u}_h \in \mathbf{V}^{\text{div}}$. As already remarked in the end of Section 4.2, finite element methods are in general exterior methods, i.e., $\mathbf{V}_h^{\text{div}} \not\subseteq \mathbf{V}^{\text{div}}$.

Consider a very special relation between the discrete velocity space $\mathbf{V}_h$ and the discrete pressure space Q_h, namely assume they fulfill

$$\nabla \cdot \mathbf{V}_h \subset Q_h. \tag{4.10}$$

The solution $\mathbf{u}_h \in \mathbf{V}_h$ of (4.1) is discretely divergence-free, i.e.,

$$b_h(\mathbf{u}_h, q_h) = 0, \ \forall q_h \in Q_h. \tag{4.11}$$

Since (4.10) implies $\nabla \cdot \mathbf{u}_h \in Q_h$ and (4.11) holds for all elements in Q_h it has to hold

$$\begin{aligned} 0 = b_h(\mathbf{u}_h, \nabla \cdot \mathbf{u}_h) &= \int_{\boldsymbol{\Omega}} \underbrace{(\nabla \cdot \mathbf{u}_h)^2}_{\geq 0} \, \mathrm{d}\mathbf{x} \\ \Longrightarrow \nabla \cdot \mathbf{u}_h = 0. & \end{aligned}$$

So the relation (4.10) is a desirable property of a finite element pair since it enforces that discretely divergence-free functions are weakly divergence-free, thus leads to an interior method.

Choosing the discontinuous finite element pressure space $Q_h = P_0 \cap L_0^2(\mathbf{\Omega})$ is another advantageous strategy, because local (elementwise) mass conservation can be expected. The elementwise constant function

$$q_h = \begin{cases} 1, & \mathbf{x} \in \mathbf{T}, \\ -\frac{|\mathbf{T}|}{|\mathbf{T}'|}, & \mathbf{x} \in \mathbf{T}', \\ 0, & else \end{cases}$$

is then an element of Q_h. Let $\mathbf{T}$, $\mathbf{T}'$ be any two mesh cells in $\mathcal{T}_h$ with $\mathbf{T} \neq \mathbf{T}'$. Testing the second equation in (4.1) with this q_h yields

$$\begin{aligned} 0 = b_h(\mathbf{u}_h, q_h) &= -\sum_{\mathbf{T} \in \mathcal{T}_h} \int\limits_{\mathbf{T}} q_h \nabla \cdot \mathbf{u}_h \, \mathrm{d}\mathbf{x} = -\int\limits_{\mathbf{T}} \nabla \cdot \mathbf{u}_h \, \mathrm{d}\mathbf{x} + \frac{|\mathbf{T}|}{|\mathbf{T}'|} \int\limits_{\mathbf{T}'} \nabla \cdot \mathbf{u}_h \, \mathrm{d}\mathbf{x} \\ \Longrightarrow \int\limits_{\mathbf{T}} \nabla \cdot \mathbf{u}_h \, \mathrm{d}\mathbf{x} &= \frac{|\mathbf{T}|}{|\mathbf{T}'|} \int\limits_{\mathbf{T}'} \nabla \cdot \mathbf{u}_h \, \mathrm{d}\mathbf{x} \\ \Longleftrightarrow \frac{1}{|\mathbf{T}|} \int\limits_{\mathbf{T}} \nabla \cdot \mathbf{u}_h \, \mathrm{d}\mathbf{x} &= \frac{1}{|\mathbf{T}'|} \int\limits_{\mathbf{T}'} \nabla \cdot \mathbf{u}_h \, \mathrm{d}\mathbf{x}. \end{aligned}$$

Fixing $\mathbf{T}$ and varying $\mathbf{T}'$ yields

$$\begin{aligned} \frac{1}{|\mathbf{T}|} \int\limits_{\mathbf{T}} \nabla \cdot \mathbf{u}_h \, \mathrm{d}\mathbf{x} &= \frac{1}{|\mathbf{T}'|} \int\limits_{\mathbf{T}'} \nabla \cdot \mathbf{u}_h \, \mathrm{d}\mathbf{x}, \qquad \forall \mathbf{T}' \in \mathcal{T}_h \\ \Longrightarrow \exists C \in \mathbb{R} : \frac{1}{|\mathbf{T}|} \int\limits_{\mathbf{T}} \nabla \cdot \mathbf{u}_h \, \mathrm{d}\mathbf{x} &= \frac{1}{|\mathbf{T}'|} \int\limits_{\mathbf{T}'} \nabla \cdot \mathbf{u}_h \, \mathrm{d}\mathbf{x} = C, \qquad \forall \mathbf{T}' \in \mathcal{T}_h. \end{aligned}$$

By the Gaussian theorem it is

$$\int\limits_{\mathbf{\Omega}} \nabla \cdot \mathbf{u}_h \, \mathrm{d}\mathbf{x} = \int\limits_{\partial \mathbf{\Omega}} \mathbf{u}_h \cdot \mathbf{n} \, \mathrm{d}\mathbf{s} = 0.$$

Therefore we obtain

$$\begin{aligned} 0 &= \int\limits_{\mathbf{\Omega}} \nabla \cdot \mathbf{u}_h \, \mathrm{d}\mathbf{x} = \sum_{\mathbf{T} \in \mathcal{T}_h} \int\limits_{\mathbf{T}} \nabla \cdot \mathbf{u}_h \, \mathrm{d}\mathbf{x} \\ &= \sum_{\mathbf{T} \in \mathcal{T}_h} |\mathbf{T}| \underbrace{\frac{1}{|\mathbf{T}|} \int\limits_{\mathbf{T}} \nabla \cdot \mathbf{u}_h \, \mathrm{d}\mathbf{x}}_{=C} = \sum_{\mathbf{T} \in \mathcal{T}_h} |\mathbf{T}| \cdot C = |\mathbf{\Omega}| \cdot C \\ &\Longleftrightarrow C = 0. \end{aligned}$$

Thus, we conclude that

$$\int\limits_{\mathbf{T}} \nabla \cdot \mathbf{u}_h \, \mathrm{d}\mathbf{x} = 0, \qquad \forall \mathbf{T} \in \mathcal{T}_h.$$

4.3.2 The Discrete Inf-Sup Condition

Some popular finite element spaces do not fulfill the discrete inf-sup condition. There is no possibility to draw the conclusion that the finite element method is inf-sup stable, from the fact that the continuous setting satisfies the inf-sup condition. To illustrate that let $\beta > 0$ be the inf-sup constant of the continuous Stokes problem. For conforming methods it is

$$\inf_{q_h \in Q_h \setminus \{0\}} \sup_{\mathbf{u} \in \mathbf{V} \setminus \{\mathbf{0}\}} \frac{b(\mathbf{u}, q_h)}{\|q_h\|_{L^2(\mathbf{\Omega})} \|\mathbf{u}\|_{\mathbf{V}}} \geq \inf_{q \in Q \setminus \{0\}} \sup_{\mathbf{u} \in \mathbf{V} \setminus \{\mathbf{0}\}} \frac{b(\mathbf{u}, q)}{\|q\|_{L^2(\mathbf{\Omega})} \|\mathbf{u}\|_{\mathbf{V}}} \geq \beta,$$

because $Q_h \subset Q$. Now the inclusion $\mathbf{V}_h \subset \mathbf{V}$ only enables an estimation such that the above term is bounded below when restricting on $\mathbf{V}_h$. Thus we cannot infer inf-sup stability of the discrete Stokes system. The discrete inf-sup condition is a compatibility condition on the discrete spaces $\mathbf{V}_h$ and Q_h.

In order to guarantee the convergence of the finite element method, the discrete inf-sup constant $\beta^\star$ has to be mesh independent. This is the case since the inverse of the discrete inf-sup constant enters the finite element error estimates. For details see [John14].

Lemma 4.3.1 *Let the pair $\mathbf{V}_h/Q_h$ fulfill the discrete inf-sup condition* (4.9)*. Then it is also fulfilled by*

1. $\widetilde{\mathbf{V}}_h/Q_h$, *if* $\widetilde{\mathbf{V}}_h \supset \mathbf{V}_h$,
2. $\mathbf{V}_h/\widetilde{Q}_h$, *if* $\widetilde{Q}_h \subset Q_h$.

Proof:
1. Taking the supremum over a larger set (superset) can only increase its value.
2. Taking the infimum over a smaller set (subset) can only increase its value.

$\square$

Hence, the enlargement of $\mathbf{V}_h$, respectively the reduction of Q_h, can only lead to an increased discrete inf-sup constant, if existing.
So given a pair of finite element spaces, which does not quite satisfy the discrete inf-sup condition, the latter idea should be applied.
In many cases, the problem is that there does not exist a positive constant $\beta^\star$ satisfying (4.9), i.e., one gets

$$\inf_{q_h \in Q_h \setminus \{0\}} \sup_{\mathbf{u}_h \in \mathbf{V}_h \setminus \{\mathbf{0}\}} \frac{b_h(\mathbf{u}_h, q_h)}{\|q_h\|_{L^2} \|\mathbf{u}_h\|_{\mathbf{V}_h}} = 0.$$

Definition 4.3.1 (Stable / Unstable finite element pair) *A pair of finite element spaces $\mathbf{V}_h/Q_h$ is called unstable or inf-sup unstable if it does not fulfill the discrete inf-sup condition* (4.9)*, otherwise the approximation by $\mathbf{V}_h/Q_h$ is said to be inf-sup stable or just stable.*

Next we will emphasize two problematic cases.

Case 1: The Locking Phenomenon
The locking phenomenon is given if $\mathbf{u}_h = \mathbf{0}$ is the only discretely divergence-free velocity field in $\mathbf{V}_h$, i.e.,

$$b_h(\mathbf{u}_h, q_h) = 0, \ \forall q_h \in Q_h \Longleftrightarrow \mathbf{u}_h = \mathbf{0}.$$

This leads to a useless discrete velocity, since $\mathbf{u}_h = \mathbf{0}$ is generally not a good approximation for the velocity in the Stokes system.

Case 2: Spurious Pressure Modes
Assume there exists a discrete pressure $\tilde{p}_h \in Q_h$ with $\tilde{p}_h \neq 0$ and

$$b_h(\mathbf{v}_h, \tilde{p}_h) = 0, \ \forall \mathbf{v}_h \in \mathbf{V}_h,$$

then there does not exist a constant $\beta^\star \neq 0$ and the discrete inf-sup condition is violated. Indeed, for such a $\tilde{p}_h \in Q_h$ it holds

$$\begin{aligned} \sup_{\mathbf{v}_h \in \mathbf{V}_h \setminus \{0\}} \frac{b_h(\mathbf{v}_h, \tilde{p}_h)}{\|\mathbf{v}_h\|_{\mathbf{V}_h}} &= 0 \\ \Longrightarrow \inf_{q_h \in Q_h \setminus \{0\}} \sup_{\mathbf{v}_h \in \mathbf{V}_h \setminus \{0\}} \frac{b_h(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{\mathbf{V}_h} \|q_h\|_{Q_h}} &\leq 0. \end{aligned}$$

Hence the discrete inf-sup condition cannot be fulfilled.

Moreover, assume $(\mathbf{u}_h, p_h)$ solves the discrete Stokes problem (4.1), then $(\mathbf{u}_h, p_h + \tilde{p}_h)$ with $b_h(\mathbf{v}_h, \tilde{p}_h) = 0$, $\forall \mathbf{v}_h \in \mathbf{V}_h$, $\tilde{p}_h \neq 0$, is a solution, too. Such a pressure $\tilde{p}_h$ is called spurious pressure mode and a finite element pair $\mathbf{V}_h/Q_h$ with a spurious pressure mode is unstable.

To summarize, the velocity finite element space should be chosen large enough to contain nontrivial discretely divergence-free functions whereas the pressure finite element space has to be small enough to not include nontrivial functions leading to a violation of the discrete inf-sup condition as presented in the second case.

The pressure finite element function is defined to be continuous on the mesh cells but at the interfaces it can be discontinuous. This is the case since derivatives of the pressure do not occur in the weak formulation. Note that the relation $Q_h \subset Q = L_0^2(\mathbf{\Omega})$ does not imply that the discrete pressure is continuous. In contrast to that, the conforming velocity finite element function is required to be continuous on the entire space, due to the appearance of the gradient.

Remark 4.3.1

There are several different techniques to prove the discrete inf-sup condition. The most useful ones are probably Fortin's trick from [Fort77] and the macroelement technique from [Sten84]. We are not going into this here, but a summarized explanation of the techniques can be found in [BoBrFor06], pp. 56.

4.3.3 Inf-Sup Unstable Pairs of FE Spaces

From the above discussion, it should be clear that a careful choice of the discretization spaces for the velocity $\mathbf{u}$ and the pressure p is necessary in order to satisfy the discrete inf-sup condition.

The probably easiest choices for finite element spaces violate the discrete inf-sup condition. In this section we will discuss two-dimensional examples of finite element pairs for the cases mentioned in the previous section, leading to instability on a simplicial triangulation.
We will sometimes abbreviate the finite element spaces for the Stokes problem, based on polynomials of order k by $\mathbf{P}_k$, respectively P_k, instead of $\mathbf{V}_h$, respectively Q_h, for simplicity and their discontinuous counterparts by P_k^{disc}.

$\mathbf{P}_1/P_0$ – The Linear-Constant Element Pair.

In the following, it is illustrated how the locking phenomenon (Case 1) can arise.

Define

$$\mathbf{V}_h := \left\{\mathbf{v}_h \in \mathbf{C}^0(\overline{\mathbf{\Omega}}) : \mathbf{v}_h|_{\mathbf{T}} \in \mathbf{P}_1(\mathbf{T}),\ \forall \mathbf{T} \in \mathcal{T}_h,\ \text{and } \mathbf{v}_h|_{\partial\mathbf{\Omega}} = \mathbf{0}\right\}$$

and

$$Q_h := \left\{ q_h \in L^2(\mathbf{\Omega}) : q_h|_{\mathbf{T}} \in P_0(\mathbf{T}),\ \forall \mathbf{T} \in \mathcal{T}_h,\ \text{and} \int\limits_{\mathbf{\Omega}} q_h \,\mathrm{d}\mathbf{x} = 0 \right\}.$$

This means that the velocity is approximated by a continuous, elementwise linear function with homogeneous Dirichlet boundary condition and for the pressure, elementwise constant functions in $L_0^2(\mathbf{\Omega})$ are used.

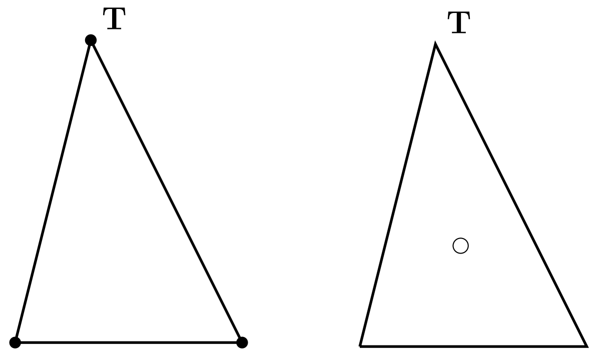


Figure 4.2: Elements of the triangulation $\mathcal{T}_h$ with the local degrees of freedom represented as filled circles for the velocity (left) and with the local degree of freedom pictured by a circle for the pressure (right), for the $\mathbf{P}_1/P_0$ element.

This choice of finite element spaces leads to a conforming method since $\mathbf{V}_h \subset \mathbf{H}_0^1(\mathbf{\Omega})$ and $Q_h \subset L_0^2(\mathbf{\Omega})$.
The divergence of a linear function is constant, hence for $\mathbf{v}_h \in \mathbf{V}_h$ we obtain $(\nabla \cdot \mathbf{v}_h)|_{\mathbf{T}} = \text{const.},\ \forall \mathbf{T} \in \mathcal{T}_h$, i.e., $(\nabla \cdot \mathbf{v}_h)|_{\mathbf{T}} \in P_0(\mathbf{T})$. Together with the homogeneous Dirichlet boundary condition this implies local incompressibility: $(\nabla \cdot \mathbf{v}_h)|_{\mathbf{T}} = 0,\ \forall \mathbf{T} \in \mathcal{T}_h$. The Gaussian theorem particularly gives

$$\int\limits_{\mathbf{\Omega}} \nabla \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} = \int\limits_{\partial\mathbf{\Omega}} \mathbf{v}_h \cdot \mathbf{n} \,\mathrm{d}\mathbf{s} = 0,\ \forall \mathbf{v}_h \in \mathbf{V}_h,$$

so that we can infer $\nabla \cdot \mathbf{v}_h \in Q_h$.
Due to the satisfaction of the property (4.10), discretely divergence-free functions are weakly divergence-free. So this finite element pair would lead to a divergence-free approximation of the velocity.

Unfortunately, for general meshes, this choice of finite element spaces is not inf-sup stable.
Consider a square domain $\mathbf{\Omega}$ subdivided into $2N^2$ triangles $\mathbf{T} \in \mathcal{T}_h$ as shown in Figure 4.3.

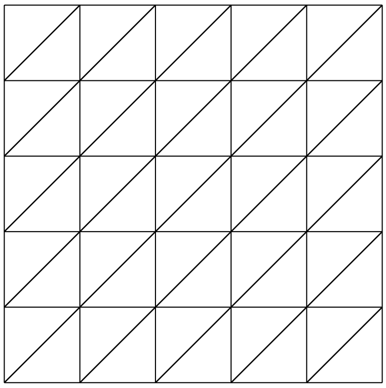


Figure 4.3: Triangulation of $\mathbf{\Omega}$ with $(N-1)^2$ inner nodes for the unstable $\mathbf{P}_1/P_0$-finite element

For the triangulation shown in Figure 4.3 it is

$$\dim(\mathbf{V}_h) = 2 \cdot \text{ \# inner nodes } = 2(N-1)^2,$$

since the homogeneous Dirichlet boundary condition fixes the values at the boundary nodes and

$$\dim(Q_h) = \text{ \# triangles } - 1 = 2N^2 - 1,$$

since the value in one element has to balance the integral mean value such that $\int\limits_{\mathbf{\Omega}} q_h \, \mathrm{d}\mathbf{x} = 0$. Hence

$$\dim(Q_h) > \dim(\mathbf{V}_h), \text{ for } N \geq 1,$$

and we have more degrees of freedom for the pressure than for the velocity. This violates the necessary condition (4.8) and the matrix B_h in (4.5) has more rows than columns and therefore a function $\mathbf{u}_h \in \mathbf{V}_h$ is overconstrained. Hence, the only discretely divergence-free velocity field then might be $\mathbf{u}_h = \mathbf{0}$. We conclude that this finite element pair is not inf-sup stable. Note that $\dim(Q_h) > \dim(\mathbf{V}_h)$ does not necessarily imply that $\mathbf{u}_h = \mathbf{0}$ is the only solution, because some rows in the matrix B_h could be linearly dependent for the chosen mesh, thus reducing $\dim(Q_h)$.

$\mathbf{P}_1/P_1$ – The Linear-Linear Element Pair.

This pair belongs to the group of equal-order finite element pairs and serves as an example for the Case 2.

The pressure and the velocity are both approximated by continuous, piecewise linear finite element functions. Consider the following model problem for the Stokes equations:
Let $\mathbf{\Omega} = (0,1)^2$ be a domain with a triangulation $\mathcal{T}_h$ and let each triangle $\mathbf{T} \in \mathcal{T}_h$ be given as the convex hull of the nodes $\{\mathbf{a}_{0,\mathbf{T}}, \mathbf{a}_{1,\mathbf{T}}, \mathbf{a}_{2,\mathbf{T}}\}$. Assume that $\tilde{p}_h$ is given such that the sum of its values at the nodes of each triangle is zero:

$$\sum_{i=0}^{2} \tilde{p}_h(\mathbf{a}_{i,\mathbf{T}}) = 0. \tag{4.12}$$

Such a setting is illustrated in the following Figure 4.4.

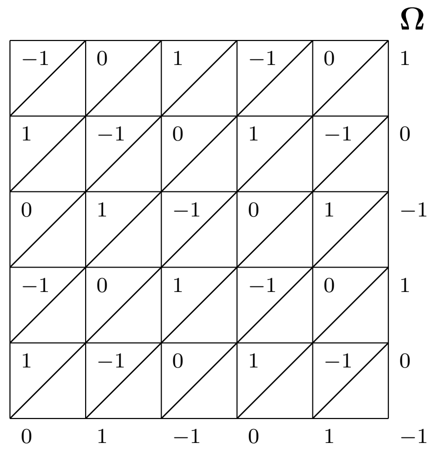


Figure 4.4: A spurious pressure mode for $\mathbf{P}_1/P_1$ with nodal pressure values.

Since $\mathbf{v}_h$ is linear on each triangle, $(\nabla \cdot \mathbf{v}_h)|_{\mathbf{T}}$ is constant. Together with (4.12), this implies

$$\begin{aligned}\int_{\mathbf{\Omega}} \tilde{p}_h \nabla \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} &= \sum_{\mathbf{T}\in\mathcal{T}_h} \int_{\mathbf{T}} \tilde{p}_h \nabla \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} = \sum_{\mathbf{T}\in\mathcal{T}_h} (\nabla \cdot \mathbf{v}_h)|_{\mathbf{T}} \int_{\mathbf{T}} \tilde{p}_h \,\mathrm{d}\mathbf{x} \\ &= \sum_{\mathbf{T}\in\mathcal{T}_h} (\nabla \cdot \mathbf{v}_h)|_{\mathbf{T}} \frac{|\mathbf{T}|}{3} \underbrace{\sum_{i=0}^{2} \tilde{p}_h(\mathbf{a}_{i,\mathbf{T}})}_{=0} = 0, \qquad \forall \mathbf{v}_h \in \mathbf{V}_h.\end{aligned}$$

For the third equality we used Gaussian quadrature, which is exact for linear functions. Hence, the $\mathbf{P}_1/P_1$ finite element pair generally can have spurious pressure modes and is therefore unstable.

$\mathbf{Q}_1/Q_0$ – The Bilinear-Constant Element Pair.

This finite element pair is another one where spurious pressure modes ruin the stability (Case 2). Again we consider $\mathbf{\Omega} = (0,1)^2$, this time subdivided into an even number $(N+1)^2$ of squares with edge length $h = \frac{1}{N+1}$, $N \geq 2$. The velocity is approximated by continuous, elementwise bilinear functions in $\mathbf{H}_0^1(\mathbf{\Omega})$ and the discrete pressure is a discontinuous, elementwise constant function contained in $L_0^2(\mathbf{\Omega})$.
Then it is

$$
\begin{aligned}
\dim(\mathbf{V}_h) &= 2 \cdot \#\text{ interior points } = 2(N)^2 \\
\dim(Q_h) &= \#\text{ mesh cells } - 1 = (N+1)^2 - 1 \\
&\Longrightarrow \dim(Q_h) \leq \dim(\mathbf{V}_h),
\end{aligned}
$$

since $N \geq 2$ and therefore this finite element pair could be inf-sup stable. But it is unstable because it has the so-called checkerboard-instability illustrated in Figure 4.5.

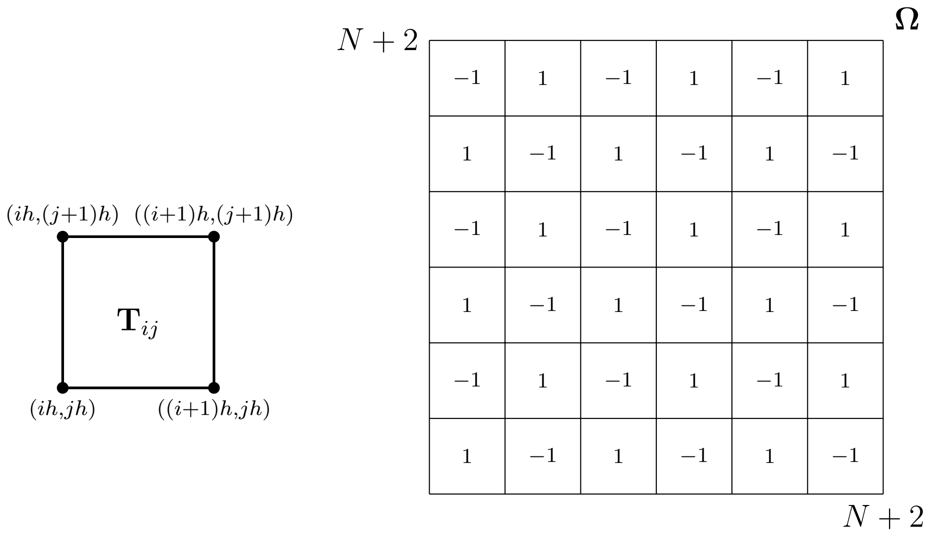


Figure 4.5: Checkerboard-instability: A spurious pressure mode for $\mathbf{Q}_1/Q_0$ with elementwise pressure values.

Denote by (ih, jh) the nodes of the triangulation starting with $(0,0)$ in the bottom left corner. Take the mesh cell $\mathbf{T}_{ij}$ having the node (ih, jh) as bottom left vertex. As a test function $\tilde{q}_h$ we choose $\tilde{q}_h|_{\mathbf{T}_{ij}} := (-1)^{i+j}$. Then $\tilde{q}_h$ is elementwise constant and belongs to $L_0^2(\mathbf{\Omega})$. With the Gaussian theorem and the trapezoidal rule it holds for each discrete velocity $\mathbf{u}_h \in \mathbf{V}_h$:

$$\int_{\mathbf{T}_{ij}} \tilde{q}_h \nabla \cdot \mathbf{u}_h \,\mathrm{d}\mathbf{x} = (-1)^{i+j} \int_{\partial \mathbf{T}_{ij}} \mathbf{u}_h \cdot \mathbf{n}_{\mathbf{T}_{ij}} \,\mathrm{d}\mathbf{s}$$
$$= (-1)^{i+j} \Big[\frac{h}{2} (\mathbf{u}_h((i+1)h, jh) + \mathbf{u}_h(ih, jh)) \cdot (-\mathbf{e}_2)$$
$$+ \frac{h}{2} (\mathbf{u}_h(ih, (j+1)h) + \mathbf{u}_h(ih, jh)) \cdot (-\mathbf{e}_1)$$
$$+ \frac{h}{2} (\mathbf{u}_h((i+1)h, jh) + \mathbf{u}_h((i+1)h, (j+1)h)) \cdot (\mathbf{e}_1)$$
$$+ \frac{h}{2} (\mathbf{u}_h((i+1)h, (j+1)h) + \mathbf{u}_h(ih, (j+1)h)) \cdot (\mathbf{e}_2) \Big].$$

Taking the sum over all mesh cells everything cancels but the values at the boundary. They are zero anyway, so we get

$$\int_{\mathbf{\Omega}} \tilde{q}_h \nabla \cdot \mathbf{u}_h \,\mathrm{d}\mathbf{x} = 0, \qquad \forall \mathbf{u}_h \in \mathbf{V}_h.$$

Thus, this pair is also an example for an unstable finite element pair for the Stokes system.

4.3.4 Inf-Sup Stable Pairs of FE Spaces

In this section we want to discuss some more interesting finite element spaces, in the sense of stability. So we analyze several finite element pairs that come up with an inf-sup constant independent of the mesh size $h > 0$.

Before talking about some inf-sup stable standard pairs, possibilities to construct stable pairs from unstable ones, by eliminating the troublesome properties, are discussed.

To get rid of spurious pressure modes the idea is to exclude them from the discrete pressure space. If the spurious pressure modes are the sole reason for the inf-sup instability, according to [Gunz89], pp. 21, and [Linke07], pp. 38, one may often get useful finite element approximations by filtering out the spurious pressure modes from the test space.

In the first example for unstable pairs, we had the problem of violating the condition $\dim(Q_h) \leq \dim(\mathbf{V}_h)$ which is necessary for the discrete inf-sup condition. To get rid of such a problem a good idea would be to enlarge $\dim(\mathbf{V}_h)$

respectively to reduce the size of Q_h accordingly. This can be achieved for instance by increasing the polynomial degree for the velocity approximation or by refining the mesh for the velocity approximation. The latter procedure is realized in the following example.

$\mathbf{P}_1$-iso-$\mathbf{P}_2/P_0$ – The Linear-Constant Element Pair on different triangulations.

In the previous section we proved the instability of $\mathbf{P}_1/P_0$ for the case of considering the same triangulation for the velocity space and the pressure space. Here, for the pressure space we take a triangulation $\mathcal{T}_h$ with mesh cells $\mathbf{T}_h$ and for the velocity space, in each cell, we connect the face barycenters with each other, pictured for dimension 2 and triangles in Figure 4.6. Denoting the resulting triangulation by $\mathcal{T}_{\frac{h}{2}}$ with elements $\mathbf{T}_{\frac{h}{2}}$, the finite element spaces are formalized to

$$\mathbf{V}_h := \left\{ \mathbf{v}_h \in \mathbf{C}^0(\overline{\mathbf{\Omega}}) : \mathbf{v}_h|_{\mathbf{T}_{\frac{h}{2}}} \in \mathbf{P}_1(\mathbf{T}_{\frac{h}{2}}),\ \forall \mathbf{T}_{\frac{h}{2}} \in \mathcal{T}_{\frac{h}{2}} \right\} \cap \mathbf{H}_0^1(\mathbf{\Omega})$$

and

$$Q_h := \left\{ q_h \in L_0^2(\mathbf{\Omega}) : q_h|_{\mathbf{T}_h} \in P_0(\mathbf{T}_h),\ \forall \mathbf{T}_h \in \mathcal{T}_h \right\}.$$

Thus, in the case of a two-dimensional domain we have 4 velocity triangles for each pressure triangle.

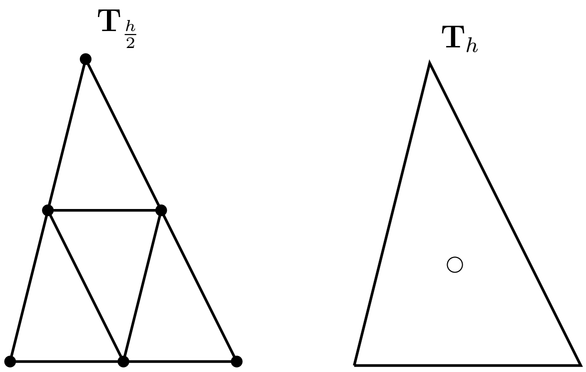


Figure 4.6: Elements of the triangulations $\mathcal{T}_{\frac{h}{2}}$ with the local degrees of freedom represented as filled circles for the velocity (left) and $\mathcal{T}_h$ with the local degree of freedom pictured by a circle for the pressure (right), for the stable $\mathbf{P}_1$-iso-$\mathbf{P}_2/P_0$-element.

Consider again a triangulation of $\mathbf{\Omega}$ into $(N+1)^2$ squares, each subdivided into two triangles by connecting the top right and the bottom left corner for the pressure space and the subdivision of it for the velocity space as described above. Therefore one obtains

$$\begin{aligned}\dim(\mathbf{V}_h) &= 2\#\text{ interior vertices} = 2\left[(2N+3)^2 - (8N+8)\right] = 8N^2+8N+1,\\ \dim(Q_h) &= \#\text{ triangles } -1 = 2(N+1)^2-1.\end{aligned}$$

Thus we get

$$\dim(Q_h) \leq \dim(\mathbf{V}_h).$$

This is a stable finite element discretization, see for example [Gunz89], p. 27, or [AuBrLov04], pp. 29.

The $\mathbf{P}_2/P_0$ Finite Element.

With this finite element pair, the velocity finite element space is in comparison to $\mathbf{P}_1/P_0$ enlarged. The velocity is approximated by continuous, piecewise quadratic vector fields and for the pressure we use piecewise constant functions.

Hence,

$$\mathbf{V}_h := \left\{\mathbf{v}_h \in \mathbf{C}^0(\overline{\mathbf{\Omega}}) : \mathbf{v}_h|_{\mathbf{T}} \in \mathbf{P}_2(\mathbf{T}),\ \forall \mathbf{T} \in \mathcal{T}_h,\ \text{and } \mathbf{v}_h|_{\partial\mathbf{\Omega}} = \mathbf{0}\right\}$$

and

$$Q_h := \left\{q_h \in L^2(\mathbf{\Omega}) : q_h|_{\mathbf{T}} \in P_0(\mathbf{T}),\ \forall \mathbf{T} \in \mathcal{T}_h,\ \text{and } \int_{\mathbf{\Omega}} q_h\,\mathrm{d}\mathbf{x} = 0\right\}.$$

The local degrees of freedom are visualized in Figure 4.7.

The enlarging leads to the satisfaction of the discrete inf-sup condition which is proven in [BoBrFor06], pp. 53.
Utilizing these finite element spaces is, however, not a satisfactory method. According to [AuBrLov04], pp. 29, the convergence rate for the velocity is suboptimal, due to the poor pressure interpolation. Furthermore, it is $\nabla \cdot \mathbf{P}_2 \subset P_1^{\text{disc}}$, the space of discontinuous, piecewise linear functions. Hence $\mathbf{V}_h^{\text{div}} \not\subseteq \mathbf{V}^{\text{div}}$ and therefore we cannot infer that discretely divergence-free vector fields are weakly divergence-free. Nevertheless, we get mass conservation elementwise since the pressure functions are constant on each mesh cell.

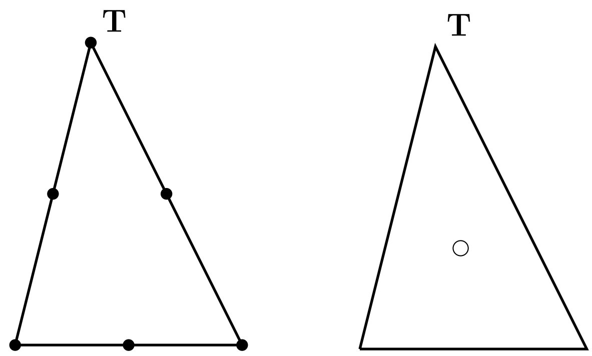


Figure 4.7: Elements of the triangulation $\mathcal{T}_h$ with the local degrees of freedom represented as filled circles for the velocity (left) and with the local degree of freedom pictured by a circle for the pressure (right), for the stable $\mathbf{P}_2/P_0$-element.

The Taylor–Hood Finite Element – $\mathbf{P}_k/P_{k-1}$.

For $k \geq 2$, the velocity is approximated elementwise by polynomials of order at most k and the discrete pressure space consists of elementwise polynomial functions with one degree less, $k-1$, at most.

$$\mathbf{V}_h^{TH} := \left\{ \mathbf{v}_h \in \mathbf{C}^0(\overline{\mathbf{\Omega}}) : \mathbf{v}_h|_{\mathbf{T}} \in \mathbf{P}_k(\mathbf{T}),\ \forall \mathbf{T} \in \mathcal{T}_h, \mathbf{v}_h|_{\partial\mathbf{\Omega}} = \mathbf{0} \right\}$$

and

$$Q_h^{TH} := \left\{ q_h \in C^0(\overline{\mathbf{\Omega}}) : q_h|_{\mathbf{T}} \in P_{k-1}(\mathbf{T}),\ \forall \mathbf{T} \in \mathcal{T}_h, \int\limits_{\mathbf{\Omega}} q_h \,\mathrm{d}\mathbf{x} = 0 \right\}.$$

Both, the discrete velocity and the discrete pressure are continuous on the entire domain. For $k \geq 2$ this pair is inf-sup stable.
The easiest Taylor–Hood element in the sense of polynomial order, $\mathbf{P}_2/P_1$, is shown in Figure 4.8.

The inf-sup stability for $k = 2$ is shown for example in [Ver84], [BrFort91], pp. 252, and [GiRa86], pp. 176. The proof for $k = 3$ can be found in [BrFa91].

The Taylor–Hood element is neither locally mass conservative, nor weakly divergence-free, since it uses continuous pressures. Therefore $\nabla \cdot \mathbf{V}_h^{TH} \not\subseteq Q_h^{TH}$ and one cannot infer weak mass conservation.

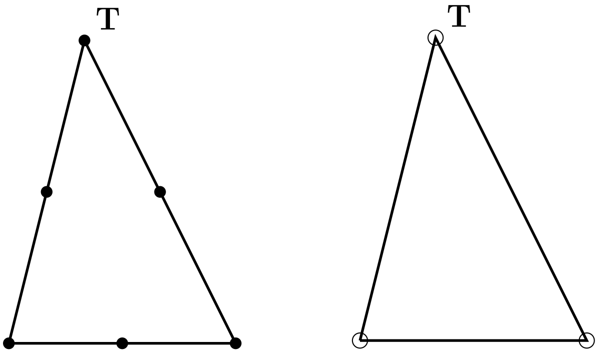


Figure 4.8: Elements of the triangulation $\mathcal{T}_h$ with the local degrees of freedom represented as filled circles for the velocity (left) and with the local degrees of freedom pictured by circles for the pressure (right), for the stable Taylor–Hood element $\mathbf{P}_2/P_1$.

The Scott–Vogelius Finite Element – $\mathbf{P}_k/P_{k-1}^{\mathrm{disc}}$.

This finite element pair is pretty similar to the Taylor–Hood element. The velocity space is the same but the crucial difference is the definition of the pressure space. In the Scott–Vogelius element the pressure is approximated by means of a discontinuous function which is elementwise a polynomial in $P_{k-1}(\mathbf{T})$. The approximation spaces are

$$\mathbf{V}_h^{SV} := \mathbf{V}_h^{TH}$$

and

$$Q_h^{SV} := \Big\{q_h \in L^2(\mathbf{\Omega}) : q_h|_{\mathbf{T}} \in P_{k-1}(\mathbf{T}),\ \forall \mathbf{T} \in \mathcal{T}_h\Big\} \cap L_0^2(\mathbf{\Omega}).$$

The degrees of freedom on a mesh cell for the cases $k = 2, 3$, $n = 2$, are pictured in Figure 4.9.

For $k \geq 4$ and $\mathbf{\Omega} \subset \mathbb{R}^2$, Scott and Vogelius have shown the inf-sup stability in [ScoVo85], Theorem 5.1, provided there are no singular vertices in the triangulation. Thereby, an interior vertex of degree 4 (in the realm of graph theory meaning that it has 4 neighboring vertices) is singular, if the four adjacent edges lie on two straight lines as in Figure 4.10.

Hence the Scott–Vogelius pair is not inf-sup stable on general meshes. A way to establish stability is to slightly perturb the singular vertices.

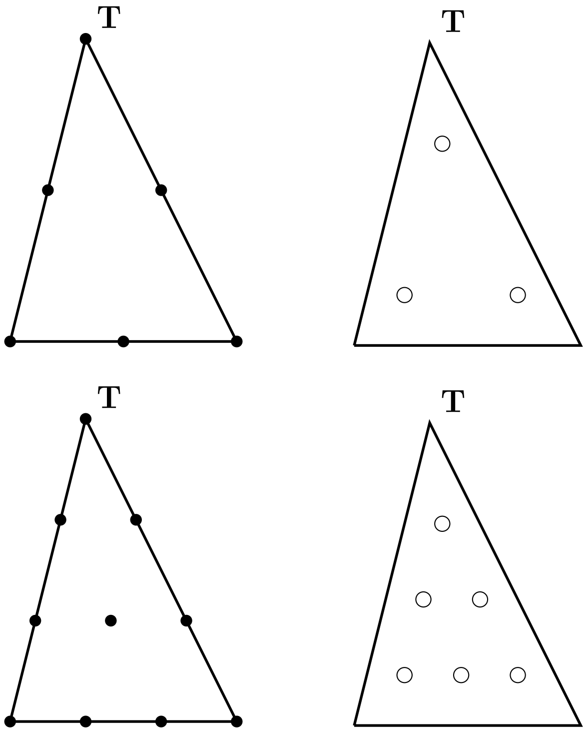


Figure 4.9: Elements of the triangulation $\mathcal{T}_h$ with the local degrees of freedom represented as filled circles for the velocity (left) and with the local degrees of freedom pictured by circles for the pressure (right), for the $\mathbf{P}_2/P_1^{\text{disc}}$-element on top and $\mathbf{P}_3/P_2^{\text{disc}}$-element beneath.

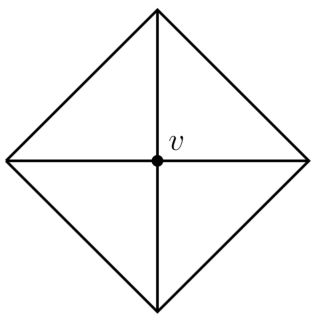


Figure 4.10: A singular vertex v.

Moreover, Qin has proven the stability for $k \geq 2$ for a special family of meshes called barycentric trisected meshes. Such a partition is obtained by connecting the vertices of each triangle with its barycenter. For further information see [Qin94].

A very nice point of these pairs is that they are weakly divergence-free due to $\nabla \cdot \mathbf{P}_k \subset P_{k-1}^{disc}$, i.e., they fulfill (4.10).

Unfortunately, the discrete inf-sup condition is not fulfilled for arbitrary triangulations.

The nonconforming Crouzeix–Raviart Finite Element - $\mathbf{P}_1^{\mathrm{nc}}/P_0$.

Up to now we where considering conforming methods, i.e., those with $\mathbf{V}_h \subset \mathbf{V}$ and $Q_h \subset Q$. The crucial point thereby is the global continuity of the discrete velocity. The usage of nonconforming methods can enable the use of low order polynomial approaches.

The simple conforming finite element pair $\mathbf{P}_1/P_0$ is not inf-sup stable, but Crouzeix and Raviart developed a nonconforming, stable version of it. The sacrifice of conformity is the point of matter. The nonconforming finite element discretization by Crouzeix and Raviart [CrRa73], also called $\mathbf{P}_1^{\mathrm{nc}}/P_0$, is here introduced as another example for an inf-sup stable pair, see [AuBrLov04], pp. 34.
The velocity finite element space $\mathbf{V}_h^{\mathrm{CR}}$ is therefore the space of Crouzeix–Raviart finite element functions defined by

$$\begin{aligned}\mathbf{V}_h^{\mathrm{CR}} := \Big\{ \mathbf{v}_h \in \mathbf{L}^2(\mathbf{\Omega}) : \ & \mathbf{v}_{h|\mathbf{T}} \in \mathbf{P}_1(\mathbf{T}),\ \forall \mathbf{T} \in \mathcal{T}_h, \\ & [\![\mathbf{v}_h]\!]_{\mathbf{F}}(\mathbf{x}_F) = \mathbf{0},\ \forall \mathbf{F} \in \mathcal{F}_h, \\ & \mathbf{v}_h(\mathbf{x}_F) = \mathbf{0},\ \forall \mathbf{F} \in \mathcal{F}_h^{\star} \setminus \mathcal{F}_h \Big\}\end{aligned}$$

and consists of elementwise linear, usually discontinuous functions in $\mathbf{L}^2(\mathbf{\Omega})$, which are continuous at the barycenters of interior faces $\mathbf{F}$ and vanish at the barycenters of boundary faces.

The pressure is approximated by elementwise constant functions with vanishing integral mean value over $\mathbf{\Omega}$:

$$Q_h^{\mathrm{CR}} := \left\{ q_h \in L_0^2(\mathbf{\Omega}) : q_h|_{\mathbf{T}} \in P_0(\mathbf{T}),\ \forall \mathbf{T} \in \mathcal{T}_h \right\}.$$

The conventional representation of the nonconforming Crouzeix–Raviart element is pictured for 2D in Figure 4.11.

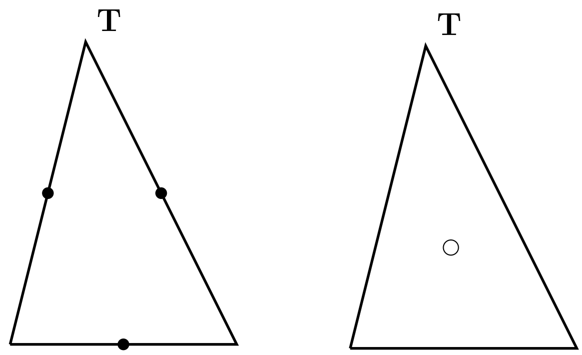


Figure 4.11: Elements of the triangulation $\mathcal{T}_h$ with the local degrees of freedom represented as filled circles for the velocity (left) and with the local degree of freedom pictured by a circle for the pressure (right), for the nonconforming Crouzeix–Raviart element $\mathbf{P}_1^{\mathrm{nc}}/P_0$.

Remark 4.3.2

1. The continuity at the barycenters of the faces is equivalent to the face jump being zero there ($[\![\mathbf{v}_h]\!]_{\mathbf{F}}(\mathbf{x_F}) = \mathbf{0}$) and the homogeneous Dirichlet boundary condition is imposed by $\mathbf{v}_h(\mathbf{x_F}) = \mathbf{0}$ for all boundary faces.
2. A function $\mathbf{v}_h \in \mathbf{V}_h^{\mathrm{CR}}$ only has to be continuous in the barycenters of the faces so in general $\mathbf{v}_h$ will have nonzero face jumps at the transition from one mesh cell to another. The space Q_h^{CR} is a subset of Q, so $\|\cdot\|_{Q_h^{\mathrm{CR}}} = \|\cdot\|_Q$. Although $Q_h^{\mathrm{CR}} \subseteq Q$, conformity is ruined by $\mathbf{V}_h^{\mathrm{CR}} \not\subseteq \mathbf{V} := \mathbf{H}_0^1(\mathbf{\Omega})$. In addition, the functions in $\mathbf{V}_h^{\mathrm{CR}}$ do not need to have a divergence in $L^2(\mathbf{\Omega})$. Thus, the Crouzeix–Raviart finite element space leads to a nonconforming method.

In fact, we are dealing with a discontinuous velocity finite element space. Thus, we cannot use the usual differential operators, but should use their elementwise defined counterparts.

Definition 4.3.2 (Broken Sobolev space) *Let $\mathcal{T}_h$ be a triangulation of the domain $\mathbf{\Omega}$. We define the broken Sobolev space $\mathbf{H}^1(\mathcal{T}_h)$ by*

$$\mathbf{H}^1(\mathcal{T}_h) := \left\{ \mathbf{v} \in \mathbf{L}^2(\mathbf{\Omega}) : \mathbf{v}|_{\mathbf{T}} \in \mathbf{H}^1(\mathbf{T}),\ \forall \mathbf{T} \in \mathcal{T}_h \right\}.$$

For such broken Sobolev spaces we want to introduce a broken divergence and a broken gradient operator.

Definition 4.3.3 (Broken divergence and broken gradient) *Let $\mathcal{T}_h$ be a triangulation of the domain $\mathbf{\Omega}$. The broken divergence is a map*

$$\nabla_h \cdot (\cdot) : \mathbf{H}^1(\mathcal{T}_h) \to L^2(\mathbf{\Omega})$$

elementwise defined by

$$(\nabla_h \cdot \mathbf{v}_h)\,|_{\mathbf{T}} := \nabla \cdot (\mathbf{v}_h|_{\mathbf{T}}), \quad \forall \mathbf{T} \in \mathcal{T}_h, \mathbf{v}_h \in \mathbf{H}^1(\mathcal{T}_h).$$

The broken gradient is a map

$$\nabla_h(\cdot) : \mathbf{H}^1(\mathcal{T}_h) \to \mathbf{L}^2(\mathbf{\Omega})$$

elementwise defined by

$$(\nabla_h \mathbf{v}_h)\,|_{\mathbf{T}} := \nabla(\mathbf{v}_h|_{\mathbf{T}}), \quad \forall \mathbf{T} \in \mathcal{T}_h, \mathbf{v} \in \mathbf{H}^1(\mathcal{T}_h).$$

This allows us to use the concept of the gradient and divergence for nonconforming finite element spaces $\mathbf{V}_h$. Then

$$\|\mathbf{v}_h\|_{1,h} := \left(\int_{\mathbf{\Omega}} \nabla_h \mathbf{v}_h : \nabla_h \mathbf{v}_h \,\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}$$

is a mesh-dependent norm in $\mathbf{V}_h^{\mathrm{CR}}$.

The associated space of discretely divergence-free functions is

$$\mathbf{V}_h^{\mathrm{div,CR}} := \left\{ \mathbf{v}_h \in \mathbf{V}_h^{\mathrm{CR}} : b_h(\mathbf{v}_h, q_h) = \int\limits_{\mathbf{\Omega}} q_h \left(\nabla_h \cdot \mathbf{v}_h\right) \,\mathrm{d}\mathbf{x} = 0, \;\; \forall q_h \in Q_h^{\mathrm{CR}} \right\}.$$

We denote the space of $\mathbf{L}^2$-functions with divergence in L^2 by

$$\mathbf{H}(\mathrm{div}, \mathbf{\Omega}) := \left\{ \mathbf{u} \in \mathbf{L}^2(\mathbf{\Omega}) : \nabla \cdot \mathbf{u} \in L^2(\mathbf{\Omega}) \right\}$$

equipped with the norm

$$\|\mathbf{u}\|^2_{\mathbf{H}(\mathrm{div},\mathbf{\Omega})} = \|\mathbf{u}\|^2_{\mathbf{L}^2(\mathbf{\Omega})} + \|\nabla \cdot \mathbf{u}\|^2_{\mathbf{L}^2(\mathbf{\Omega})}.$$

Lemma 4.3.2 *Let $\mathcal{T}_h$ be a triangulation of $\mathbf{\Omega}$ and $\mathbf{u}$ a finite element function, i.e., elementwise in $\mathbf{C}^\infty$. Then it holds $\mathbf{u} \in \mathbf{H}(\mathrm{div}, \mathbf{\Omega})$, if $\mathbf{u} \cdot \mathbf{n}_{\mathbf{F}}$ is continuous on each $\mathbf{F} \in \mathcal{F}_h$.*

Proof: Assume that $\mathbf{u} \in \mathbf{L}^2(\mathbf{\Omega})$ and let $\mathcal{T}_h$ be a triangulation of the domain $\mathbf{\Omega}$. By Definition 2.2.4, $\nabla \cdot \mathbf{u} \in L^2(\mathbf{\Omega})$ if and only if there exists an element $s \in L^2(\mathbf{\Omega})$ with

$$-\int_{\mathbf{\Omega}} \mathbf{u} \cdot \nabla \phi \, \mathrm{d}\mathbf{x} = \int_{\mathbf{\Omega}} s\phi \, \mathrm{d}\mathbf{x}, \qquad \forall \phi \in C_0^\infty(\mathbf{\Omega}).$$

For all $\phi \in C_0^\infty(\mathbf{\Omega})$, using integration by parts, we compute

$$\begin{aligned} -\int_{\mathbf{\Omega}} \mathbf{u} \cdot \nabla \phi \, \mathrm{d}\mathbf{x} = -\sum_{\mathbf{T} \in \mathcal{T}_h} \int_{\mathbf{T}} \mathbf{u} \cdot \nabla \phi \, \mathrm{d}\mathbf{x} &= \sum_{\mathbf{T} \in \mathcal{T}_h} \left[\int_{\mathbf{T}} \nabla \cdot \mathbf{u} \phi \, \mathrm{d}\mathbf{x} - \int_{\partial \mathbf{T}} \phi \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}\mathbf{s} \right] \\ &= \sum_{\mathbf{T} \in \mathcal{T}_h} \int_{\mathbf{T}} \nabla \cdot \mathbf{u} \phi \, \mathrm{d}\mathbf{x} - \sum_{\mathbf{T} \in \mathcal{T}_h} \sum_{\mathbf{F} \in \mathcal{F}_{\mathbf{T}}} \int_{\mathbf{F}} \phi \mathbf{u} \cdot \mathbf{n}_{\mathbf{T},\mathbf{F}} \, \mathrm{d}\mathbf{s} \\ &= \int_{\mathbf{\Omega}} \nabla_h \cdot \mathbf{u} \phi \, \mathrm{d}\mathbf{x} - \sum_{\mathbf{F} \in \mathcal{F}_h} \int_{\mathbf{F}} \phi [\![\mathbf{u} \cdot \mathbf{n}_{\mathbf{F}}]\!]_{\mathbf{F}} \, \mathrm{d}\mathbf{s} \\ &\quad + \underbrace{\sum_{\mathbf{F} \in \mathcal{F}_h^\star \setminus \mathcal{F}_h} \int_{\mathbf{F}} \phi \mathbf{u} \cdot \mathbf{n}_{\mathbf{T},\mathbf{F}} \, \mathrm{d}\mathbf{s}}_{=0}. \end{aligned}$$

The last summand vanishes since $\phi \in C_0^\infty(\mathbf{\Omega})$. We see that $\mathbf{u}$ has a divergence in $L^2(\mathbf{\Omega})$ if also the summand before last is zero. Therefore the jumps of the normal components have to vanish on every interior face, which is equivalent to the requirement that the normal component of $\mathbf{u}$ is continuous at the boundaries of the mesh cells. $\square$

Remark 4.3.3
It is $\mathbf{V}_h^{\mathrm{div,CR}} \not\subseteq H(\mathrm{div}, \mathbf{\Omega})$, since discretely divergence-free functions in $\mathbf{V}_h^{\mathrm{CR}}$ do in general not have a divergence in $L^2(\mathbf{\Omega})$. Therefore they are usually not weakly divergence-free.

Lemma 4.3.3 *It holds*

$$\mathbf{V}_h^{\mathrm{div,CR}} = \left\{ \mathbf{v}_h \in \mathbf{V}_h^{\mathrm{CR}} : \nabla_h \cdot \mathbf{v}_h = 0 \right\}.$$

Proof: $\underline{\subseteq}$:
A function $\mathbf{v}_h \in \mathbf{V}_h^{\mathrm{div,CR}}$ is by definition piecewise linear since $\mathbf{V}_h^{\mathrm{div,CR}} \subset \mathbf{V}_h^{\mathrm{CR}}$. Therefore, $(\nabla \cdot \mathbf{v}_h)|_{\mathbf{T}} = \text{const.}$ for all $\mathbf{T} \in \mathcal{T}_h$ implying $\nabla_h \cdot \mathbf{v}_h \in$

$L^2(\mathbf{\Omega})$. The midpoint rule yields

$$\begin{aligned}
\sum_{\mathbf{T}\in\mathcal{T}_h}\int_{\mathbf{T}}\nabla\cdot\mathbf{v}_h\,\mathrm{d}\mathbf{x} &= \sum_{\mathbf{T}\in\mathcal{T}_h}\sum_{\mathbf{F}\in\mathcal{F}_{\mathbf{T}}}\int_{\mathbf{F}}\mathbf{v}_h\cdot\mathbf{n}_{\mathbf{T},\mathbf{F}}\,\mathrm{d}\mathbf{s}\\
&= \sum_{\mathbf{T}\in\mathcal{T}_h}\sum_{\mathbf{F}\in\mathcal{F}_{\mathbf{T}}}|\mathbf{F}|\mathbf{v}_h(\mathbf{x}_{\mathbf{F}})\cdot\mathbf{n}_{\mathbf{T},\mathbf{F}}\\
&= \sum_{\mathbf{F}\in\mathcal{F}_h}|\mathbf{F}|\underbrace{[\![\mathbf{v}_h\cdot\mathbf{n}_{\mathbf{F}}]\!]_{\mathbf{F}}(\mathbf{x}_{\mathbf{F}})}_{=0}+\sum_{\mathbf{F}\in\mathcal{F}_h^*\setminus\mathcal{F}_h}|\mathbf{F}|\underbrace{\mathbf{v}_h(\mathbf{x}_{\mathbf{F}})}_{=\mathbf{0}}\cdot\mathbf{n}_{\mathbf{F}}\\
&= 0.
\end{aligned}$$

The terms vanish in the face barycenters by the definition of the Crouzeix–Raviart element. It follows

$$\int_{\mathbf{\Omega}}\nabla_h\cdot\mathbf{v}_h\,\mathrm{d}\mathbf{x}=0\Longrightarrow\nabla_h\cdot\mathbf{v}_h\in Q_h^{\mathrm{CR}}.$$

By assumption we can now choose $q_h=\nabla_h\cdot\mathbf{v}_h$ leading to

$$0=b_h\left(\mathbf{v}_h,\nabla_h\cdot\mathbf{v}_h\right)=-\sum_{\mathbf{T}\in\mathcal{T}_h}\underbrace{\int_{\mathbf{T}}\left(\nabla\cdot\mathbf{v}_h\right)^2\,\mathrm{d}\mathbf{x}}_{\geq 0}\Longrightarrow\nabla_h\cdot\mathbf{v}_h=0.$$

$\supseteq$:
Assume $\mathbf{v}_h\in\mathbf{V}_h^{\mathrm{CR}}$ with $\nabla_h\cdot\mathbf{v}_h=0$. Then, it is

$$\begin{aligned}
0&=-\int_{\mathbf{\Omega}}q_h\left(\nabla_h\cdot\mathbf{v}_h\right)\,\mathrm{d}\mathbf{x}=-\sum_{\mathbf{T}\in\mathcal{T}_h}\int_{\mathbf{T}}q_h\left(\nabla\cdot\mathbf{v}_h\right)\,\mathrm{d}\mathbf{x}\\
&=b_h\left(\mathbf{v}_h,q_h\right),\qquad\forall q_h\in Q_h^{\mathrm{CR}}.
\end{aligned}$$

□

Now the task is to solve the problem (4.1) for the above introduced finite element spaces. As we have seen in Theorem 4.2.3, the discrete problem has a unique solution if $a_h(\cdot,\cdot)$ is $\mathbf{V}_h^{\mathrm{div,CR}}$-elliptic and the spaces $\mathbf{V}_h^{\mathrm{CR}}$ and Q_h^{CR} fulfill the discrete inf-sup condition.

Theorem 4.3.1 *There is a constant $C>0$ such that*

$$a_h(\mathbf{v}_h,\mathbf{v}_h)\geq C\|\mathbf{v}_h\|_{1,h}^2,\qquad\forall\mathbf{v}_h\in\mathbf{V}_h^{\mathrm{div,CR}},$$

i.e., $a_h(\cdot,\cdot)$ is $\mathbf{V}_h^{\mathrm{div,CR}}$-elliptic and there exists a constant $\beta^\star>0$ independent of h with

$$\beta^\star\leq\inf_{q_h\in Q_h^{\mathrm{CR}}\setminus\{0\}}\sup_{\mathbf{v}_h\in\mathbf{V}_h^{\mathrm{CR}}\setminus\{\mathbf{0}\}}\frac{b_h(\mathbf{v}_h,q_h)}{\|\mathbf{v}\|_{1,h}\|q_h\|_{L^2(\mathbf{\Omega})}}.$$

Additionally $\mathbf{V}_h^{\mathrm{div,CR}}\neq\{\mathbf{0}\}$.

Proof: See [Ver98] pp. 71. □

Thus, $\mathbf{V}_h^{\mathrm{CR}}/Q_h^{\mathrm{CR}}$ is inf-sup stable.

Using the Crouzeix–Raviart finite element for the velocity and approximating the pressure by piecewise constant functions, one obtains the following error estimates.

Theorem 4.3.2 (Error estimates for the CR method) *Let $(\mathbf{u}, p) \in \mathbf{H}_0^1(\boldsymbol{\Omega}) \times L_0^2(\boldsymbol{\Omega})$ be the unique weak solution of the Stokes problem with $\mathbf{u} \in \mathbf{H}^2(\boldsymbol{\Omega})$ and $p \in H^1(\boldsymbol{\Omega})$. Moreover, let $(\mathbf{u}_h, p_h) \in \mathbf{V}_h^{\mathrm{CR}} \times Q_h^{\mathrm{CR}}$ be the unique solution of the discrete Stokes problem. Then there are constants $C_1, C_2 \in \mathbb{R}$ such that one gets the error estimates*

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,h} + \|p - p_h\|_{L^2(\boldsymbol{\Omega})} \leq C_1 h \left\{ |\mathbf{u}|_{\mathbf{H}^2(\boldsymbol{\Omega})} + |p|_{H^1(\boldsymbol{\Omega})} \right\}$$

and for convex domains $\boldsymbol{\Omega}$

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\boldsymbol{\Omega})} \leq C_2 h^2 \left\{ |\mathbf{u}|_{\mathbf{H}^2(\boldsymbol{\Omega})} + |p|_{H^1(\boldsymbol{\Omega})} \right\}. \tag{4.13}$$

Proof: The proof is stated in [Ver98], pp. 76. □

Remark 4.3.4

This theorem illustrates the qualitative dependence of the finite element velocity approximation on the pressure. The bigger the pressure p is, the bigger the velocity error can get. This will also be seen analyzing the Helmholtz decomposition for irrotational translations of the forces. So, modeling certain physical systems using this discretization method might not provide an authentic simulation of the reality.
It is desirable to use a method which on the one hand preserves the order of convergence stated in Theorem 4.3.2 and on the other hand produces a discrete velocity solution, such that the velocity error is independent of the pressure. Moreover, it is desirable to get a weakly divergence-free approximation. This is the subject of Chapter 5 and Chapter 6.

5 A Divergence-Free Reconstruction for the Crouzeix–Raviart Element

This chapter will cover a modified version of the nonconforming Crouzeix–Raviart finite element method proposed by Linke in [Linke14]. The Crouzeix–Raviart element fulfills the inf-sup condition and is locally divergence-free, but its nonconformity is reflected in $\mathbf{V}_h^{\mathrm{CR}} \not\subseteq H(\mathrm{div})$, i.e., in particular discretely divergence-free vector fields do not necessarily have a divergence in L^2. Consequently the utilization of this finite element pair results in an exterior method. In fact most of the finite element methods for the Stokes problem are exterior methods.

Via the Helmholtz decomposition, we will investigate that a special property, the invariance property, is fulfilled for the weak formulation, but not necessarily for the discretized system. Typically, exterior methods do not fulfill this invariance property whereas divergence-free methods as the Scott–Vogelius element do. The fact that the Crouzeix–Raviart element does not inherit this property is reflected in the dependence of the velocity error (4.13) on the pressure. This is equivalent to the change of the discrete velocity under translations of the right-hand side in the discrete Stokes problem (4.1) by gradient forces. Thus irrotational translations of the force field affect the velocity error. The reason for that is the lack of L^2-orthogonality for discretely divergence-free vector fields and irrotational vector fields, in contrast to divergence-free vector fields.

The presented method modifies the finite element formulation for the Crouzeix–Raviart element in a way that the right-hand side $\mathbf{f}$ is only tested by projections on the lowest order Raviart–Thomas space. This Raviart–Thomas projection maps discretely divergence-free vector fields onto weakly divergence-free vector fields. Therefore, by L^2-orthogonality of divergence-free and irrotational vector fields, the pressure uncouples from the velocity error (4.13) for this modified method and the invariance property is established.
Additionally, this Raviart–Thomas projection provides a possibility to recover a divergence-free velocity approximation.

5.1 The Helmholtz Decomposition for Vector Fields in $\mathbf{L}^2(\mathbf{\Omega})$ and the Helmholtz Projection

The Helmholtz decomposition is a fundamental functional analytic result. Remembering the fact mentioned in the beginning of Chapter 2 about the gradient of a sufficiently smooth scalar field being irrotational, it states that each $\mathbf{L}^2$-vector field can be decomposed into a divergence-free and an irrotational vector field.

Lemma 5.1.1 (Helmholtz decomposition) *Every vector field* $\mathbf{f} \in \mathbf{L}^2(\mathbf{\Omega})$ *can be uniquely decomposed into*

$$\mathbf{f} = \nabla\phi + \mathbf{w},$$

where $\phi \in H^1(\mathbf{\Omega})/\mathbb{R}$ *is the solution of*

$$\int_{\mathbf{\Omega}} \nabla\phi \cdot \nabla\chi \mathrm{d}\mathbf{x} = \int_{\mathbf{\Omega}} \mathbf{f} \cdot \nabla\chi \mathrm{d}\mathbf{x}, \quad \forall\chi \in H^1(\mathbf{\Omega})/\mathbb{R}. \tag{5.1}$$

Thereby

1. $\nabla\phi$ *is irrotational and* $\mathbf{w}$ *is divergence-free,*
2. $\mathbf{w}$ *and* $\nabla\phi$ *are orthogonal in* $\mathbf{L}^2(\mathbf{\Omega})$*, i.e.,*

$$\int_{\mathbf{\Omega}} \mathbf{w} \cdot \nabla\phi \mathrm{d}\mathbf{x} = 0.$$

Proof: 1. Since the gradient of a scalar field is irrotational (see Lemma 2.1.1), the first part is trivially true. To show that $\mathbf{w}$ is divergence-free we first see that it is an element of $\mathbf{L}^2(\mathbf{\Omega})$:

$$\mathbf{w} = \underbrace{\mathbf{f}}_{\in\mathbf{L}^2(\mathbf{\Omega})} - \underbrace{\nabla\phi}_{\in\mathbf{L}^2(\mathbf{\Omega})} \implies \mathbf{w} \in \mathbf{L}^2(\mathbf{\Omega}).$$

For all $\psi \in C_0^\infty(\mathbf{\Omega})$ it holds

$$\int_{\mathbf{\Omega}} \mathbf{w} \cdot \nabla\psi \mathrm{d}\mathbf{x} = \int_{\mathbf{\Omega}} \mathbf{f} \cdot \nabla\psi \mathrm{d}\mathbf{x} - \int_{\mathbf{\Omega}} \nabla\phi \cdot \nabla\psi \mathrm{d}\mathbf{x} \overset{(5.1)}{=} 0.$$

Using Definition 2.2.4 yields

$$\begin{aligned} 0 = -\int_{\mathbf{\Omega}} \mathbf{w} \cdot \nabla\phi \, \mathrm{d}\mathbf{x} &= \int_{\mathbf{\Omega}} s\phi \, \mathrm{d}\mathbf{x}, \quad \forall\phi \in C_0^\infty(\mathbf{\Omega}) \\ \implies s &= 0 \\ \iff \nabla \cdot \mathbf{w} &= 0. \end{aligned}$$

2. To show the orthogonality we compute

$$\int\limits_{\boldsymbol{\Omega}} \mathbf{w} \cdot \nabla\phi \mathrm{d}\mathbf{x} = \int\limits_{\boldsymbol{\Omega}} (\mathbf{f} - \nabla\phi) \cdot \nabla\phi \mathrm{d}\mathbf{x} \overset{(5.1)}{=} 0.$$

□

Remark 5.1.1

1. The quotient of $H^1(\boldsymbol{\Omega})$ by $\mathbb{R}$ is denoted by $H^1(\boldsymbol{\Omega})/\mathbb{R}$.
2. Problem (5.1) is the weak formulation of

$$-\Delta\phi \,(= -\nabla \cdot (\nabla\phi + \mathbf{w})) = -\nabla \cdot \mathbf{f}, \qquad \text{in } \boldsymbol{\Omega}, \tag{5.2a}$$

$$(\mathbf{f} - \nabla\phi) \cdot \mathbf{n} = 0, \qquad \text{on } \partial\boldsymbol{\Omega}. \tag{5.2b}$$

Multiplying (5.2a) with a test function χ and integration utilizing integration by parts and Green's first formula yields

$$\begin{aligned} & -\int\limits_{\boldsymbol{\Omega}} (\Delta\phi)\chi \mathrm{d}\mathbf{x} = -\int\limits_{\boldsymbol{\Omega}} (\nabla \cdot \mathbf{f})\chi \mathrm{d}\mathbf{x} \\ \iff & \int\limits_{\boldsymbol{\Omega}} \nabla\phi \cdot \nabla\chi \mathrm{d}\mathbf{x} - \int\limits_{\partial\boldsymbol{\Omega}} (\nabla\phi \cdot \mathbf{n})\chi \mathrm{d}\mathbf{s} = \int\limits_{\boldsymbol{\Omega}} \mathbf{f} \cdot \nabla\chi \mathrm{d}\mathbf{x} - \int\limits_{\partial\boldsymbol{\Omega}} (\mathbf{f} \cdot \mathbf{n})\chi \mathrm{d}\mathbf{s} \\ \iff & \int\limits_{\boldsymbol{\Omega}} \nabla\phi \cdot \nabla\chi \mathrm{d}\mathbf{x} + \int\limits_{\partial\boldsymbol{\Omega}} \underbrace{(\mathbf{f} - \nabla\phi) \cdot \mathbf{n}}_{\overset{(5.2b)}{=} 0 \text{ on } \partial\boldsymbol{\Omega}} \chi \mathrm{d}\mathbf{s} = \int\limits_{\boldsymbol{\Omega}} \mathbf{f} \cdot \nabla\chi \mathrm{d}\mathbf{x}. \end{aligned}$$

3. Irrotational vector fields and divergence-free vector fields are orthogonal in L^2, provided that one of them vanishes at the boundary. For a detailed analysis see [Sohr01], pp. 81.

Definition 5.1.1 (Helmholtz projection) *Let* $\mathbf{f} = \mathbf{w} + \nabla\psi$ *be the Helmholtz decomposition of* $\mathbf{f} \in \mathbf{L}^2(\boldsymbol{\Omega})$ *as in Lemma* 5.1.1. *Then the map*

$$\mathbb{P} \text{ with } \mathbb{P}(\mathbf{f}) = \mathbf{w}$$

is a bounded, linear operator called the Helmholtz projection of $\mathbf{L}^2(\boldsymbol{\Omega})$.

5.2 Implications for the Stokes Equations

5.2.1 The Continuous Problem

The Stokes system can be decomposed into two problems decoupling the velocity from the pressure. Firstly, a problem in the space of weakly divergence-free functions has to be solved, leading to the velocity solution $\mathbf{u}$. Secondly,

this velocity solution is used to find a pressure solution for the problem stated in the orthogonal complement $\mathbf{V}^{\mathrm{div},\perp}$.
We have seen that in order to determine the velocity solution $\mathbf{u} \in \mathbf{V}^{\mathrm{div}}$ of the Stokes problem for a given $\mathbf{f} \in \mathbf{L}^2(\boldsymbol{\Omega})$, it suffices to consider (3.10):

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &= f(\mathbf{v}), && \forall \mathbf{v} \in \mathbf{V}^{\mathrm{div}} \\ \Longleftrightarrow \int_{\boldsymbol{\Omega}} \nabla \mathbf{u} : \nabla \mathbf{v} \,\mathrm{d}\mathbf{x} &= \int_{\boldsymbol{\Omega}} \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x}, && \forall \mathbf{v} \in \mathbf{V}^{\mathrm{div}}. \end{aligned}$$

When searching for the pressure $p \in L_0^2(\boldsymbol{\Omega})$ for a given $\mathbf{f} \in \mathbf{L}^2(\boldsymbol{\Omega})$, we have to solve (3.12):

$$\begin{aligned} b(\mathbf{v},p) &= f(\mathbf{v}), && \forall \mathbf{v} \in \mathbf{V}^{\mathrm{div},\perp} \\ \Longleftrightarrow -\int_{\boldsymbol{\Omega}} (\nabla \cdot \mathbf{v})\, p \,\mathrm{d}\mathbf{x} &= \int_{\boldsymbol{\Omega}} \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x}, && \forall \mathbf{v} \in \mathbf{V}^{\mathrm{div},\perp}. \end{aligned}$$

Using the Helmholtz projection of $\mathbf{f}$, (3.10) can be equivalently reformulated as

$$\int_{\boldsymbol{\Omega}} \nabla \mathbf{u} : \nabla \mathbf{v} \,\mathrm{d}\mathbf{x} = \int_{\boldsymbol{\Omega}} \mathbb{P}(\mathbf{f}) \cdot \mathbf{v} \,\mathrm{d}\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{V}^{\mathrm{div}}. \tag{5.3}$$

Remark 5.2.1

To understand why (5.3) is an equivalent reformulation of (3.10) one has to use the Helmholtz decomposition of $\mathbf{f}$. According to that, $\mathbf{f} \in \mathbf{L}^2(\boldsymbol{\Omega})$ can be written as the sum of a divergence-free vector field $\mathbf{w}$ and an irrotational vector field $\nabla\psi$. Hence we can replace $\mathbf{f}$ in (3.10) by its decomposition $\mathbf{f} = \mathbf{w} + \nabla\psi$ and use Remark 5.1.1(3) and Definition 5.1.1 to get

$$\begin{aligned} \int_{\boldsymbol{\Omega}} \nabla \mathbf{u} : \nabla \mathbf{v} \,\mathrm{d}\mathbf{x} &= \int_{\boldsymbol{\Omega}} (\mathbf{w} + \nabla\psi) \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} = \int_{\boldsymbol{\Omega}} \mathbf{w} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} + \underbrace{\int_{\boldsymbol{\Omega}} \nabla\psi \cdot \mathbf{v} \,\mathrm{d}\mathbf{x}}_{=0} \\ &= \int_{\boldsymbol{\Omega}} \mathbf{w} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} \overset{Def. 5.1.1}{=} \int_{\boldsymbol{\Omega}} \mathbb{P}(\mathbf{f}) \cdot \mathbf{v} \,\mathrm{d}\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{V}^{\mathrm{div}}. \end{aligned}$$

An interesting aspect to study is the impact of changing the right-hand side $\mathbf{f}$ on the solution. We will analyze the effect of adding an irrotational vector field $\nabla\psi$ to the right-hand side on the solution $(\mathbf{u}, p)$.

Theorem 5.2.1 (Invariance Property) *Let* $\mathbf{f} \in \mathbf{L}^2(\boldsymbol{\Omega})$ *and* $(\mathbf{u}, p)$ *be the solution of the Stokes problem* (3.8). *Then an irrotational translation of the* *force field is totally balanced by the gradient of the pressure, i.e., it has no influence on the velocity solution.*

$$\mathbf{f} \mapsto \mathbf{f} + \nabla\psi \Longrightarrow \mathbf{u} \mapsto \mathbf{u}. \tag{5.4}$$

Proof: We will show the implication

$$\mathbf{f} \mapsto \mathbf{f} + \nabla\psi \Longrightarrow (\mathbf{u}, p) \mapsto (\mathbf{u}, p + \psi).$$

1. As already described in Remark 5.2.1 it holds

   $$\int_{\mathbf{\Omega}} (\mathbf{f} + \nabla\psi) \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} = \int_{\mathbf{\Omega}} \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{V}^{\mathrm{div}}.$$

   Using this equality we obtain

   $$\int_{\mathbf{\Omega}} \nabla\mathbf{u} : \nabla\mathbf{v} \,\mathrm{d}\mathbf{x} = \int_{\mathbf{\Omega}} \mathbb{P}(\mathbf{f} + \nabla\psi) \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} = \int_{\mathbf{\Omega}} \mathbb{P}(\mathbf{f}) \cdot \mathbf{v} \,\mathrm{d}\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{V}^{\mathrm{div}}.$$

   The homogeneous Dirichlet boundary condition of the test function $\mathbf{v}$ was used to show $\mathcal{P}(\nabla\psi) = \mathbf{0}$.
   Hence, the solution $\mathbf{u}$ of the problem (3.10) does not change for $\mathbf{f} \mapsto \mathbf{f} + \nabla\psi$.
2. For the pressure approximation we have to solve (3.12):

   $$b(\mathbf{v}, p) = f(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}^{\mathrm{div},\perp}.$$

   For $\mathbf{f} \mapsto \mathbf{f} + \nabla\psi$ we get with integration by parts

   $$\begin{aligned} -\int_{\mathbf{\Omega}} (\nabla \cdot \mathbf{v}) p \,\mathrm{d}\mathbf{x} &= \int_{\mathbf{\Omega}} (\mathbf{f} + \nabla\psi) \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} \overset{(Ibp)}{=} \int_{\mathbf{\Omega}} \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} \\ &\quad - \int_{\mathbf{\Omega}} \psi \,(\nabla \cdot \mathbf{v}) \,\mathrm{d}\mathbf{x} + \int_{\partial\mathbf{\Omega}} \underbrace{(\mathbf{v} \cdot \mathbf{n})}_{=0 \text{ on } \partial\mathbf{\Omega}} \psi \,\mathrm{d}\mathbf{s} \\ &= \int_{\mathbf{\Omega}} \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} - \int_{\mathbf{\Omega}} \psi \,(\nabla \cdot \mathbf{v}) \,\mathrm{d}\mathbf{x}, \ \forall \mathbf{v} \in \mathbf{V}^{\mathrm{div},\perp}. \end{aligned}$$

   This implies that with this translation, p changes to $p + \psi$.

□

To summarize, we have seen that the weak Stokes problem has the property that the translation of the right-hand side $\mathbf{f} \mapsto \mathbf{f} + \nabla\psi$ only leads to a translation of the pressure in the solution of the form $(\mathbf{u}, p) \mapsto (\mathbf{u}, p + \psi)$. This can be interpreted as the pressure gradient completely balancing an additional irrotational force field.

5.2.2 The Discretized Problem

For the discretized system, this invariance property (5.4) is not necessarily satisfied. In the following, we will analyze the Crouzeix–Raviart element and come to the conclusion that the discrete velocity varies under irrotational translations of the force $\mathbf{f}$.

Theorem 5.2.2 (Variance Property) *Let $(\mathbf{u}_h, p_h)$ be the solution of the discretized Stokes problem* (4.1) *for the Crouzeix–Raviart element. Then an irrotational translation of the force field affects the discrete velocity solution $\mathbf{u}_h$.*

Proof: The discrete version of (3.10) for the Crouzeix–Raviart element reads as:

Find $\mathbf{u}_h \in \mathbf{V}_h^{\mathrm{div,CR}}$ such that

$$\sum_{\mathbf{T}\in\mathcal{T}_h} \int_{\mathbf{T}} \nabla\mathbf{u}_h : \nabla\mathbf{v}_h \,\mathrm{d}\mathbf{x} = \sum_{\mathbf{T}\in\mathcal{T}_h} \int_{\mathbf{T}} \mathbf{f}\cdot\mathbf{v}_h \,\mathrm{d}\mathbf{x}, \qquad \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathrm{div,CR}}.$$

For $\mathbf{f} \mapsto \mathbf{f} + \nabla\psi$ we obtain

$$\begin{aligned}\sum_{\mathbf{T}\in\mathcal{T}_h} \int_{\mathbf{T}} \nabla\mathbf{u}_h : \nabla\mathbf{v}_h \,\mathrm{d}\mathbf{x} &= \sum_{\mathbf{T}\in\mathcal{T}_h} \int_{\mathbf{T}} (\mathbf{f}+\nabla\psi)\cdot\mathbf{v}_h \,\mathrm{d}\mathbf{x} \\ &= \sum_{\mathbf{T}\in\mathcal{T}_h} \int_{\mathbf{T}} \mathbf{f}\cdot\mathbf{v}_h \,\mathrm{d}\mathbf{x} + \sum_{\mathbf{T}\in\mathcal{T}_h} \int_{\mathbf{T}} \nabla\psi\cdot\mathbf{v}_h \,\mathrm{d}\mathbf{x},\end{aligned}$$

$\forall \mathbf{v}_h \in \mathbf{V}_h^{\mathrm{div,CR}}$.

Of course, the properties the continuous setting possesses are desired to be satisfied by the discretized system, too. Hence, one would like to have the invariance property as in the continuous setting, i.e., the second summand $\sum_{\mathbf{T}\in\mathcal{T}_h} \int_{\mathbf{T}} \nabla\psi\cdot\mathbf{v}_h \,\mathrm{d}\mathbf{x}$ should vanish.

Unfortunately, this is not the case. Discretely divergence-free vector fields and irrotational vector fields are in general not orthogonal in the L^2-scalar product. Applying the Gaussian theorem to the last summand yields

$$\sum_{\mathbf{T}\in\mathcal{T}_h}\int_{\mathbf{T}}\nabla\psi\cdot\mathbf{v}_h\,\mathrm{d}\mathbf{x} \overset{Lem.2.1.1(ii)}{=} \underbrace{-\sum_{\mathbf{T}\in\mathcal{T}_h}\int_{\mathbf{T}}\psi\nabla\cdot\mathbf{v}_h\,\mathrm{d}\mathbf{x}}_{=0 \text{ for } \mathbf{v}_h\in\mathbf{V}_h^{\mathrm{div,CR}}} + \sum_{\mathbf{T}\in\mathcal{T}_h}\int_{\mathbf{T}}\nabla\cdot(\psi\mathbf{v}_h)\,\mathrm{d}\mathbf{x}$$

$$= \sum_{\mathbf{T}\in\mathcal{T}_h}\int_{\mathbf{T}}\nabla\cdot(\psi\mathbf{v}_h)\,\mathrm{d}\mathbf{x}$$

$$= \sum_{\mathbf{F}\in\mathcal{F}_h}\int_{\mathbf{F}}[\![\psi\mathbf{v}_h\cdot\mathbf{n}_{\mathbf{F}}]\!]_{\mathbf{F}}\,\mathrm{d}\mathbf{s} + \sum_{\mathbf{F}\in\mathcal{F}_h^*\setminus\mathcal{F}_h}\int_{\mathbf{F}}\psi\mathbf{v}_h\cdot\mathbf{n}_{\mathbf{F}}\,\mathrm{d}\mathbf{s}.$$

The term $-\sum_{\mathbf{T}\in\mathcal{T}_h}\int_{\mathbf{T}}\psi\nabla\cdot\mathbf{v}_h\,\mathrm{d}\mathbf{x}$ is zero since by Lemma 4.3.3 it is $\mathbf{v}_h\in\mathbf{V}_h^{\mathrm{div,CR}}$ if and only if $\nabla_h\cdot\mathbf{v}_h=0$.

The two resulting terms will usually not vanish. As a matter of fact, the definition of the Crouzeix–Raviart element only assures that the jumps, respectively values, in the barycenters of inner faces, respectively boundary faces, are zero. Unfortunately, the vanishing of the jumps of the normal components on the whole faces is required in order to satisfy the invariance property. For details see [Kod14]. □

Thus, the separation of irrotational and divergence-free forces in the discrete case for the Crouzeix–Raviart element is disproved. This dependence can be seen in Theorem 4.3.2, where the *a priori* error estimate for the velocity depends on the pressure p.

Remark 5.2.2

A special case points out the dramatic consequences this variance property may cause.

Assume the given force field is zero, i.e., $\mathbf{f}=\mathbf{0}$. Then $(\mathbf{u},p)=(\mathbf{0},0)$ is the solution of the corresponding Stokes problem with homogeneous Dirichlet boundary conditions. An irrotational translation of the form $\mathbf{f}\mapsto\mathbf{f}+\nabla\psi$, i.e., $\mathbf{0}\mapsto\mathbf{0}+\nabla\psi$ yields the following change of the solution:

$$(\mathbf{0},0)\mapsto(\mathbf{0},\psi),$$

see Theorem 5.2.1. Thus, the velocity solution for an irrotational forcing equals the velocity solution for $\mathbf{f}=\mathbf{0}$. Using the finite element method based on the Crouzeix–Raviart element, things look different. This was analyzed in Theorem 5.2.2. The natural expectation is that the bigger the irrotational part in the force field is, the bigger the velocity error will be.

5.2.3 The Raviart–Thomas Projection

In this section we want to present a modification which eliminates the just explained dependency. This can be achieved via the Raviart–Thomas element:

Definition 5.2.1 (Raviart–Thomas element) *The lowest order Raviart–Thomas element is defined by*

$$\mathbf{V}_h^{\mathrm{RT}} := \Big\{ \mathbf{v}_h \in \mathbf{L}^2(\mathbf{\Omega}) : \mathbf{v}_h|_{\mathbf{T}}(\mathbf{x}) = \mathbf{a}_{\mathbf{T}} + \frac{b_{\mathbf{T}}}{n}(\mathbf{x} - \mathbf{x}_{\mathbf{T}}), \mathbf{T} \in \mathcal{T}_h, \mathbf{a}_{\mathbf{T}} \in \mathbb{R}^n, b_{\mathbf{T}} \in \mathbb{R},$$
$$[\![\mathbf{v}_h \cdot \mathbf{n}_{\mathbf{F}}]\!]_{\mathbf{F}}(\mathbf{x}_{\mathbf{F}}) = 0, \ \forall \mathbf{F} \in \mathcal{F}_h,$$
$$\mathbf{v}_h(\mathbf{x}_{\mathbf{F}}) \cdot \mathbf{n}_{\mathbf{F}} = 0, \ \forall \mathbf{F} \in \mathcal{F}_h^* \setminus \mathcal{F}_h \Big\}.$$

The degrees of freedom for $\mathbf{V}_h^{\mathrm{RT}}$ can be seen in Figure 5.1.

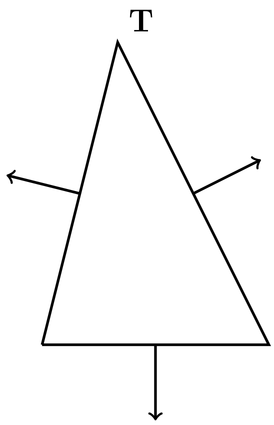


Figure 5.1: An element of the triangulation $\mathcal{T}_h$ with the local degrees of freedom represented as normal vectors in the face barycenters for the velocity, for the Raviart–Thomas space $\mathbf{V}_h^{\mathrm{RT}}$.

Remark 5.2.3

For $\mathbf{v}_h \in \mathbf{V}_h^{\mathrm{RT}}$ it is $\mathbf{v}_h|_{\mathbf{T}} \in \mathbf{H}(\mathrm{div}, \mathbf{T})$. In order to be in $\mathbf{H}(\mathrm{div}, \mathbf{\Omega})$ by Lemma 4.3.2, it suffices that the normal components $\mathbf{v}_h \cdot \mathbf{n}_{\mathbf{F}}$ are continuous at the transition from one cell to another cell. The normal component $\mathbf{v}_h(\mathbf{x}) \cdot \mathbf{n}_{\mathbf{F}}$ of a Raviart–Thomas field $\mathbf{v}_h \in \mathbf{V}_h^{\mathrm{RT}}$ is constant along each edge $\mathbf{F} \in \mathcal{F}_{\mathbf{T}}$. This can be seen in Figure 5.2 by considering the decomposition of $\mathbf{x} - \mathbf{x}_{\mathbf{T}}$ into its normal part $(\mathbf{x} - \mathbf{x}_{\mathbf{T}})_{\mathrm{normal}}$ and its tangential part $(\mathbf{x} - \mathbf{x}_{\mathbf{T}})_{\mathrm{tangential}}$. Using the fact that $(\mathbf{x} - \mathbf{x}_{\mathbf{T}})_{\mathrm{tangential}}$

and $\mathbf{n_F}$ are orthogonal and that $(\mathbf{x}-\mathbf{x_T})_{\text{normal}}$ is the same for all $\mathbf{x} \in \mathbf{F}$, yields

$$\begin{aligned}\mathbf{v}_h|_{\mathbf{T}}(\mathbf{x}) \cdot \mathbf{n_F} &= \underbrace{\mathbf{a_T} \cdot \mathbf{n_F}}_{=\text{const.}} + \underbrace{\frac{b_{\mathbf{T}}}{n} (\mathbf{x} - \mathbf{x_T}) \cdot \mathbf{n_F}}_{=\text{const.}} \\ &= \text{const.}, \qquad \forall \mathbf{x} \in \mathbf{F}.\end{aligned}$$

This implies that

$$[\![\mathbf{v}_h \cdot \mathbf{n_F}]\!]_{\mathbf{F}}(\mathbf{x_F}) = 0 \Longrightarrow [\![\mathbf{v}_h \cdot \mathbf{n_F}]\!]_{\mathbf{F}}(\mathbf{x}) = 0, \qquad \forall \mathbf{x} \in \mathbf{F}.$$

Therefore the normal components of a Raviart–Thomas field $\mathbf{v}_h \in \mathbf{V}_h^{\mathrm{RT}}$ are continuous on the edges $\mathbf{F} \in \mathcal{F}_h$ which implies by Lemma 4.3.2

$$\mathbf{V}_h^{\mathrm{RT}} \subset \mathbf{H}(\mathrm{div}, \mathbf{\Omega}).$$

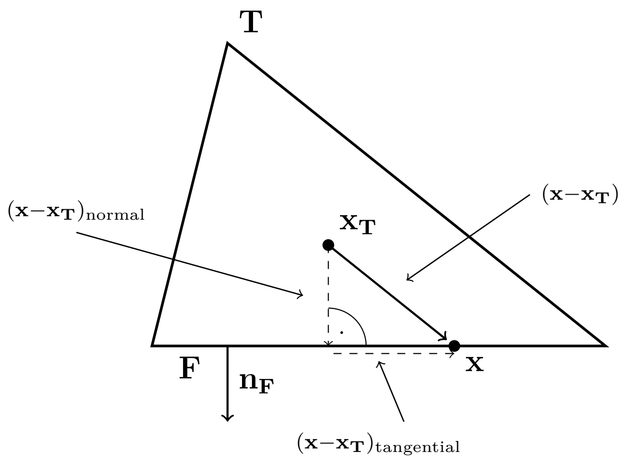


Figure 5.2: Visualization of the face-constant normal components of Raviart–Thomas functions.

Definition 5.2.2 (Raviart–Thomas interpolation operator) *The map* $\boldsymbol{\pi}_h^{\mathbf{RT}} : \mathbf{V} \cup \mathbf{V}_h^{\mathbf{CR}} \to \mathbf{V}_h^{\mathrm{RT}}$ *is defined by*

$$\left(\boldsymbol{\pi}_h^{\mathbf{RT}} \mathbf{v}\right)(\mathbf{x_F}) \cdot \mathbf{n_F} := \begin{cases} \frac{1}{|\mathbf{F}|} \int\limits_{\mathbf{F}} \mathbf{v} \cdot \mathbf{n_F} \, \mathrm{d}\mathbf{s}, & \mathbf{F} \in \mathcal{F}_h, \\ 0, & \mathbf{F} \in \mathcal{F}_h^* \setminus \mathcal{F}_h. \end{cases}$$

Remark 5.2.4

This interpolation operator is defined such that the normal component of the resulting field on a face $\mathbf{F}$ equals the mean value of the normal component of the original vector field over $\mathbf{F}$.

One has to determine the Raviart–Thomas reconstruction of a Crouzeix–Raviart vector field $\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{CR}}$ on each triangle $\mathbf{T} \in \mathcal{T}_h$, i.e., one seeks for $\mathbf{a_T}$ and $b_\mathbf{T}$. Thereby, the reconstruction shall have
(i) constant normal components on the faces:

$$\left(\boldsymbol{\pi}_h^{\mathbf{RT}} \mathbf{v}_h\right)|_\mathbf{F} \cdot \mathbf{n_F} = \mathbf{v}_h(\mathbf{x_F}) \cdot \mathbf{n_F}$$

and
(ii) the elementwise divergence has to be preserved:

$$\left(\nabla \cdot \boldsymbol{\pi}_h^{\mathbf{RT}} \mathbf{v}_h\right)|_\mathbf{T} = (\nabla_h \cdot \mathbf{v}_h)\,|_\mathbf{T}.$$

It suffices to restrict to the basis functions $\{\boldsymbol{\phi}_i(\mathbf{x})\}$ of $\mathbf{V}_h^{\mathrm{CR}}$. So we try to find the coefficients of

$$\boldsymbol{\pi}_h^{\mathbf{RT}} \boldsymbol{\phi}_i(\mathbf{x}) = \mathbf{a_T} + \frac{b_\mathbf{T}}{n}(\mathbf{x} - \mathbf{x_T}). \tag{5.5}$$

The divergence of the reconstruction on an element $\mathbf{T} \in \mathcal{T}_h$ is

$$\nabla \cdot \boldsymbol{\pi}_h^{\mathbf{RT}} \boldsymbol{\phi}_i(\mathbf{x}) = \frac{b_\mathbf{T}}{n} \sum_{i=1}^{n} \frac{\partial x_i}{\partial x_i} = b_\mathbf{T}. \tag{5.6}$$

The ambition to create a reconstruction which is elementwise divergence preserving yields

$$\nabla \cdot \boldsymbol{\phi}_i(\mathbf{x}) \overset{(ii)}{=} \nabla \cdot \boldsymbol{\pi}_h^{\mathbf{RT}} \boldsymbol{\phi}_i(\mathbf{x}) \overset{(5.6)}{=} b_\mathbf{T}.$$

Each basis function $\boldsymbol{\phi}_i(\mathbf{x})$ is elementwise linear, hence $\nabla \cdot \boldsymbol{\phi}_i(\mathbf{x})$ is elementwise constant. The Gaussian theorem and the midpoint rule yield

$$\begin{aligned} \nabla \cdot \boldsymbol{\phi}_i(\mathbf{x}) &= \frac{1}{|\mathbf{T}|} \int_\mathbf{T} \nabla \cdot \boldsymbol{\phi}_i(\mathbf{x})\,\mathrm{d}\mathbf{x} = \frac{1}{|\mathbf{T}|} \sum_{\mathbf{F} \in \mathcal{F}_\mathbf{T}} \int_\mathbf{F} \boldsymbol{\phi}_i(\mathbf{s}) \cdot \mathbf{n}_{\mathbf{T},\mathbf{F}}\,\mathrm{d}\mathbf{s} \\ &= \frac{1}{|\mathbf{T}|} \sum_{\mathbf{F} \in \mathcal{F}_\mathbf{T}} |\mathbf{F}| \boldsymbol{\phi}_i(\mathbf{x_F}) \cdot \mathbf{n}_{\mathbf{T},\mathbf{F}}. \end{aligned} \tag{5.7}$$

It remains to determine $\mathbf{a_T}$. Therefore we take advantage of the characterization of the barycenter by an integral. For the barycenter $\mathbf{x_T}$ of a mesh cell $\mathbf{T}$, using the midpoint rule, it holds

$$|\mathbf{T}|\mathbf{x_T} = \int_{\mathbf{T}} \mathbf{x}\,\mathrm{d}\mathbf{x},\ \forall \mathbf{T} \in \mathcal{T}_h \Longleftrightarrow \mathbf{x_T} = \frac{1}{|\mathbf{T}|}\int_{\mathbf{T}} \mathbf{x}\,\mathrm{d}\mathbf{x},\ \forall \mathbf{T} \in \mathcal{T}_h. \tag{5.8}$$

Utilizing (5.8) and (5.5) yields

$$\begin{aligned}
\frac{1}{|\mathbf{T}|}\int_{\mathbf{T}} \boldsymbol{\pi}_h^{\mathbf{RT}}\boldsymbol{\phi}_i\,(\mathbf{x})\ \mathrm{d}\mathbf{x} &= \frac{1}{|\mathbf{T}|}\int_{\mathbf{T}} \mathbf{a_T} + \frac{b_{\mathbf{T}}}{n}(\mathbf{x}-\mathbf{x_T})\,\mathrm{d}\mathbf{x} \\
&= \frac{1}{|\mathbf{T}|}\left[\int_{\mathbf{T}} \underbrace{\mathbf{a_T}}_{=\text{const.}}\ \mathrm{d}\mathbf{x} + \int_{\mathbf{T}} \underbrace{\frac{b_{\mathbf{T}}}{n}}_{=\text{const.}}\ \mathbf{x}\,\mathrm{d}\mathbf{x} - \int_{\mathbf{T}} \underbrace{\frac{\mathbf{b_T}}{n}\mathbf{x_T}}_{=\text{const.}}\ \mathrm{d}\mathbf{x}\right] \\
&= \mathbf{a_T}\frac{1}{|\mathbf{T}|}\int_{\mathbf{T}}\ \mathrm{d}\mathbf{x} + \frac{b_{\mathbf{T}}}{n}\frac{1}{|\mathbf{T}|}\int_{\mathbf{T}} \mathbf{x}\,\mathrm{d}\mathbf{x} - \frac{b_{\mathbf{T}}}{n}\mathbf{x_T}\frac{1}{|\mathbf{T}|}\int_{\mathbf{T}}\ \mathrm{d}\mathbf{x} \\
&= \mathbf{a_T} + \frac{b_{\mathbf{T}}}{n}(\mathbf{x_T}-\mathbf{x_T}) = \mathbf{a_T}.
\end{aligned}$$

Using the product rule we obtain

$$\begin{aligned}
\nabla\cdot\left(x_j\boldsymbol{\pi}_h^{\mathbf{RT}}\boldsymbol{\phi}_i\right) &= \sum_{k=1}^{n}\frac{\partial\left(x_j\boldsymbol{\pi}_h^{\mathbf{RT}}\boldsymbol{\phi}_i\right)_k}{\partial x_k} = \sum_{k=1}^{n}\frac{\partial\left(x_j\left(\boldsymbol{\pi}_h^{\mathbf{RT}}\boldsymbol{\phi}_i\right)_k\right)}{\partial x_k} \\
&= \sum_{k=1}^{n}\frac{\partial(x_j)}{\partial x_k}\left(\boldsymbol{\pi}_h^{\mathbf{RT}}\boldsymbol{\phi}_i\right)_k + \sum_{k=1}^{n} x_j\frac{\partial(\boldsymbol{\pi}_h^{\mathbf{RT}}\boldsymbol{\phi}_i)_k}{\partial x_k} \\
&= \left(\boldsymbol{\pi}_h^{\mathbf{RT}}\boldsymbol{\phi}_i\right)_j + x_j\nabla\cdot\boldsymbol{\pi}_h^{\mathbf{RT}}\boldsymbol{\phi}_i.
\end{aligned}$$

This equality is now used in the componentwise representation of $\mathbf{a_T}$:

$$(\mathbf{a_T})_j = \frac{1}{|\mathbf{T}|}\int_{\mathbf{T}}\left(\boldsymbol{\pi}_h^{\mathbf{RT}}\boldsymbol{\phi}_i\right)_j = \frac{1}{|\mathbf{T}|}\int_{\mathbf{T}}\nabla\cdot\left(x_j\boldsymbol{\pi}_h^{\mathbf{RT}}\boldsymbol{\phi}_i\right) - \frac{1}{|\mathbf{T}|}\int_{\mathbf{T}} x_j\nabla\cdot\boldsymbol{\pi}_h^{\mathbf{RT}}\boldsymbol{\phi}_i.$$

Applying the Gaussian theorem, (i), (ii), and the midpoint rule yields

$$\begin{aligned}
(\mathbf{a_T})_j &= \frac{1}{|\mathbf{T}|}\sum_{\mathbf{F}\in\mathcal{F}_{\mathbf{T}}}\int_{\mathbf{F}} x_j\,\underbrace{\boldsymbol{\pi}_h^{\mathbf{RT}}\boldsymbol{\phi}_i\cdot\mathbf{n}_{\mathbf{T},\mathbf{F}}}_{\text{const.}} - \frac{1}{|\mathbf{T}|}\int_{\mathbf{T}} x_j\,\underbrace{\nabla\cdot\boldsymbol{\pi}_h^{\mathbf{RT}}\boldsymbol{\phi}_i}_{\text{const.}} \\
&\overset{(i),(ii)}{=} \frac{1}{|\mathbf{T}|}\sum_{\mathbf{F}\in\mathcal{F}_{\mathbf{T}}}\boldsymbol{\phi}_i(\mathbf{x_F})\cdot\mathbf{n}_{\mathbf{T},\mathbf{F}}\int_{\mathbf{F}} x_j - \frac{1}{|\mathbf{T}|}\nabla\cdot\boldsymbol{\phi}_i\int_{\mathbf{T}} x_j \\
&= \frac{1}{|\mathbf{T}|}\sum_{\mathbf{F}\in\mathcal{F}_{\mathbf{T}}}\boldsymbol{\phi}_i(\mathbf{x_F})\cdot\mathbf{n}_{\mathbf{T},\mathbf{F}}|\mathbf{F}|(x_{\mathbf{F}})_j - (x_{\mathbf{T}})_j\,\nabla\cdot\boldsymbol{\phi}_i.
\end{aligned}$$

Finally, for the second summand we use (5.7) to get

$$\begin{aligned}
\mathbf{a_T} &= \frac{1}{|\mathbf{T}|} \sum_{\mathbf{F}\in\mathcal{F}_\mathbf{T}} \boldsymbol{\phi}_i(\mathbf{x_F}) \cdot \mathbf{n_{T,F}} |\mathbf{F}| \mathbf{x_F} - \mathbf{x_T} \frac{1}{|\mathbf{T}|} \sum_{\mathbf{F}\in\mathcal{F}_\mathbf{T}} |\mathbf{F}| \boldsymbol{\phi}_i(\mathbf{x_F}) \cdot \mathbf{n_{T,F}} \\
&= \frac{1}{|\mathbf{T}|} \sum_{\mathbf{F}\in\mathcal{F}_\mathbf{T}} |\mathbf{F}| \boldsymbol{\phi}_i(\mathbf{x_F}) \cdot \mathbf{n_{T,F}} \left(\mathbf{x_F} - \mathbf{x_T}\right).
\end{aligned}$$

So the conditions (i) and (ii) uniquely determine the coefficients $\mathbf{a_T}$ and $b_\mathbf{T}$ for a mesh cell $\mathbf{T} \in \mathcal{T}_h$ as

$$\begin{aligned}
\mathbf{a_T} &= \frac{1}{|\mathbf{T}|} \sum_{\mathbf{F}\in\mathcal{F}_\mathbf{T}} |\mathbf{F}| \boldsymbol{\phi}_i(\mathbf{x_F}) \cdot \mathbf{n_{T,F}} \left(\mathbf{x_F} - \mathbf{x_T}\right), \\
b_\mathbf{T} &= \frac{1}{|\mathbf{T}|} \sum_{\mathbf{F}\in\mathcal{F}_\mathbf{T}} |\mathbf{F}| \boldsymbol{\phi}_i(\mathbf{x_F}) \cdot \mathbf{n_{T,F}}.
\end{aligned}$$

Indeed, the normal components of the reconstruction $\boldsymbol{\pi}_h^{\mathbf{RT}} \mathbf{v}_h$ are constant on each face leading to vanishing jumps. Therefore $\boldsymbol{\pi}_h^{\mathbf{RT}} \mathbf{v}_h \in \mathbf{H}(\mathrm{div}, \boldsymbol{\Omega})$.

The way we have chosen the coefficients $\mathbf{a_T}$ and $b_\mathbf{T}$ guarantees that a discretely divergence-free field $\mathbf{v}_h \in \mathbf{V}_h^{\mathrm{div,CR}} \subset \mathbf{V}_h^{\mathrm{CR}}$ is mapped by the operator $\boldsymbol{\pi}_h^{\mathbf{RT}}$ to a weakly divergence-free field.

The modified CR finite element formulation:

We introduce the bilinear forms $a_h : \mathbf{V}_h^{\mathrm{CR}} \times \mathbf{V}_h^{\mathrm{CR}} \to \mathbb{R}$ and $b_h : \mathbf{V}_h^{\mathrm{CR}} \times Q_h^{\mathrm{CR}} \to \mathbb{R}$ with

$$\begin{aligned}
a_h(\mathbf{u}_h, \mathbf{v}_h) &:= \int_{\boldsymbol{\Omega}} \nabla_h \mathbf{u}_h : \nabla_h \mathbf{v}_h \, \mathrm{d}\mathbf{x}, \\
b_h(\mathbf{u}_h, q_h) &:= \int_{\boldsymbol{\Omega}} q_h \left(\nabla_h \cdot \mathbf{u}_h\right) \, \mathrm{d}\mathbf{x},
\end{aligned}$$

and the linear form $\tilde{f}_h : \mathbf{V}_h^{\mathrm{CR}} \to \mathbb{R}$ with

$$\tilde{f}_h(\mathbf{v}_h) := \int_{\boldsymbol{\Omega}} \mathbf{f} \cdot \pi_h^{\mathbf{RT}} \mathbf{v}_h \, \mathrm{d}\mathbf{x}.$$

The resulting modified discrete Stokes problem is:

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h^{\mathrm{CR}} \times Q_h^{\mathrm{CR}}$ such that

$$\begin{aligned}
a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) &= \tilde{f}_h(\mathbf{v}_h), \qquad \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathrm{CR}}, \\
b_h(\mathbf{u}_h, q_h) &= 0, \qquad\qquad \forall q_h \in Q_h^{\mathrm{CR}}.
\end{aligned} \tag{5.9}$$

Theorem 5.2.3 *Let* $(\mathbf{u}_h, p_h)$ *be the solution of the modified Stokes problem* (5.9)*. Then an irrotational translation of the force field* $\mathbf{f}$ *does not change the discrete velocity* $\mathbf{u}_h$*.*

Proof: For $\mathbf{v}_h \in \mathbf{V}_h^{\mathrm{div,CR}}$ in (5.9) one obtains

$$\begin{aligned}\int_{\mathbf{\Omega}} \mathbf{f} \cdot \boldsymbol{\pi}_h^{\mathbf{RT}} \mathbf{v}_h \,\mathrm{d}\mathbf{x} &= \int_{\mathbf{\Omega}} (\mathbf{w} + \nabla\psi) \cdot \boldsymbol{\pi}_h^{\mathbf{RT}} \mathbf{v}_h \,\mathrm{d}\mathbf{x} \\ &= \int_{\mathbf{\Omega}} \mathbf{w} \cdot \boldsymbol{\pi}_h^{\mathbf{RT}} \mathbf{v}_h \,\mathrm{d}\mathbf{x} + \underbrace{\int_{\mathbf{\Omega}} \nabla\psi \cdot \boldsymbol{\pi}_h^{\mathbf{RT}} \mathbf{v}_h \,\mathrm{d}\mathbf{x}}_{=0} \\ &= \int_{\mathbf{\Omega}} \mathbf{w} \cdot \boldsymbol{\pi}_h^{\mathbf{RT}} \mathbf{v}_h \,\mathrm{d}\mathbf{x} = \int_{\mathbf{\Omega}} \mathbb{P}(\mathbf{f}) \cdot \boldsymbol{\pi}_h^{\mathbf{RT}} \mathbf{v}_h \,\mathrm{d}\mathbf{x}.\end{aligned}$$

The term $\int_{\mathbf{\Omega}} \nabla\psi \cdot \boldsymbol{\pi}_h^{\mathbf{RT}} \mathbf{v}_h \,\mathrm{d}\mathbf{x}$ vanishes, since $\boldsymbol{\pi}_h^{\mathbf{RT}}$, maps discretely divergence-free vector fields onto weakly divergence-free vector fields yielding $\mathbf{L}^2$-orthogonality. We conclude that for $\mathbf{f} \mapsto \mathbf{f} + \nabla\psi$ the modified Stokes problem yields $\mathbf{u}_h \mapsto \mathbf{u}_h$. □

Consequently, the modified method (5.9) inherits the desired invariance property $\mathbf{f} \mapsto \mathbf{f} + \nabla\psi \Rightarrow \mathbf{u}_h \mapsto \mathbf{u}_h$. Note that the modification affects the right-hand side only.

Now, it would be great to prove the independence of the velocity error estimate from the pressure, for the modified scheme.

Theorem 5.2.4 (Error estimates for the modified CR method) *Let* $(\mathbf{u}, p)$ *be the solution of the continuous Stokes problem* (3.8) *and assume that* $(\mathbf{u}, p) \in \mathbf{H}^2(\mathbf{\Omega}) \times H^1(\mathbf{\Omega})$*. Then for the solution* $(\mathbf{u}_h, p_h)$ *of the modified discrete problem* (5.9) *the following error estimates hold:*

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_h\|_{1,h} &\leq Ch|\mathbf{u}|_{\mathbf{H}^2(\mathbf{\Omega})}, \\ \|p - p_h\|_{L^2(\mathbf{\Omega})} &\leq Ch\left(|\mathbf{u}|_{\mathbf{H}^2(\mathbf{\Omega})} + |p|_{H^1(\mathbf{\Omega})}\right).\end{aligned}$$

Proof: The proof can be found in [Linke14], pp. 790. □

Remark 5.2.5

Applying the presented Raviart–Thomas projection to the discrete velocity solution obtained by the usual Crouzeix–Raviart element yields a divergence-free approximation.

6 A Divergence-Free Post-Processing for a Pressure-Stabilized Formulation

In this chapter an approach from [BaVa11] shall be presented. The lack of inf-sup stability, e.g., for $\mathbf{P}_1/P_0$, requires some stabilization. The idea is to add pressure-stabilizing terms in order to be able to assure the uniqueness of the pressure and therefore to circumvent this instability.
We will discuss a post-processing for a stabilized $\mathbf{P}_1/P_0$ discretization leading to a divergence-free velocity field located in $\mathbf{P}_1 + \mathbf{V}_h^{\mathrm{RT}}$, with $\mathbf{V}_h^{\mathrm{RT}}$ being the lowest order Raviart–Thomas finite element space.

The finite element pair $\mathbf{P}_1/P_0$ is not inf-sup stable (see Section 4.3.3), so the pressure will not be unique. In order to stabilize the pair, the discrete Stokes problem is modified accordingly by adding stabilizing terms.
The resulting problem is then well-posed but, unfortunately, the stabilizing terms cause a perturbation of the conservation of mass that has to be corrected.

In this section it is still $\mathbf{V} := \mathbf{H}_0^1(\mathbf{\Omega})$ and $Q := L_0^2(\mathbf{\Omega})$. Let us denote the underlying pair of finite element spaces by $\mathbf{V}_h/Q_h$. The velocity field $\mathbf{u}$ is approximated by a continuous function which is elementwise a linear polynomial $\mathbf{u}_h|_{\mathbf{T}} \in \mathbf{P}_1(\mathbf{T})$ with $\mathbf{u}_h|_{\partial\mathbf{\Omega}} = \mathbf{0}$ and the pressure field p is approximated by a function which is elementwise a constant polynomial $p_h|_{\mathbf{T}} \in P_0(\mathbf{T})$ with vanishing mean value in $\mathbf{\Omega}$:

$$\mathbf{V}_h := \Big\{ \mathbf{v}_h \in \mathbf{C}^0(\overline{\mathbf{\Omega}}) : \mathbf{v}_h|_{\mathbf{T}} \in \mathbf{P}_1(\mathbf{T}),\ \forall \mathbf{T} \in \mathcal{T}_h \text{ and } \mathbf{v}_h|_{\partial\mathbf{\Omega}} = \mathbf{0} \Big\}$$

and

$$Q_h := \Big\{ q_h \in L_0^2(\mathbf{\Omega}) : q_h|_{\mathbf{T}} \in P_0(\mathbf{T}),\ \forall \mathbf{T} \in \mathcal{T}_h \Big\}. \tag{6.1}$$

We have seen in Section 4.3.3 that this finite element pair is unstable.

As in the previous chapter, we utilize the notation introduced in Section 4.1. Furthermore, a mesh cell is assumed to be a simplex and $(\cdot,\cdot)_{\mathbf{D}}$ denotes the inner product in $L^2(\mathbf{D})$.

The aim is to find an approximate solution of the weak Stokes problem and by adding the two equations in (3.6) the following equivalent formulation is obtained:

Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$(\nabla \mathbf{u}, \nabla \mathbf{v})_{\mathbf{\Omega}} - (p, \nabla \cdot \mathbf{v})_{\mathbf{\Omega}} + (q, \nabla \cdot \mathbf{u})_{\mathbf{\Omega}} = (\mathbf{f}, \mathbf{v})_{\mathbf{\Omega}}, \quad \forall (\mathbf{v}, q) \in \mathbf{V} \times Q.$$

Adding two stabilizing extra terms, a least squares control and the weak form of the momentum equation tested with a special test function on each element as described in [RoStTo08], pp. 468, results in the following pressure-stabilized finite element formulation:

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned}
&(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_{\mathbf{\Omega}} - (p_h, \nabla \cdot \mathbf{v}_h)_{\mathbf{\Omega}} + (q_h, \nabla \cdot \mathbf{u}_h)_{\mathbf{\Omega}} + \mu(\nabla \cdot \mathbf{u}_h, \nabla \cdot \mathbf{v}_h)_{\mathbf{\Omega}} \\
&+ \sum_{\mathbf{F} \in \mathcal{F}_h} \nu_{\mathbf{F}}([\![p_h]\!]_{\mathbf{F}}, [\![q_h]\!]_{\mathbf{F}})_{\mathbf{F}} + \sum_{\mathbf{T} \in \mathcal{T}_h} \delta_{\mathbf{T}}(\nabla p_h, \nabla q_h)_{\mathbf{T}} - \sum_{\mathbf{T} \in \mathcal{T}_h} \delta_{\mathbf{T}}(-\Delta \mathbf{u}_h, \nabla q_h)_{\mathbf{T}} \\
&= \sum_{\mathbf{T} \in \mathcal{T}_h} (\mathbf{f}, \nabla q_h)_{\mathbf{T}} + (\mathbf{f}, \mathbf{v}_h)_{\mathbf{\Omega}}, \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h,
\end{aligned}$$

where $\mu \geq 0$, $\delta_{\mathbf{T}} > 0$, and $\nu_{\mathbf{F}} := \gamma |\mathbf{F}|$ with $\gamma > 0$.

For the previously defined finite element pair $\mathbf{V}_h / Q_h$ three terms vanish automatically:

$$\sum_{\mathbf{T} \in \mathcal{T}_h} \delta_{\mathbf{T}}(\nabla p_h, \nabla q_h)_{\mathbf{T}} = 0$$

and

$$\sum_{\mathbf{T} \in \mathcal{T}_h} (\mathbf{f}, \nabla q_h)_{\mathbf{T}} = 0,$$

since p_h, $q_h \in Q_h = P_0$, i.e., they are constant on a mesh cell which implies $(\nabla p_h)|_{\mathbf{T}} = (\nabla q_h)|_{\mathbf{T}} = \mathbf{0}$. Moreover,

$$\sum_{\mathbf{T} \in \mathcal{T}_h} \delta_{\mathbf{T}}(-\Delta \mathbf{u}_h, \nabla q_h)_{\mathbf{T}} = 0$$

since $\mathbf{u}_h \in \mathbf{V}_h = \mathbf{P}_1$ implies $(\Delta \mathbf{u}_h)|_{\mathbf{T}} = \mathbf{0}$.
This yields the

Pressure-stabilized Stokes problem for $\mathbf{P}_1/P_0$:

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_{\mathbf{\Omega}} - (p_h, \nabla \cdot \mathbf{v}_h)_{\mathbf{\Omega}} + (q_h, \nabla \cdot \mathbf{u}_h)_{\mathbf{\Omega}} + \mu(\nabla \cdot \mathbf{u}_h, \nabla \cdot \mathbf{v}_h)_{\mathbf{\Omega}} \\ + \sum_{\mathbf{F} \in \mathcal{F}_h} \nu_{\mathbf{F}} \left([\![p_h]\!]_{\mathbf{F}}, [\![q_h]\!]_{\mathbf{F}} \right)_{\mathbf{F}} = (\mathbf{f}, \mathbf{v}_h)_{\mathbf{\Omega}}, \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h, \tag{6.2}$$

where $\mu \geq 0$ and $\nu_{\mathbf{F}} := \gamma |\mathbf{F}|$ with $\gamma > 0$.

The resulting velocity field $\mathbf{u}_h$ is not locally mass conservative anymore but we will discuss a computationally easy way to post-process it accordingly.

Define the Raviart–Thomas field $\mathbf{u}_h^{\mathrm{RT}}$ by

$$\mathbf{u}_h^{\mathrm{RT}} := \sum_{\mathbf{F} \in \mathcal{F}_h} \nu_{\mathbf{F}} \frac{1}{|\mathbf{F}|} \int_{\mathbf{F}} [\![p_h]\!]_{\mathbf{F}} \, \mathrm{ds} \; \varphi_{\mathbf{F}}(\mathbf{x}) \tag{6.3}$$

with

$$\varphi_{\mathbf{F}}(\mathbf{x}) := \pm \left(\frac{|\mathbf{F}|}{2|\mathbf{T}|} \right) (\mathbf{x} - \mathbf{x}_{\mathbf{F}})$$

being the local basis function for the lowest order Raviart–Thomas finite element space $\mathbf{V}_h^{\mathrm{RT}}$ with $\mathbf{x}_{\mathbf{F}}$ denoting here the vertex opposite to the face $\mathbf{F}$, see Figure 6.1.

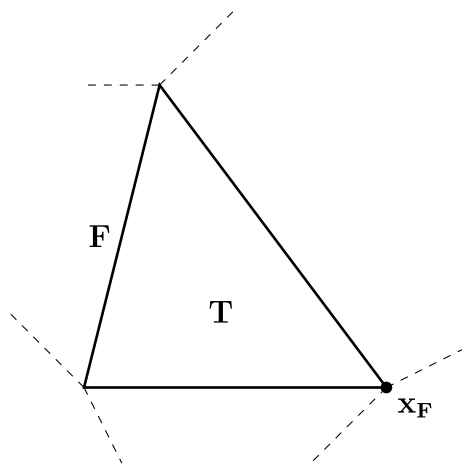


Figure 6.1: The illustration of $\mathbf{x}_{\mathbf{F}}$ in $\varphi_{\mathbf{F}}(\mathbf{x})$.

Remark 6.0.6

1. The function $\varphi_{\mathbf{F}}(\mathbf{x}) := +\left(\frac{|\mathbf{F}|}{2|\mathbf{T}|} \right) (\mathbf{x} - \mathbf{x}_{\mathbf{F}})$ if $\mathbf{n}_{\mathbf{T},\mathbf{F}}$ points outward $\mathbf{T}$, else it has a negative sign.

2. The discrete pressure p_h is in Q_h, i.e., it is constant on each mesh cell and hence the pressure jump across each inner $(n-1)$-face, $[\![p_h]\!]_{\mathbf{F}}$, is constant:
$$[\![p_h]\!]_{\mathbf{F}} = \text{const.}, \qquad \forall p_h \in Q_h,\, \mathbf{F} \in \mathcal{F}_h. \tag{6.4}$$

3. For $p_h \in Q_h$ defined by (6.1) it holds
$$\begin{aligned}\mathbf{u}_h^{\mathrm{RT}} &:= \sum_{\mathbf{F}\in\mathcal{F}_h} \nu_{\mathbf{F}} \frac{1}{|\mathbf{F}|} \int_{\mathbf{F}} [\![p_h]\!]_{\mathbf{F}}\, \mathrm{ds}\ \varphi_{\mathbf{F}}(\mathbf{x}) \\ &\overset{(6.4)}{=} \sum_{\mathbf{F}\in\mathcal{F}_h} \nu_{\mathbf{F}} [\![p_h]\!]_{\mathbf{F}} \underbrace{\frac{1}{|\mathbf{F}|} \underbrace{\int_{\mathbf{F}} 1\, \mathrm{ds}}_{=|\mathbf{F}|}}_{=1}\ \varphi_{\mathbf{F}}(\mathbf{x}) = \sum_{\mathbf{F}\in\mathcal{F}_h} \nu_{\mathbf{F}} [\![p_h]\!]_{\mathbf{F}}\ \varphi_{\mathbf{F}}(\mathbf{x}).\end{aligned} \tag{6.5}$$

Theorem 6.0.5 *Let* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *be the solution of* (6.2). *Then*
$$\tilde{\mathbf{u}}^h := \mathbf{u}_h + \mathbf{u}_h^{\mathrm{RT}}$$
fulfills
$$(\nabla \cdot \tilde{\mathbf{u}}^h)|_{\mathbf{T}} = 0,\ \ \forall \mathbf{T} \in \mathcal{T}_h.$$

Proof: Let $\mathbf{T}$ and $\mathbf{T}'$ be two mesh cells in $\mathcal{T}_h$ and let $\mathbf{T}'$ be fixed. We want to show that
$$(\nabla \cdot \tilde{\mathbf{u}}^h)|_{\mathbf{T}} = 0, \quad \forall \mathbf{T} \in \mathcal{T}_h.$$

Assume that the normals $\mathbf{n}_{\mathbf{T},\mathbf{F}}$ are fixed such that they point outwards $\mathbf{T}$. Then $\varphi_{\mathbf{F}}(\mathbf{x})$ has a positive sign.

The fact that $(\mathbf{u}_h, p_h)$ fulfills (6.2) for all test functions in $\mathbf{V}_h \times Q_h$ implies that the same holds for the specific choice $(\mathbf{0}, q_h) \in \mathbf{V}_h \times Q_h$ with q_h defined by
$$q_h := \begin{cases} 1, & \text{if } \mathbf{x} \in \mathbf{T}', \\ -\frac{|\mathbf{T}'|}{|\mathbf{T}|}, & \text{if } \mathbf{x} \in \mathbf{T}, \\ 0, & \text{else.} \end{cases}$$

The function q_h is in fact an element of Q_h since it is elementwise constant, thus in $L^2(\mathbf{\Omega})$ and
$$\int_{\mathbf{\Omega}} q_h\, \mathrm{d}\mathbf{x} = \int_{\mathbf{T}'} 1\, \mathrm{d}\mathbf{x} - \int_{\mathbf{T}} \frac{|\mathbf{T}'|}{|\mathbf{T}|} \mathrm{d}\mathbf{x} = |\mathbf{T}'| - \frac{|\mathbf{T}'|}{|\mathbf{T}|} |\mathbf{T}| = 0.$$

For $(\mathbf{0}, q_h)$, (6.2) reformulates to

$$(\nabla \cdot \mathbf{u}_h, q_h)_{\mathbf{\Omega}} + \sum_{\mathbf{F} \in \mathcal{F}_h} \nu_{\mathbf{F}} ([\![p_h]\!]_{\mathbf{F}}, [\![q_h]\!]_{\mathbf{F}})_{\mathbf{F}} = 0. \tag{6.6}$$

<u>**Claim:**</u> $\quad 0 = (\nabla \cdot \tilde{\mathbf{u}}^h, q_h)_{\mathbf{T}} + (\nabla \cdot \tilde{\mathbf{u}}^h, q_h)_{\mathbf{T}'}, \qquad \forall \mathbf{T}, \mathbf{T}' \in \mathcal{T}_h.$

We treat the different possibilities of intersections between the mesh cells in the following three cases:

Case 1: $\mathbf{T} \cap \mathbf{T}' = \emptyset$
By the definition of q_h it is

$$\begin{aligned}(\nabla \cdot \mathbf{u}_h, q_h)_{\mathbf{\Omega}} &= (\nabla \cdot \mathbf{u}_h, q_h)_{\mathbf{T}' \cup \mathbf{T}} + \underbrace{(\nabla \cdot \mathbf{u}_h, q_h)_{\mathbf{\Omega} \setminus (\mathbf{T}' \cup \mathbf{T})}}_{=0} \\ &= (\nabla \cdot \mathbf{u}_h, q_h)_{\mathbf{T}'} + (\nabla \cdot \mathbf{u}_h, q_h)_{\mathbf{T}}.\end{aligned}$$

Using (6.4) leads to

$$\begin{aligned}\sum_{\mathbf{F} \in \mathcal{F}_h} \nu_{\mathbf{F}} ([\![p_h]\!]_{\mathbf{F}}, [\![q_h]\!]_{\mathbf{F}})_{\mathbf{F}} &= \sum_{\mathbf{F} \in \mathcal{F}_h} \nu_{\mathbf{F}} \int_{\mathbf{F}} [\![p_h]\!]_{\mathbf{F}} [\![q_h]\!]_{\mathbf{F}} \, \mathrm{d}\mathbf{x} \\ &= \sum_{\mathbf{F} \in \mathcal{F}_h} \nu_{\mathbf{F}} [\![p_h]\!]_{\mathbf{F}} [\![q_h]\!]_{\mathbf{F}} \int_{\mathbf{F}} 1 \, \mathrm{d}\mathbf{x} \\ &= \sum_{\mathbf{F} \in \mathcal{F}_h} \nu_{\mathbf{F}} |\mathbf{F}| [\![p_h]\!]_{\mathbf{F}} [\![q_h]\!]_{\mathbf{F}}.\end{aligned}$$

For this setting, (6.6) is equivalent to

$$\begin{aligned}(\nabla \cdot \mathbf{u}_h, q_h)_{\mathbf{T}'} + (\nabla \cdot \mathbf{u}_h, q_h)_{\mathbf{T}} &+ \sum_{\mathbf{F} \in \partial \mathbf{T}'} \nu_{\mathbf{F}} |\mathbf{F}| [\![p_h]\!]_{\mathbf{F}} [\![q_h]\!]_{\mathbf{F}} \\ &+ \sum_{\mathbf{F} \in \partial \mathbf{T}} \nu_{\mathbf{F}} |\mathbf{F}| [\![p_h]\!]_{\mathbf{F}} [\![q_h]\!]_{\mathbf{F}} = 0.\end{aligned} \tag{6.7}$$

Let us start to investigate $\tilde{\mathbf{u}}^h$:

$$
\begin{aligned}
(\nabla \cdot \tilde{\mathbf{u}}^h, q_h)_{\mathbf{T}'} &= (\nabla \cdot \mathbf{u}_h, 1)_{\mathbf{T}'} + (\nabla \cdot \mathbf{u}_h^{\mathrm{RT}}, 1)_{\mathbf{T}'} \\
&\overset{(6.5)}{=} (\nabla \cdot \mathbf{u}_h, 1)_{\mathbf{T}'} + \left(\sum_{\mathbf{F} \in \partial \mathbf{T}'} \nu_{\mathbf{F}} [\![p_h]\!]_{\mathbf{F}} \nabla \cdot \varphi_{\mathbf{F}}(\mathbf{x}), 1 \right)_{\mathbf{T}'} \\
&\overset{(6.4)}{=} (\nabla \cdot \mathbf{u}_h, 1)_{\mathbf{T}'} + \sum_{\mathbf{F} \in \partial \mathbf{T}'} \nu_{\mathbf{F}} [\![p_h]\!]_{\mathbf{F}} \left(\nabla \cdot \varphi_{\mathbf{F}}(\mathbf{x}), 1 \right)_{\mathbf{T}'} \\
&\overset{\text{(Gaussian Thm.)}}{=} (\nabla \cdot \mathbf{u}_h, 1)_{\mathbf{T}'} + \sum_{\mathbf{F} \in \partial \mathbf{T}'} \nu_{\mathbf{F}} [\![p_h]\!]_{\mathbf{F}} \int_{\mathbf{F}} \varphi_{\mathbf{F}} \cdot \mathbf{n}_{\mathbf{F}} \, \mathrm{ds} \\
&= (\nabla \cdot \mathbf{u}_h, 1)_{\mathbf{T}'} \\
&\quad + \sum_{\mathbf{F} \in \partial \mathbf{T}'} \nu_{\mathbf{F}} [\![p_h]\!]_{\mathbf{F}} \left[\underbrace{+ \frac{|\mathbf{F}|}{2|\mathbf{T}|} \overbrace{\int_{\mathbf{F}} (\mathbf{x} - \mathbf{x}_{\mathbf{F}}) \cdot \mathbf{n}_{\mathbf{F}} \, \mathrm{ds}}^{=2|\mathbf{T}|}}_{=|\mathbf{F}|} \right] \\
&= (\nabla \cdot \mathbf{u}_h, 1)_{\mathbf{T}'} + \sum_{\mathbf{F} \in \partial \mathbf{T}'} \nu_{\mathbf{F}} [\![p_h]\!]_{\mathbf{F}} |\mathbf{F}|
\end{aligned}
$$

$$
\Longrightarrow (\nabla \cdot \mathbf{u}_h, 1)_{\mathbf{T}'} = (\nabla \cdot \tilde{\mathbf{u}}^h, q_h)_{\mathbf{T}'} - \sum_{\mathbf{F} \in \partial \mathbf{T}'} \nu_{\mathbf{F}} [\![p_h]\!]_{\mathbf{F}} |\mathbf{F}|. \tag{6.8}
$$

Almost the same calculations are done for the next term, so we omit the otherwise repeating intermediate steps, which is indicated by dots:

$$
\begin{aligned}
(\nabla \cdot \tilde{\mathbf{u}}^h, q_h)_{\mathbf{T}} &= -\frac{|\mathbf{T}'|}{|\mathbf{T}|} (\nabla \cdot \tilde{\mathbf{u}}^h, 1)_{\mathbf{T}} \\
&\overset{(6.5)}{=} -\frac{|\mathbf{T}'|}{|\mathbf{T}|} \left[(\nabla \cdot \mathbf{u}_h, 1)_{\mathbf{T}} + \left(\sum_{\mathbf{F} \in \partial \mathbf{T}} \nu_{\mathbf{F}} [\![p_h]\!]_{\mathbf{F}} \nabla \cdot \varphi_{\mathbf{F}}(\mathbf{x}), 1 \right)_{\mathbf{T}} \right] \\
&= \cdots = -\frac{|\mathbf{T}'|}{|\mathbf{T}|} (\nabla \cdot \mathbf{u}_h, 1)_{\mathbf{T}} - \frac{|\mathbf{T}'|}{|\mathbf{T}|} \sum_{\mathbf{F} \in \partial \mathbf{T}} \nu_{\mathbf{F}} [\![p_h]\!]_{\mathbf{F}} |\mathbf{F}|
\end{aligned}
$$

$$
\Longrightarrow -\frac{|\mathbf{T}'|}{|\mathbf{T}|} (\nabla \cdot \mathbf{u}_h, 1)_{\mathbf{T}} = (\nabla \cdot \tilde{\mathbf{u}}^h, q_h)_{\mathbf{T}} + \frac{|\mathbf{T}'|}{|\mathbf{T}|} \sum_{\mathbf{F} \in \partial \mathbf{T}} \nu_{\mathbf{F}} [\![p_h]\!]_{\mathbf{F}} |\mathbf{F}|. \tag{6.9}
$$

We insert (6.8) and (6.9) into (6.7) resulting in

$$
\begin{aligned}
0 =&(\nabla \cdot \mathbf{u}_h, q_h)_{\mathbf{T}'} + (\nabla \cdot \mathbf{u}_h, q_h)_{\mathbf{T}} \\
&+ \sum_{\mathbf{F}\in\partial\mathbf{T}'} \nu_{\mathbf{F}}|\mathbf{F}|[\![p_h]\!]_{\mathbf{F}}[\![q_h]\!]_{\mathbf{F}} + \sum_{\mathbf{F}\in\partial\mathbf{T}} \nu_{\mathbf{F}}|\mathbf{F}|[\![p_h]\!]_{\mathbf{F}}[\![q_h]\!]_{\mathbf{F}} \\
\overset{(6.8),(6.9)}{=}& (\nabla \cdot \tilde{\mathbf{u}}^h, q_h)_{\mathbf{T}'} - \cancel{\sum_{\mathbf{F}\in\partial\mathbf{T}'} \nu_{\mathbf{F}}[\![p_h]\!]_{\mathbf{F}}|\mathbf{F}|} \\
&+ (\nabla \cdot \tilde{\mathbf{u}}^h, q_h)_{\mathbf{T}} + \cancel{\frac{|\mathbf{T}'|}{|\mathbf{T}|}\sum_{\mathbf{F}\in\partial\mathbf{T}} \nu_{\mathbf{F}}[\![p_h]\!]_{\mathbf{F}}|\mathbf{F}|} \\
&+ \cancel{\sum_{\mathbf{F}\in\partial\mathbf{T}'} \nu_{\mathbf{F}}|\mathbf{F}|[\![p_h]\!]_{\mathbf{F}} \cdot 1} + \cancel{\sum_{\mathbf{F}\in\partial\mathbf{T}} \nu_{\mathbf{F}}|\mathbf{F}|[\![p_h]\!]_{\mathbf{F}} \cdot \left(-\frac{|\mathbf{T}'|}{|\mathbf{T}|}\right)}
\end{aligned}
$$

$$
\iff 0 = (\nabla \cdot \tilde{\mathbf{u}}^h, q_h)_{\mathbf{T}} + (\nabla \cdot \tilde{\mathbf{u}}^h, q_h)_{\mathbf{T}'}.
$$

Case 2: $\mathbf{T} = \mathbf{T}'$
It holds

$$
0 = (\nabla \cdot \tilde{\mathbf{u}}^h, q_h)_{\mathbf{T}} - (\nabla \cdot \tilde{\mathbf{u}}^h, q_h)_{\mathbf{T}'}.
$$

Case 3: $\mathbf{T} \cap \mathbf{T}' = \mathbf{E}$, $\mathbf{E} \in \mathcal{F}_h$

In this case, equation (6.6) is equivalent to

$$
\begin{aligned}
&(\nabla \cdot \mathbf{u}_h, q_h)_{\mathbf{T}} + (\nabla \cdot \mathbf{u}_h, q_h)_{\mathbf{T}'} + \sum_{\mathbf{F}\in\partial\mathbf{T}\setminus\mathbf{E}} \nu_{\mathbf{F}}|\mathbf{F}|[\![p_h]\!]_{\mathbf{F}} \underbrace{[\![q_h]\!]_{\mathbf{F}}}_{=-\frac{|\mathbf{T}'|}{|\mathbf{T}|}} \\
+& \sum_{\mathbf{F}\in\partial\mathbf{T}'\setminus\mathbf{E}} \nu_{\mathbf{F}}|\mathbf{F}|[\![p_h]\!]_{\mathbf{F}} \underbrace{[\![q_h]\!]_{\mathbf{F}}}_{=1} + \nu_{\mathbf{E}}|\mathbf{E}|[\![p_h]\!]_{\mathbf{E}}[\![q_h]\!]_{\mathbf{E}} = 0.
\end{aligned}
\tag{6.10}
$$

The computations leading to (6.8) and (6.9) do not differ for this case, so combining them with (6.10) and treating the common face $\mathbf{E}$ separately, one obtains

$$
\begin{aligned}
0 =& (\nabla \cdot \tilde{\mathbf{u}}^h, q_h)_{\mathbf{T}} + \cancel{\frac{|\mathbf{T}'|}{|\mathbf{T}|}\sum_{\mathbf{F}\in\partial\mathbf{T}\setminus\mathbf{E}} \nu_{\mathbf{F}}[\![p_h]\!]_{\mathbf{F}}|\mathbf{F}|} + \cancel{\frac{|\mathbf{T}'|}{|\mathbf{T}|}\nu_{\mathbf{E}}[\![p_h]\!]_{\mathbf{E}}|\mathbf{E}|} \\
&+ (\nabla \cdot \tilde{\mathbf{u}}^h, q_h)_{\mathbf{T}'} - \cancel{\sum_{\mathbf{F}\in\partial\mathbf{T}'\setminus\mathbf{E}} \nu_{\mathbf{F}}[\![p_h]\!]_{\mathbf{F}}|\mathbf{F}|} - \cancel{\nu_{\mathbf{E}}[\![p_h]\!]_{\mathbf{E}}|\mathbf{E}|} \\
&+ \cancel{\sum_{\mathbf{F}\in\partial\mathbf{T}\setminus\mathbf{E}} \nu_{\mathbf{F}}|\mathbf{F}|[\![p_h]\!]_{\mathbf{F}} \underbrace{[\![q_h]\!]_{\mathbf{F}}}_{=-\frac{|\mathbf{T}'|}{|\mathbf{T}|}}} + \cancel{\sum_{\mathbf{F}\in\partial\mathbf{T}'\setminus\mathbf{E}} \nu_{\mathbf{F}}|\mathbf{F}|[\![p_h]\!]_{\mathbf{F}} \underbrace{[\![q_h]\!]_{\mathbf{F}}}_{=1}} \\
&+ \cancel{\nu_{\mathbf{E}}|\mathbf{E}|[\![p_h]\!]_{\mathbf{E}} \underbrace{[\![q_h]\!]_{\mathbf{E}}}_{=1-\frac{|\mathbf{T}'|}{|\mathbf{T}|}}}.
\end{aligned}
$$

Thus

$$0 = (\nabla \cdot \tilde{\mathbf{u}}^h, q_h)_{\mathbf{T}} + (\nabla \cdot \tilde{\mathbf{u}}^h, q_h)_{\mathbf{T}'}.$$

Conclusion:
We have shown for all mesh cells $\mathbf{T}, \mathbf{T}' \in \mathcal{T}_h$:

$$(\nabla \cdot \tilde{\mathbf{u}}^h, 1)_{\mathbf{T}} = \frac{|\mathbf{T}|}{|\mathbf{T}'|}(\nabla \cdot \tilde{\mathbf{u}}^h, 1)_{\mathbf{T}'}.$$

Summing up this expression for all mesh cells results in

$$\begin{aligned}(\nabla \cdot \tilde{\mathbf{u}}^h, 1)_{\mathbf{\Omega}} &= \sum_{\mathbf{T} \in \mathcal{T}_h} (\nabla \cdot \tilde{\mathbf{u}}^h, 1)_{\mathbf{T}} = \sum_{\mathbf{T} \in \mathcal{T}_h} \frac{|\mathbf{T}|}{|\mathbf{T}'|}(\nabla \cdot \tilde{\mathbf{u}}^h, 1)_{\mathbf{T}'} \\ &= \frac{1}{|\mathbf{T}'|}(\nabla \cdot \tilde{\mathbf{u}}^h, 1)_{\mathbf{T}'} \underbrace{\sum_{\mathbf{T} \in \mathcal{T}_h} |\mathbf{T}|}_{=|\mathbf{\Omega}|}.\end{aligned} \tag{6.11}$$

With integration by parts we conclude

$$(\nabla \cdot \tilde{\mathbf{u}}^h, 1)_{\mathbf{\Omega}} = \int_{\mathbf{\Omega}} \nabla \cdot \tilde{\mathbf{u}}^h = \int_{\partial\mathbf{\Omega}} \underbrace{\tilde{\mathbf{u}}^h \cdot \mathbf{n}}_{=0 \text{ on } \partial\mathbf{\Omega}} = 0$$

$$\begin{aligned}&\overset{(6.11)}{\Longleftrightarrow} 0 = \frac{|\mathbf{\Omega}|}{|\mathbf{T}'|}(\nabla \cdot \tilde{\mathbf{u}}^h, 1)_{\mathbf{T}'} \\ &\Longleftrightarrow 0 = (\nabla \cdot \tilde{\mathbf{u}}^h, 1)_{\mathbf{T}'}.\end{aligned}$$

The elementwise linearity of the velocity implies $\nabla \cdot \tilde{\mathbf{u}}^h = \text{const.}$ on each mesh cell, so

$$\begin{aligned}&\Longrightarrow 0 = (\underbrace{\nabla \cdot \tilde{\mathbf{u}}^h}_{=\text{const.}}, 1)_{\mathbf{T}'} = (\nabla \cdot \tilde{\mathbf{u}}^h)|_{\mathbf{T}'} \cdot \underbrace{\int_{\mathbf{T}'} 1 \, \mathrm{d}\mathbf{x}}_{=|\mathbf{T}'|} \\ &\Longrightarrow 0 = (\nabla \cdot \tilde{\mathbf{u}}^h)|_{\mathbf{T}'}, \quad \forall \mathbf{T}' \in \mathcal{T}_h.\end{aligned}$$

□

Remark 6.0.7
This theorem states that the addition of the Raviart–Thomas field $\mathbf{u}_h^{\mathrm{RT}}$ defined in (6.3) to the discrete velocity obtained for the stabilized problem (6.2) is a possibility to reconstruct the local mass conservation. Moreover, by the $\mathbf{H}(\mathrm{div})$-conformity of $\mathbf{P}_1 + \mathbf{V}_h^{\mathbf{RT}}$ the discrete velocity solution $\tilde{\mathbf{u}}_h$ is weakly divergence-free. This post-processing does not undermine the convergence of the method. For details see [BaVa11], pp. 807.

7 Numerical Studies

Last but not least we want to analyze the two methods we introduced in the previous chapters numerically.
For both, a possibility to reconstruct mass conservation is presented and they are therefore very interesting for applications where mass conservation is of particular importance. In addition, the modified CR method has the property, that irrotational forces do not influence the velocity solution.
We test their implementations in C++ with two examples.

In both examples below we take the domain to be the unit square

$$\mathbf{\Omega} = (0,1)^2$$

and suppose that it is decomposed uniformly by means of a triangular mesh. The initial grid, i.e., level zero ($h = \frac{1}{\sqrt{2}}$), is visualized in Figure 7.1.

The results, obtained using the following four methods, are compared with each other:

- the nonconforming Crouzeix–Raviart method (CR),
- the modified Crouzeix–Raviart method (ModCR),
- the pressure-stabilized $\mathbf{P}1/P0$ method (PSP1P0),
- the post-processed, pressure-stabilized $\mathbf{P}1/P0$ method (PPPSP1P0).

The absolute values of the velocity in level 4 are visualized using the application Paraview, version 4.1.0. For simplicity the parameter μ, used in the pressure-stabilized $\mathbf{P}1/P0$ formulation (6.2), is set equal to zero and $\gamma = 0.5$.

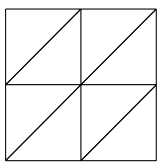


Figure 7.1: The grid for level 0.

7.1 Example 1 – The Vortex

Consider the stream function

$$\psi(\mathbf{x}) = 100x_1^2(1-x_1)^2x_2^2(1-x_2)^2.$$

We choose the data such that the exact velocity field is given by

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} u_1(\mathbf{x}) \\ u_2(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \frac{\partial\psi}{\partial x_2} \\ -\frac{\partial\psi}{\partial x_1} \end{pmatrix} = \begin{pmatrix} 200x_1^2(1-x_1)^2x_2(1-x_2)(1-2x_2) \\ -200x_2^2(1-x_2)^2x_1(1-x_1)(1-2x_1) \end{pmatrix}$$

with homogeneous Dirichlet boundary conditions, i.e.,

$$\mathbf{u}|_{\partial\boldsymbol{\Omega}} = \mathbf{0},$$

see Figure 7.2.

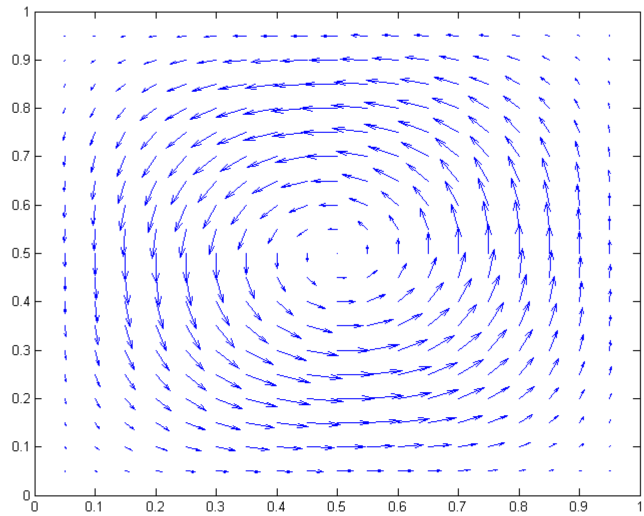


Figure 7.2: The velocity field $\mathbf{u}$ for the vortex.

The exact pressure field is defined by

$$p(\mathbf{x}) = 10\left(x_1 - \frac{1}{2}\right)^3 x_2^2 + (1-x_1)^3\left(x_2 - \frac{1}{2}\right)^3.$$

The function $p(\mathbf{x})$ is indeed an element of $L_0^2(\mathbf{\Omega})$:

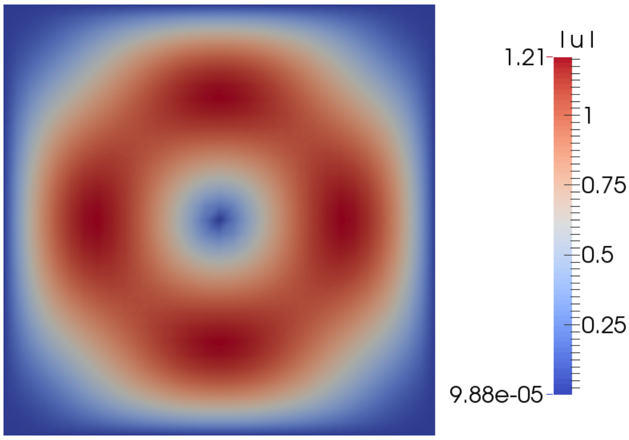
$$
\begin{aligned}
\int\limits_{\mathbf{\Omega}} p(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \int\limits_{(0,1)^2} 10\left(x_1-\frac{1}{2}\right)^3 x_2^2 + (1-x_1)^3\left(x_2-\frac{1}{2}\right)^3 \mathrm{d}\mathbf{x} \\
&= \int\limits_{(0,1)}\int\limits_{(0,1)} 10\left(x_1-\frac{1}{2}\right)^3 x_2^2 + (1-x_1)^3\left(x_2-\frac{1}{2}\right)^3 \mathrm{d}x_1\,\mathrm{d}x_2 \\
&= 0.
\end{aligned}
$$

The divergence of the velocity $\mathbf{u}$ is zero:

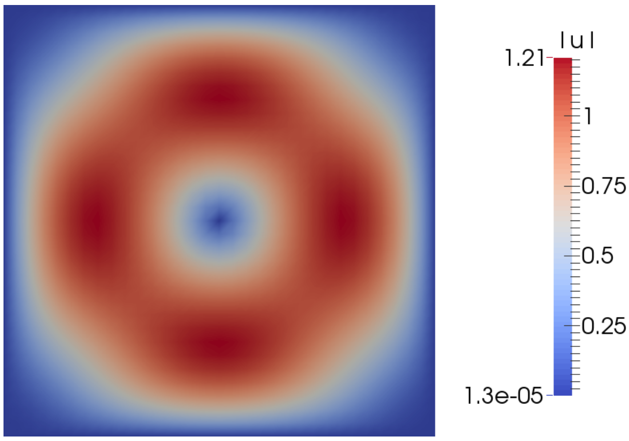
$$
\begin{aligned}
\nabla\cdot\mathbf{u}(\mathbf{x}) &= \frac{\partial u_1(\mathbf{x})}{\partial x_1} + \frac{\partial u_2(\mathbf{x})}{\partial x_2} \\
&= 200x_2(1-x_2)(1-2x_2)\left[2x_1(1-x_1)^2 - x_1^2 2(1-x_1)\right] \\
&\quad - 200x_1(1-x_1)(1-2x_1)\left[2x_2(1-x_2)^2 - x_2^2 2(1-x_2)\right] = 0.
\end{aligned}
$$

In Figure 7.3, one cannot recognize a significant difference between the visualizations of the velocity magnitudes. The only noticeable difference is that the PSP1P0 method yields a solution which is exactly zero at the boundary while the other methods are just nearly zero, but this is a direct consequence of the definition of the Crouzeix–Raviart space and the functionality of Paraview. As a matter of fact, Paraview visualizes the values in the vertices, and for nonconforming methods it uses the means of the values in the vertices of the neighboring mesh cells. Therefore the values at the boundary are not exactly zero.

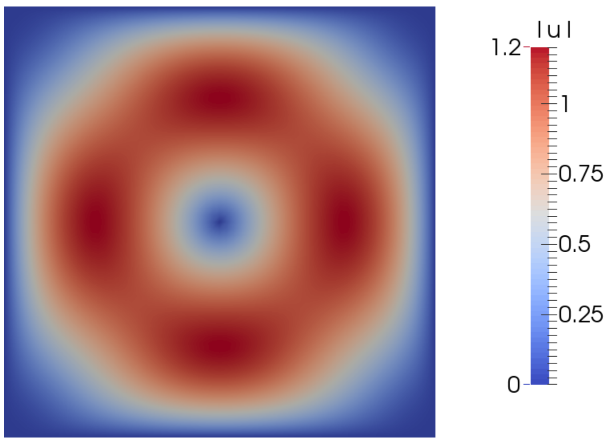
For this example, both, the velocity errors and the pressure errors have the orders of convergence, predicted by the numerical analysis, see Figure 7.4.



(a) The CR method 6.7.

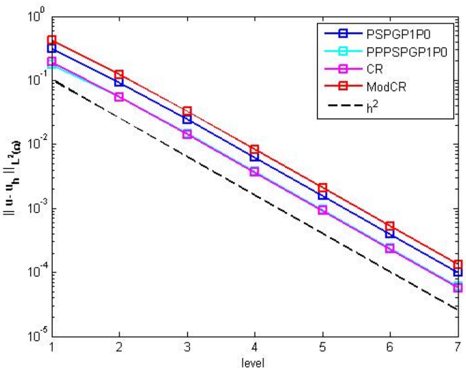


(b) The modified CR method (5.9).

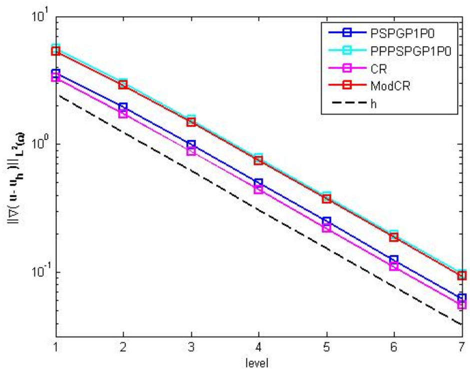


(c) The pressure-stabilized $\mathbf{P}1/P0$ method (6.2).

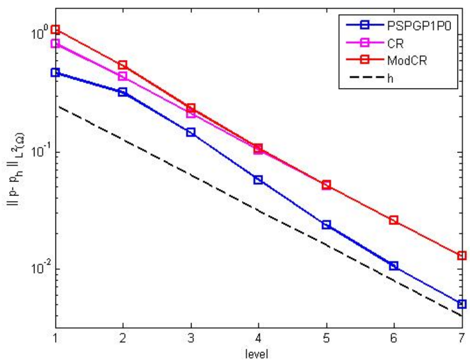
Figure 7.3: The magnitude of the velocity approximation in level 4 for the vortex example.



(a) The velocity error in the L^2-norm.



(b) The velocity error in the H^1-seminorm.



(c) The pressure error in the L^2-norm.

Figure 7.4: The errors in different norms for the vortex problem.

7.2 Example 2 – No Flow

The velocity field is given by

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} u_1(\mathbf{x}) \\ u_2(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The pressure is defined by

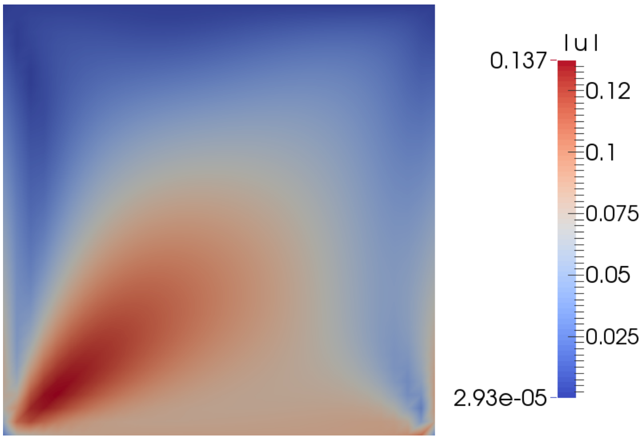
$$p(\mathbf{x}) = -\frac{Ra}{2}y^2 + Ra \cdot y - \frac{Ra}{3}$$

with $Ra := 1000$.

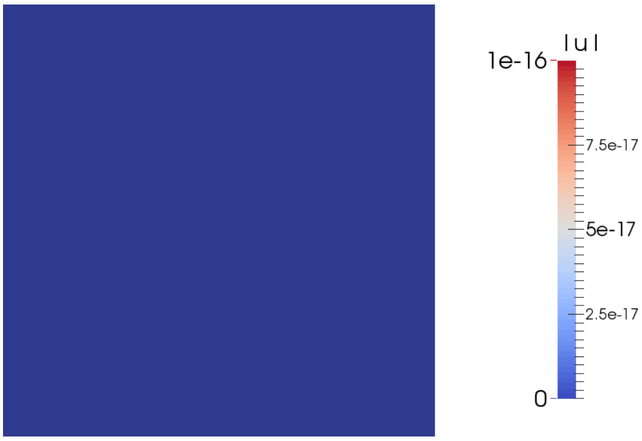
Figure 7.5 shows that the ModCR method (b) approximates the solution significantly better than the others. In fact, the approximation is almost zero. This behavior was predicted in Remark 5.2.2.
While the CR method and the PSP1P0 method do not inherit the invariance property, the ModCR method does. Therefore, the velocity solution for the totally irrotational force field in this example, is the zero field (apart from rounding errors).
This is not the case for the other methods. There is only a slight difference between the CR method (a) and the PSP1P0 method (c) but the magnitude of the velocity solution using the latter method is smaller.

The errors in different norms are presented in Figure 7.6. For the ModCR method, the velocity error is almost constant during the mesh refinement and its value is almost zero. The order of convergence in the L^2-norm agrees with the one predicted by the numerical analysis but using PPPSP1P0 and CR results in a smaller error than for PSP1P0. In the H^1-seminorm, the velocity error using PPPSP1P0 and PSP1P0 is decreasing faster when refining the mesh (order 3/2 instead of 1), than when using the CR method.

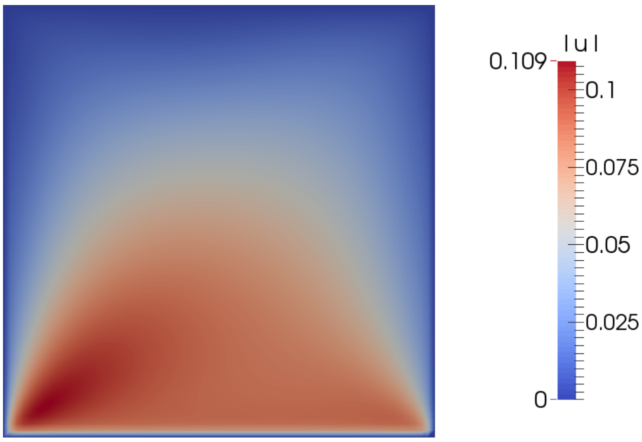
A difference between the pressure errors is only visible for the coarsest grids, where the ModCR method produces the smallest error value closely followed by the CR method.



(a) The CR method $\mathbf{V}_h^{\mathrm{CR}}/Q_h^{\mathrm{CR}}$.

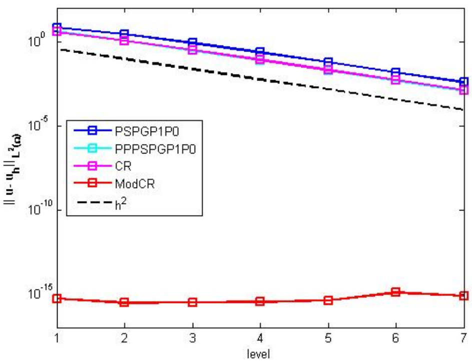


(b) The modified CR (5.9).

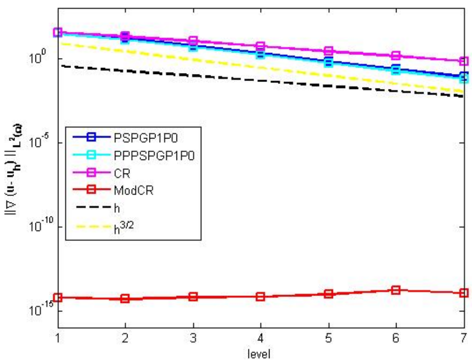


(c) The pressure-stabilized $\mathbf{P}1/P0$ method (6.2).

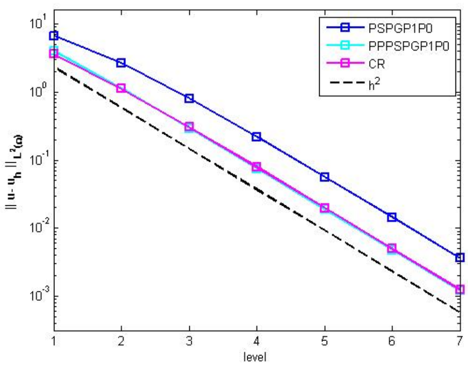
Figure 7.5: The magnitude of the velocity approximation in level 4 for the no flow example.



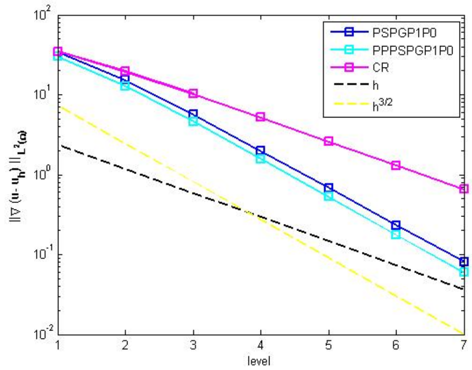
(a) The velocity error in the L^2-norm.



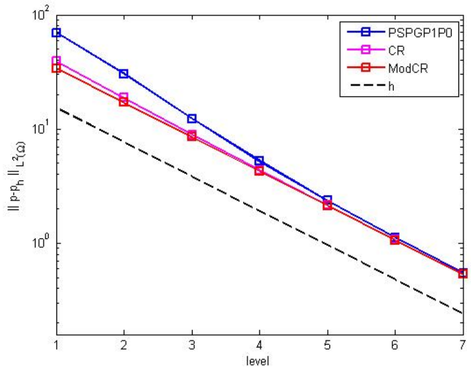
(b) The velocity error in the H^1-seminorm.



(c) The velocity error in the L^2-norm zoomed.



(d) The velocity error in the H^1-seminorm zoomed.



(e) The pressure error in the L^2-norm.

Figure 7.6: The errors in different norms for the no flow problem.

8 Summary and Conclusion

The subject of this project was to analyze the approach to use the finite element method for solving the Stokes equations.
On the one hand, the inf-sup condition imposes a restriction on the choice of the finite element spaces that should be satisfied in order to guarantee unique solvability. On the other hand, during the last years it has become of big interest to create methods such that the resulting approximations fulfill the qualitative properties assumed by the system.

To that effect, two methods were introduced. One of them is a pressure-stabilized version of the $\mathbf{P}1/P0$ finite element method, with a post-processing for the velocity approximation. This post-processing guarantees that the velocity is divergence-free.
The other one is the modified Crouzeix–Raviart method which uses a projection into the space of lowest order Raviart–Thomas functions. The effect is that discretely divergence-free vector fields are projected onto weakly divergence-free vector fields. Thus, L^2-orthogonality is established. The consequence is that the modified method satisfies a property which is fulfilled by the original formulation of the Stokes problem. This property is called invariance property throughout this thesis and it is typically inherited by divergence-free methods as the Scott–Vogelius element. The invariance property assures that the velocity approximation does not depend on irrotational forces, thus the velocity error is independent of the pressure.
In order to compare these methods with each other and their well known counterparts without post-processing and modification, they were applied to two examples. While the first example, the vortex, did not provoke any significant differences between the methods, the second example, no flow, did.
The results show that the modified Crouzeix–Raviart method is in the no flow problem the best choice since the velocity error is much smaller than for the other methods. In fact, the velocity error is almost zero.
This perfectly fits with the theoretical analysis because thanks to the invariance property, the velocity error of this method does not depend on irrotational forces. So the advantage of utilizing the modified Crouzeix–Raviart method in applications, where the force field has a large, respectively dominating irrotational part is theoretically proven and confirmed by applications.

An alternative reconstruction fulfilling the invariance property uses the lowest order Brezzi-Douglas-Marini element (BDM). In two dimensions, it was derived in [BrDoMa85]. In addition to the velocity error in the H^1-seminorm, the velocity error in the L^2-norm is proven to be independent from the pressure for this BDM reconstruction, too.
For further information, [BreLiMeSchö14] is recommended.

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I declare that I have authored this thesis independently and that I have not used other than the declared sources/resources. Moreover, I declare that this thesis has not been submitted to any examination procedure before.

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