Turbulence Modelling of the Navier-Stokes Equations using the NS-α approach

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Declaration of Originality

Herewith I declare that this thesis is the result of my independent work. All sources and auxiliary materials used by me in this thesis are cited completely.

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Contents

Introduction

Partial differential equations (PDEs) arise from a variety of problems in physics, engineering, ecology, economics and all other areas in which mathematical methods are necessary. These PDEs describe certain aspects of phenomena in a mathematical way. These aspects vary from typical processes such as heat diffusion and populations balances, to crash tests, fluid dynamics and weather forecast. The resulting PDEs are models for these phenomena of interest. They take the characteristics into account and simplify the behavior into an abstract model, such that the problem can be investigated on a theoretical- and application-oriented basis. This approach leads to numerical simulations as well, since it is possible to find discrete approximations to these PDEs and calculate an approximated solution via high performance computers.

The theoretical handling of PDEs can be very challenging, since slight modifications of the equations can cause huge mathematical issues.

In this thesis we consider the theoretical machinery behind the Navier-Stokes equations (NSE) which describe the motion of a homogeneous incompressible Newtonian fluid. In **Chapter 1** we will derive the Navier-Stokes equation using an approach from continuum mechanics. A brief introduction in turbulent flows and Kolmogorv's theory is also given here. **Chapter 2** will cover the theoretical basics and notation to prove three important theorems. We will discuss the difficulties we encounter during that process. **Chapter 3** is a slightly application oriented chapter, where we discuss the challenge of solving the NSE with direct numerical methods and their limitation. We will motivate turbulence modeling and the Navier-Stokes-α (NS-α) approach will be introduced. Furthermore, we will show that a solution of NS-α will converge in some sense to a solution of NSE, which is a desired property to have. A convergence statement in the context of finite-elements will be shown in **Chapter 4**. The last chapter will discuss explicit results using numerical computations.

1 The Navier-Stokes Equation

The fundamental equations of fluid dynamics are the so called Navier-Stokes equations. These equations describe the behavior of a fluid from a mathematical point of view under certain assumptions. In the following section, we will derive the NSE using an approach from continuum mechanics.

1.1 Derivation

The equations essentially come from the conservation of mass and momentum (Newton's second law). Let's recall the Gauss' divergence theorem and the transport theorem first.

Theorem 1.1 (Gauss' divergence theorem)**.** *Let $\Omega \subset \mathbb{R}^3$ be a compact set with piece wise smooth boundary and $\vec{n}$ be the outer unit field on $\partial\Omega$. If $f : G \to \mathbb{R}^3$ is a continuously differentiable function on G with $\Omega \subset G$, then we have:*

$$\int\limits_{\Omega} div(\vec{f}(x))dx = \int\limits_{\partial\Omega} \vec{f}(s) \cdot \vec{n}\, ds.$$

Proof. The proof can be found in any analysis book. □

Let $\omega_0 \subset \Omega$ be a bounded subdomain and let

$$\omega(t) = \Psi(\cdot, t)(\omega_0) = \{\Psi(x,t)|x \in \omega_0\}$$

be the image of ω_0 under Ψ. The mapping $\Psi(x,t)$ is called a trajectory of a fluid particle $x \in \omega_0$ in the velocity field v. Furthermore we observe that

$$\Psi(x,0) = x,$$
$$\Psi_t(x,t) = v(\Psi(x,t),t).$$

Theorem 1.2 (Transport theorem)**.** *If $f \in C^1(\Omega)$ then it holds:*

$$\frac{d}{dt}\int\limits_{\omega(t)} f(x,t)dx = \int\limits_{\omega(t)} f_t(x,t) + \nabla \cdot (f(x,t)v(x,t))\, dx.$$

Proof. Using the Wronskian determinant, the substitution rule and some straightforward calculations lead to the above statement. □

1.1.1 Mass conservation and continuity equation

The continuity equation emerges from the mass conservation. Consider herefore a control volume $\omega_0 \subset \Omega$ of a fluid with density ρ, then it holds for all t:

$$m(t) = \int\limits_{\omega(t)} \rho(t,x)\,dx. \tag{1.1}$$

The law of mass conservation states that the mass in a closed system stays constant, which means it has a vanishing derivative.

$$\frac{dm}{dt}(t) = 0.$$

Using the transport theorem we reach the following conclusion:

$$\begin{aligned}\frac{dm}{dt}(t) &= \frac{d}{dt}\int\limits_{\omega(t)} \rho(t,x)\,dx \\ &= \int\limits_{\omega(t)} \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho v)\,dx.\end{aligned}$$

Through the arbitrary choice of ω_0 and a localization argument, we conclude the continuity equation:

$$\partial_t\rho + \nabla\cdot(\rho v) = 0. \tag{1.2}$$

1.1.2 Momentum conservation and Newton's second law

Now we will reformulate Newton's second law in the context of fluid dynamics. It states:

$$force = mass \times acceleration = change\ of\ momentum.$$

Considering the right-hand side first, we state:

$$momentum(t) = mass \times velocity = \int\limits_{\omega(t)} \rho v\,dx.$$

We now apply the transport theorem to the time derivative of each component of the term above:

$$\left(\frac{d}{dt}\int\limits_{\omega(t)} \rho v\,dx\right)_j = \frac{d}{dt}\int\limits_{\omega(t)} \rho v_j\,dx = \int\limits_{\omega(t)} \partial_t(\rho v_j) + \nabla\cdot(\rho v_j v)\,dx.$$

We put all the components back together and get the following identity:

$$\frac{d}{dt}\int\limits_{\omega(t)} \rho v \, dx = \int\limits_{\omega(t)} \partial_t(\rho v) + \nabla \cdot (\rho v \otimes v^t) \, dx,$$

where v^t is the transpose of v and $(a \otimes b)_{i,j} = a_i b_j$ is a tensor. Inserting this identity into Newton's second law results in:

$$force = \int\limits_{\omega(t)} \partial_t(\rho v) + \nabla \cdot (\rho v \otimes v^t) \, dx.$$

Let us consider the left-hand side now, namely the force term acting on a control volume, $\omega(t)$. We distinguish between external and internal forces. Interneral forces are forces the fluid exerts on itself like pressure and viscous drag. External forces are forces acting on the fluid from outside, such as gravity. We get the following expression for the force term:

$$F_{total} = \underbrace{\int\limits_{\omega(t)} F(t,x) \, dx}_{\text{external forces}} + \underbrace{\int\limits_{\partial\omega(t)} \sigma \cdot \vec{n} \, ds}_{\text{internal forces}} = \int\limits_{\omega(t)} F(t,x) \, dx + \int\limits_{\omega(t)} \nabla \cdot \sigma \, dx,$$

where σ is the Cauchy stress tensor. Inserting this identity into Newton's second law results in:

$$\int\limits_{\omega(t)} F + \nabla \cdot \sigma \, dx = \int\limits_{\omega(t)} \partial_t(\rho v) + \nabla \cdot (\rho v \otimes v^t) \, dx.$$

Through the arbitrary choice of ω_0 and a localisation argument, we conclude the following representation of Newton's second law:

$$F + \nabla \cdot \sigma = \partial_t(\rho v) + \nabla \cdot (\rho v \otimes v^t).$$

We now have a representation of conservation of mass and momentum, but further modeling is necessary. The Cauchy stress tensor needs to be expressed in terms that are known or can be calculated.

1.1.3 The Cauchy stress tensor

One fundamental property of a stationary fluid is that the tension acts in the opposite direction than the outer normal. This means that for a stationary fluid the Cauchy stress tensor σ only depends on the thermodynamic pressure p and it holds:

$$\sigma = -p \, Id.$$

If the fluid is being moved, we can decompose the stress tensor into its components, namely the pressure p and the viscosity:

$$\sigma = -p \, Id + \tau,$$

where τ is the viscous stress tensor. However, we still need to find a suitable representation for τ.
Assuming we have an isotropic, Newtonian fluid, we can express the viscous stress tensor τ linearly by the gradient of the velocity field:

$$\tau = \lambda(\nabla \cdot v)Id + \mu(\nabla v + (\nabla v)^T).$$

The resulting representation is a fundamental assumption for modeling the NSE. The variables λ and μ are only dependent on thermodynamical variables (pressure, temperature) but not on the velocity v, they are called Lamé constants.

1.1.4 The homogeneous, incompressible NSE for isotropic, Newtonian fluids

If we assume we have an isotropic, Newtonian fluid and use the approximation of the Chauchy stress tensor τ from our previous section, we can state Newton's second law in the following way:

$$\begin{aligned} F + \nabla \cdot \sigma &= \partial_t(\rho v) + \nabla \cdot (\rho v\, v^t), \\ F + \nabla \cdot \Big(- pId + \lambda(\nabla \cdot v)Id + \mu(\nabla v + (\nabla v)^T)\Big) &= \partial_t(\rho v) + \nabla \cdot (\rho v\, v^t). \end{aligned} \tag{1.3}$$

Before we continue, let us simplify first. Consider the deformation tensor $\mathbb{D}(v) = \frac{1}{2}\big(\nabla v + (\nabla v)^T\big)$. Assuming enough regularity of v we can state:

$$\begin{aligned} \nabla \cdot \big(\nabla v + (\nabla v)^T\big) &= \Delta v + \nabla \cdot (\nabla v)^T \\ &= \Delta v + \nabla \cdot \begin{pmatrix} \partial_1 v_1 & \cdots & \partial_n v_1 \\ \vdots & & \vdots \\ \partial_1 v_n & \cdots & \partial_n v_n \end{pmatrix}^T \\ &= \Delta v + \nabla \cdot \begin{pmatrix} \partial_1 v_1 & \cdots & \partial_1 v_n \\ \vdots & & \vdots \\ \partial_n v_1 & \cdots & \partial_n v_n \end{pmatrix} \\ &= \Delta v + \begin{pmatrix} \partial_1 \partial_1 v_1 & + \cdots + & \partial_n \partial_1 v_n \\ \vdots & & \vdots \\ \partial_1 \partial_n v_1 & + \cdots + & \partial_n \partial_n v_n \end{pmatrix} \\ &= \Delta v + \begin{pmatrix} \partial_1 \partial_1 v_1 & + \cdots + & \partial_1 \partial_n v_n \\ \vdots & & \vdots \\ \partial_n \partial_1 v_1 & + \cdots + & \partial_n \partial_n v_n \end{pmatrix} \\ &= \Delta v + \begin{pmatrix} \partial_1 (\nabla \cdot v_1) \\ \vdots \\ \partial_n (\nabla \cdot v_n) \end{pmatrix} \\ &= \Delta v + \nabla(\nabla \cdot v). \end{aligned} \tag{1.4}$$

On the right-hand side of equation (1.3) we can extract the continuity equation (1.2). Before we do that we consider:

$$
\begin{aligned}
\big(\nabla \cdot (\rho v\, v^t)\big)_j &= \nabla \cdot (\rho v_j v) = \sum_i \partial_i(\rho v_j v_i) \\
&= \sum_i \left(v_j \partial_i(\rho v_i) + \rho v_i \partial_i v_j\right) \\
&= \left(v \nabla \cdot (\rho v) + \rho (v \cdot \nabla) v\right)_j .
\end{aligned}
$$

Using this identity, we can consider the right-hand side of (1.3) now:

$$
\begin{aligned}
\partial_t(\rho v) + \nabla \cdot (\rho v\, v^t) &= \partial_t \rho v + \rho \partial_t v + \rho(\nabla \cdot v)v + v \nabla \cdot (\rho v) \\
&= v \underbrace{(\partial_t \rho + \nabla \cdot (\rho v))}_{=0} + \rho \partial_t v + (\nabla \cdot v)\rho v \\
&= \rho \partial_t v + \rho(\nabla \cdot v) v.
\end{aligned}
\tag{1.5}
$$

We insert (1.4) and (1.5) into (1.3) and get the following representation of the NSE:

$$
\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho v) &= 0, && \text{(Continuity Equation)} \\
F - \nabla p + \lambda \nabla \cdot (\nabla \cdot v) + \mu \Delta v + \mu \nabla(\nabla \cdot v) &= \rho \partial_t v + \rho(\nabla \cdot v)v. && \text{(Newton's second law)}
\end{aligned}
\tag{1.6}
$$

Given we have a homogeneous, incompressible fluid, the density is constant everywhere, $\rho \equiv const$. This reduces the continuity equation to just a constraint on the velcoity field v:

$$
\nabla \cdot v = 0. \tag{1.7}
$$

Inserting (1.7) into (1.6) results in the following formulation of the NSE:

$$
\begin{aligned}
\nabla \cdot v &= 0 \\
F - \nabla p + \mu \Delta v &= \rho \partial_t v + \rho (v \cdot \nabla) v.
\end{aligned}
$$

We divide by ρ and reorder the terms.

$$
\begin{aligned}
\partial_t v - \nu \Delta v + (v \cdot \nabla) v + \nabla P &= \tilde{F}, \\
\nabla \cdot v &= 0,
\end{aligned}
\tag{1.8}
$$

where $\nu = \frac{1}{\rho}\mu$ is the viscosity constant, $P = \frac{p}{\rho}$ is the scaled pressure and $\tilde{F} = \frac{1}{\rho}F$ are scaled inner forces.
So far we considered physical quantities with units, but to continue describing and solving the problem from a mathematical point of view we must rescale the equations to eliminate the units. To do so, we need the following units:

- $L[m]$: characteristic length,
- $U[\frac{m}{s}]$: characteristic velocity scale,

- $T^*[s]$: characteristic time,

and the corresponding variable transformations of the previous variables $(t', x')[s, m]$:

- $x = \dfrac{x'}{L}$,
- $u = \dfrac{v}{U}$,
- $t = \dfrac{t'}{T^*}$.

Reformulating and simplifying (1.8) with the new variables (t, x) we get:

$$\frac{L}{UT^*}\partial_t u - \frac{\nu}{UL}\nu\Delta u + (u\cdot\nabla)u + \nabla\frac{P}{U^2} = \frac{L}{U^2}\tilde{F},$$
$$\nabla\cdot u = 0.$$

For simplification we normalize the characteristic time to $1s$:

$$T^* = \frac{L}{U} = 1s,$$

and obtain:

$$\partial_t u - \frac{1}{Re}\Delta u + (u\cdot\nabla)u + \nabla p = f$$
$$\nabla\cdot u = 0,$$

where:

$$Re = \frac{UL}{\nu}, \quad p = \frac{P}{U^2}, \quad f = \frac{\tilde{F}L}{U^2}.$$

We now have a dimensionless representation of the Navier-Stokes equations:

$$NSE \quad \begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f & \text{in } (0,T]\times\Omega, \\ \nabla\cdot u = 0 & \text{in } (0,T]\times\Omega. \end{cases}$$

1.1.5 Initial and boundary condition

The NSE are partial differential equations of first order in time and second order in space. Consequently we have to add an initial condition (IC) at time $t = 0$ and boundary conditions (BC) at the boundary $\Gamma = \partial\Omega$ to the system of equations. The chosen initial and boundary conditions should naturally satisfy a compatibility condition for $t \to 0, t > 0$.
A suitable IC would be a divergence free velocity field u_0 such that:

$$
\begin{aligned}
u(0,x) &= u_0(x) \qquad \text{in } \Omega \\
\nabla \cdot u_0 &= 0.
\end{aligned}
$$

When choosing a BC we have many alternatives. We can split the boundary Γ into smaller disjoint segments and prescribe different boundary conditions on each of them. Naturally they should be compatible with each other. The following are typical types of BCs:

- Dirichlet BC: fixed inflow and outflow

$$u(t,x) = g(t,x) \qquad \text{in } (0,T] \times \Gamma_{dirichlet}.$$

- No-slip BC: no penetration and no slip

$$u(t,x) = 0 \qquad \text{in } (0,T] \times \Gamma_{noslip}.$$

- Free-slip BC: no friction

$$u(t,x) \cdot n = g(t,x) \qquad \text{in } (0,T] \times \Gamma_{slip}.$$

Remark *1. When we prescribe Dirichlet BC on $\Gamma_{dirichlet} = \Gamma$ an additional issue arises. Because there is no constraint on the pressure at the boundary, it is only determined up to a constant. Adding a condition to the pressure for example, an integral mean*

$$\int_{\Omega} p(t,x)\,dx = 0 \qquad t \in (0,T],$$

is sufficient to resolve this issue.

2. Since the continuity equation degenerates to a simple divergence free constraint on u, we need an additional condition on the BC:

$$0 = \int_{\Omega} \nabla \cdot u(t,x)\,dx = \int_{\Gamma} (u \cdot n)(t,x)\,dx.$$

This identity has to be satisfied at all times.

1.2 Laminar and turbulent flows

Osbourne Reynolds discovered through coloring experiments of a pipe flow that the occurrence of vortices and chaotic movements are highly dependent on the Reynolds number. In general, we can make the following observations: for small Reynolds numbers, the flow reaches a steady state. It circulates around obstacles and adapts to it. It's time independent and forms a laminar flow. Enlarging the Reynolds number results in the flow becoming time dependent and the vortices and eddies start to form. The size of these vortices and eddies depends positively on the Reynolds number. If the Reynolds number gets very large, the flow becomes chaotic and irregular, a so-called turbulent flow.
Turbulent flow differs from laminar flow basically in the following points:

- A turbulent flow is irregular, chaotic and unpredictable.
- Small perturbations are amplified greatly by the non-linear character of the NSE.
- Eddies occur. Their size range from that of the domain to very small scales. The largest eddies hold the most energy. In terms of the mean, these large eddies transfer their energy to smaller eddies until the smallest dissipates the energy as heat. Therefore, the system consumes energy to maintain its turbulent behavior. A constant inflow of energy is needed to maintain the turbulent flow, for example in the form of a fixed inflow with a high velocity.
- The rate of diffusion, i.e., the velocity in which mass, heat and momentum propagate, is significantly higher than in the laminar case.

1.3 Kolmogorov's theory and Energy cascade

As briefly discussed in the previous section, energy is transferred in a mean sense from large eddies to the smallest ones; this decay of energy is called energy cascade. From an application-oriented point of view, it is important to know how large these smallest eddies are, because if our standard discretization is too coarse we lose information. Naturally, the discretization should be fine enough to resolve the smallest eddies. Kolmogorov's theory describes the magnitude of the smallest eddies in comparison to the largest ones.

Kolmogorov's hypothesis
Let us consider a turbulent flow with a high Reynolds number,

$$Re = \frac{LU}{\nu}.$$

We can interpret turbulence as an interaction between eddies of different sizes. A large domain with a large eddy can consist of many smaller eddies. Consider the size of an eddy l and its characteristic velocity $u(l)$. The size of the largest eddies are of magnitude of the characteristic length L and have characteristic velocity $u(L) = U$. We define a size dependent Reynolds number $Re(l)$

$$Re(l) = \frac{u(l)\, l}{\nu}.$$

In terms of the mean, the largest eddies are unstable, collapse and transfer their energy to smaller eddies; this process goes on for as long as the eddies are unstable. At the smallest scale the eddies become stable and dissipates their energy as heat. The energy lost through heat dissipation only occurs in the last step, namely at the smallest eddies. The cascade proceeds to smaller and smaller scales until the Reynolds number $Re(l)$ is small enough ($Re(l) \approx O(1)$) for dissipation to be effective. The rate of transfer of energy ε from the largest to the smallest eddies can be estimated. Therefore, we state that the energy at the largest scales is $\frac{1}{2}U^2$ with its corresponding timescale $\tau = \frac{L}{U}$. For ε it holds

$$\varepsilon \sim \frac{U^2}{\tau} = \frac{U^3}{L}.$$

This is not enough to conclude any information about the smallest eddies. We need Kolmogorov's hypothesis of local isotropy and his first similarity hypotheses.
The first hypothesis refers to the statistically isotropic behavior for the smallest eddies. In general, the flow is anisotropic and depends on the boundary of the domain, but for the smallest eddies, directional information is lost during the energy cascade. Kolmogorov argued that all information about the geometry of the large eddies is also lost. As a consequence, the statistics of the small scale motion are even, which is a different way of saying the smallest eddies are locally isotropic. This relation only holds for large Reynolds numbers.
Let l_{EL} be the threshold of the eddy sizes where energy transfer dominates ($l > l_{EL}$) and where heat dissipation dominates ($l < l_{EL}$). Kolmogorov's first similarity hypothesis states that the statistics of the small scale motion considering a sufficiently high Reynolds number is uniquely determined by the kinematic viscosity ν and the rate of transfer of energy ε. Let us assume ν and ε are given. We want to define a unique (up to a multiplicative constant) length, velocity and time scales that ensure the Reynolds number to be 1. These new scales will be denoted with a subscript η.

We can calculate the following:

$$1 = Re_\eta = \frac{l_\eta \, u_\eta}{\nu} \qquad \Longrightarrow \qquad \begin{aligned} u_\eta &= \frac{\nu}{l_\eta} \\ l_\eta &= \frac{\nu}{u_\eta} \end{aligned}$$

and using $\tau_\eta = \frac{l_\eta}{u_\eta}$ results in

$$\varepsilon = \frac{u_\eta^2}{\tau_\eta} = \frac{u_\eta^3}{l_\eta} \qquad \Longrightarrow \qquad u_\eta = (\varepsilon l_\eta)^{\frac{1}{3}}.$$

Using these identities we can determine the desired scales:

$$\begin{aligned} l_\eta &= \frac{\nu}{u_\eta} = \frac{\nu}{\varepsilon^{\frac{1}{3}} l_\eta^{\frac{1}{3}}} && \Longrightarrow && l_\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}} && \text{length scale,} \\ u_\eta &= \frac{\nu}{l_\eta} = \frac{\nu}{\left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}}} && \Longrightarrow && u_\eta = (\varepsilon\nu)^{\frac{1}{4}} && \text{velocity scale,} \\ \tau_\eta &= \frac{l_\eta}{u_\eta} = \frac{\left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}}}{(\varepsilon\nu)^{\frac{1}{4}}} && \Longrightarrow && \tau_\eta = \left(\frac{\nu}{\varepsilon}\right)^{\frac{1}{2}} && \text{time scale.} \end{aligned}$$

These scales are called Kolmogorov scales. Let us recall the main issue of this section. We want to find the size of the smallest eddies. We can do so by inserting $\varepsilon \approx \frac{U^3}{L}$ in the length scale, resulting in:

$$u_\eta \sim \left(\frac{U^3}{L}\,\nu\right)^{\frac{1}{4}} = \left(\frac{U^4}{L\,U}\,\nu\right)^{\frac{1}{4}} = U\,Re^{-\frac{1}{4}} \qquad \Longrightarrow \qquad \frac{u_\eta}{U} \sim Re^{-\frac{1}{4}}$$

For a more in depth treating of the Kolmogorov scales and a more precise description of the value l_{EL} see [Pop00]

2 Theoretical approach to the Navier-Stokes-Equation

The NSE are nonlinear partial differential equations. This nonlinearity in particular is difficult and challenging to handle on a mathematical level. In this section we will briefly state a few fundamental statements from functional analysis that are needed to show existence and uniqueness of a solution.

2.1 Definitions

Definition (Lebesgue spaces)**.** Let $\Omega \subset \mathbb{R}^n$ then we define

$$L^p(\Omega) = \left\{ f \text{ Lebesgue measurable} : \int\limits_{\Omega} |f(x)|^p \, dx < \infty \right\} \qquad 1 \leq p < \infty,$$
$$L^\infty(\Omega) = \left\{ f \text{ Lebesgue measurable} : \underset{x \in \Omega}{ess\ sup} f(x) < \infty \right\} \qquad p = \infty.$$

Lebesgue spaces with exponent p.

Remark *Lebesgue spaces are normed vector spaces with norm*

$$||f||_{L^p(\Omega)} = \left(\int\limits_{\Omega} |f(x)|^p \, dx \right)^{\frac{1}{p}}, \qquad 1 \leq p < \infty.$$

Remark *The Lebesgue space L^p for $p = 2$ is a Hilbert space with scalar product*

$$(f, g)_{L^2(\Omega)} = \int\limits_{\Omega} f g \, dx, \qquad (f, f)_{L^2(\Omega)} = ||f||^2_{L^2(\Omega)}.$$

Definition (Space of testfunctions)**.** Let $\Omega \subset \mathbb{R}^n$ then we define

$$C_c^\infty(\Omega) = \{\varphi \text{ smooth} : supp(\varphi) \subset \Omega\}.$$

Definition (Weak derivative)**.** Let the following identity hold for $u, v \in L^p(\Omega)$

$$\int\limits_{\Omega} u\varphi' \, dx = -\int\limits_{\Omega} v\varphi \, dx \qquad \forall \varphi \in C_c^\infty(\Omega)$$

then we say that v is the **weak derivative** of u. We write u' or Du in the same way as if we consider strong derivatives, although we mean the weak derivatives.

Definition (Sobolev space)**.** For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$ we define $W^{k,p}(\Omega)$ as elements of $L^p(\Omega)$ such that the kth weak derivative exists. The kth weak derivative should be an element of L^p:

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) : \forall\alpha \text{ multiindex}, |\alpha| = k \text{ the } \alpha th \text{ derivative } v^\alpha \text{ is in } L^p(\Omega)\}.$$

Remark *It is* $L^p(\Omega) = W^{0,p}$*.*

Remark *We denote only* $H^k(\Omega)$ *instead of* $W^{k,2}(\Omega)$*.*

Remark *Sobolev spaces are normed vector spaces with norm*

$$||f||_{W^{k,p}(\Omega)} = \left(\sum_{i=0}^{k} ||D^i f||^p_{L^p(\Omega)}\right)^{\frac{1}{p}}.$$

Remark *Analogously to the Lebesgue spaces it also holds that for* $p = 2$ *the Sobolev spaces* $W^{k,2}(\Omega)$ *are Hilbert spaces with scalar product*

$$(f, g)_{H^k(\Omega)} = \sum_{i=0}^{k}(D^i f, D^i g)_{L^2(\Omega)}, \qquad (f, f)_{H^k(\Omega)} = ||f||^2_{H^k(\Omega)}.$$

Remark *In addition we will use the spaces* $H^k_0(\Omega)$*. These are all the functions of* $H^k(\Omega)$ *who vanish on the boundary of* Ω*.*

Definition (Time dependent Sobolev spaces)**.** Similar to the time independant case we define the **time dependant Sobolev spaces** by

$$L^s(0, T; W^{k,p}(\Omega)) = \left\{ v(t) \in W^{k,p}(\Omega) : \int_0^T ||v(t)||^s_{W^{k,p}(\Omega)}\, dt < \infty \right\},$$

where $v(t)$ is only defined for almost every $t \in [0, T]$.

Definition (Sobolev-Slobodedzki spaces)**.** Let X be an arbitrary Sobolev space. For $1 \leq p < \infty$ und $0 < \sigma < 1$ we define the **Sobolev-Slobodedzki spaces**

$$W^{\sigma,p}(0, T; X) = \left\{v \in L^p(0, T; X) : |v|_{W^{\sigma,p}(0,T;X)} < \infty\right\},$$

where

$$|v|_{W^{\sigma,p}(0,T;X)} = \left(\int_0^T\int_0^T |s - t|^{-(1+\sigma p)}||v(s) - v(t)||^p_X\, ds\, dt\right)^{\frac{1}{p}}$$

Remark *Similar to Sobolev spaces we denote* $W^{\sigma,2}(0,T;X) = H^\sigma(0,T;X)$.

Definition (Seperable spaces)**.** A topological space is called **seperable**, if it contains a countably, dense subset.

Remark *Lebesgue spaces* $L^p(\Omega)$ *for an open, bounded* Ω *and* $1 \leq p < \infty$ *are seperable.* $L^\infty(\Omega)$ *is not seperable.*

Definition (Dual pair)**.** Let X be a normed vector space and X' the corresponding dual space. We denote the acting of $x' \in X'$ on $x \in X$ as the **dual pairing**

$$\langle x', x\rangle_{X' \times X}.$$

Definition (Reflexive space)**.** A normed space $(X, ||.||_X)$ is called a **reflexive space**, if the canonical embedding onto its bidual space is an isometrical isomorphism, that means the mapping $J : X \to X''$ with

$$< Jx, x' >_{X'' \times X'} = < x', x >_{X' \times X}$$

is surjective.

Remark *Because of the Riesz representation theorem every Hilbert space is a reflexive space.*

Remark $L^p(\Omega)$ *and* $W^{k,p}(\Omega)$ *are for open domains* Ω *and* $1 < p < \infty$ *reflexive. The spaces* $L^1(\Omega)$ *and* $L^\infty(\Omega)$ *are not reflexive.*

Definition (Continuously, compactly and densely embedded)**.** Let X,Y be normed vector spaces with norms $|| \cdot ||_X, || \cdot ||_Y$.

- X is **continuously embedded** in Y, $(X \hookrightarrow Y)$ if

$$||x||_Y \leq C||x||_X \qquad \forall x \in X.$$

- X is **densely embedded** in Y, $(X \overset{d}{\hookrightarrow} Y)$, if X is continuously embedded in Y and X is dense in Y.
- X is **compactly embedded** in Y, $(X \overset{c}{\hookrightarrow} Y)$ if X is continuously embedded in Y and the identity mapping is a compact operator, that is, for every sequence in a bounded set of X there exists a converging subsequence in Y.

Definition (Gelfand triple)**.** Let V be a real, reflexive, seperable Banach space, that is continuously and densly embedded in a Hilbert space H. Then the following relation is called a **Gelfand triple**

$$V \subseteq H \cong H' \subseteq V',$$

where H', V' are the respective dual spaces.

Remark *In the context of the NSE we will have* $V = H_0^1(\Omega), H = L^2(\Omega), V' = H^{-1}(\Omega)$.

Definition (Weak and weak* convergence)**.** A sequence $(x_k)_{k\in\mathbb{N}}$ in X **converges weakly** to $x \in X$ ($x_k \rightharpoonup x$ in X), if

$$< x', x_k >_{X'\times X} \longrightarrow < x', x >_{X'\times X} \qquad \forall x' \in X'.$$

A sequence $(x'_k)_{k\in\mathbb{N}}$ in X' **converges weak*** to $x' \in X'$ ($x'_k \overset{*}{\rightharpoonup} x'$ in X'), if

$$< x'_k, x >_{X'\times X} \longrightarrow < x', x >_{X'\times X} \qquad \forall x \in X.$$

Remark *The weak limit is unique.*

Remark *Let H be a Hilbert space. For a bounded sequence $(v_k)_{k\in\mathbb{N}} \in H$, there exists a subsequence v_{k_l} converging weakly in H.*

2.2 Inequalities

Theorem 2.1 (Hölder inequality)**.** *Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $p, q \in [1, \infty)$, then it holds*

$$\int_\Omega |fg| \, dx < ||f||_{L^p(\Omega)} \cdot ||g||_{L^q(\Omega)}.$$

Theorem 2.2 (General Hölder inequality)**.** *Let $f_i \in L^{p_i}(\Omega), i = 1, ..., n$ with $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, then it holds*

$$\int_\Omega \left| \prod_{i=1}^{n} f_i \right| dx < \prod_{i=1}^{n} ||f_i||_{L^{p_i}(\Omega)}.$$

Theorem 2.3 (Young inequality). *For $a, b \geq 0$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ it holds*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Remark *We will use a scaled version of the Young inequality, namely*

$$ab = \big((p\varepsilon)^{\frac{1}{p}} a\big)\big((p\varepsilon)^{-\frac{1}{p}} b\big) \leq \varepsilon a^p + C_\varepsilon b^q, \qquad C_\varepsilon = \frac{(p\varepsilon)^{-\frac{q}{p}}}{q}.$$

Theorem 2.4. *Let Ω be a domain with finite measure, then all $u \in L^r(\Omega)$ and all $1 \leq s \leq r$ satisfy*

$$||u||_{L^s(\Omega)} \leq c||u||_{L^r(\Omega)},$$

this means that the $L^p(\Omega)$ spaces are nested

$$L^\infty(\Omega) \subset ... \subset L^4(\Omega) \subset L^3(\Omega) \subset L^2(\Omega) \subset L^1(\Omega).$$

Theorem 2.5 (Minkowski inequality, triangle inequality for L^p spaces). *For $f, g \in L^p(\Omega)$, $1 \leq p \leq \infty$, it holds*

$$||f + g||_{L^p(\Omega)} \leq ||f||_{L^p(\Omega)} + ||g||_{L^p(\Omega)}.$$

Theorem 2.6 (Poincaré inequality). *Let $1 \leq p \leq \infty$ and Ω be a domain with finite measure, then for all $u \in (H^1(\Omega))^d$ the inequality*

$$||u||_{L^p(\Omega)} \leq C(d, \Omega)||\nabla u||_{L^p(\Omega)}$$

is satisfied.

Theorem 2.7 (Interpolation inequality). *Let Ω be a domain, then for all $u \in H_0^1(\Omega)$ the inequality*

$$||u||_{L^4} \leq c||\nabla u||_{L^2}^{\frac{3}{4}}||u||_{L^2}^{\frac{1}{4}}$$

holds in three dimensions.

Remark *For two dimensions it holds*

$$||u||_{L^4} \leq c||\nabla u||_{L^2}^{\frac{1}{2}}||u||_{L^2}^{\frac{1}{2}}.$$

Lemma 2.8 (Gronwall). *Let $T > 0$ and $t_0 \in [0, T)$, $a \in W^{1,1}(t_0, T)$ and $g, \lambda \in L^1(t_0, T)$. Starting from*

$$a'(t) \leq g(t) + \lambda(t)a(t) \qquad \text{almost everywhere in } (t_0, T)$$

the inequality

$$a(t) \leq e^{\Lambda(t)}a(t_0) + \int_{t_0}^{t} e^{\Lambda(t)-\lambda(s)}g(s)\,ds$$

holds, where $\Lambda(t) := \int_{t_0}^{t} \lambda(s)\,ds$.

Theorem 2.9 (Lions-Aubin). *Let V, H, W be Banach spaces with $V \overset{c}{\hookrightarrow} H \hookrightarrow W$ and $1 \leq p < \infty$, then it holds $\forall \sigma \in (0, 1)$*

$$L^p(0, T; V) \cap W^{\sigma,q}(0, T; W) \overset{c}{\hookrightarrow} L^p(0, T; H).$$

This means that for every bounded sequence in $L^p(0, T; V) \cap W^{\sigma,q}(0, T; W)$ there exists a subsequence converging strongly in $L^p(0, T; H)$.

Remark *Instead of seeking a fractional derivative we can demand a full time derivative in $L^q(0, T; W)$ and the same holds as well. Defining X to be:*

$$X = \{u \in L^p(0, T; V) : u' \in L^q(0, T; W)\}$$

we can state another version of the Lions-Aubin theorem:

$$X \overset{c}{\hookrightarrow} L^p(0, T; H).$$

2.3 Preliminaries

This section will consider the Navier-Stokes equations in the context of functional analysis. The needed functionals will be defined and their properties will be described. Let us recall the NSE:

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f & \text{in } (0,T]\times\Omega, \\ \nabla\cdot u = 0 & \text{in } (0,T]\times\Omega, \\ u = 0 & \text{in } \partial\Omega\times(0,T), \\ u(0,\cdot) = u_0 & \text{in } \Omega\times 0. \end{cases}$$

We will consider additional functional spaces

$$V = \{v \in H_0^1(\Omega)^d : \nabla\cdot v = 0\},$$

where the divergence has to be understood in a weak sense and

$$H = \{v \in L^2(\Omega)^d : \nabla\cdot v = 0 \text{ and } v|_{\partial\Omega}\cdot n = 0\},$$

where the divergence has to be understood in a distributional sense.

The spaces V and H are closed subspaces of $H_0^1(\Omega)^d$ and $L^2(\Omega)^d$, they form a Gelfand triple $V \subset H \cong H' \subset V'$.

Remark *It holds that V is densely embedded in H and H' is densely embedded in V'.*

Remark *We will abbreviate the norms in V with only $||\cdot||$ and the norm in H with only $|\cdot|$. Since $v\in V$ vanishes on the boundary the norm of V can be considered only to be the seminorm in $H^1(\Omega)$, $||\cdot|| = |\nabla\cdot|$.*

Before we go ahead and note the weak formulation of our problem, we consider the product of the pressure gradient with $v \in V$ first.

$$(\nabla p, v)_V = (p, \nabla\cdot v)_V = 0.$$

This holds because v is divergence free, therefore the pressure term drops out of the weak formulation.

We define

$$a : V\times V \to \mathbb{R}, \qquad a(u,v) = ((u,v)) = \int_\Omega \nabla u : \nabla v\,dx = \int_\Omega \sum_{i,j=1}^d \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i},$$

$$b : V\times V\times V \to \mathbb{R}, \qquad b(u,v,w) = \int_\Omega (u\cdot\nabla)v\cdot w\,dx.$$

Remark *Properties of a and b:*

- *$a(\cdot,\cdot)$ is well-defined, bilinear and bounded, thus contuniuous.*
- *$b(\cdot,\cdot,\cdot)$ is well-defined, trilinear and bounded, thus continuous.*

- $b(\cdot,\cdot,\cdot)$ *is antisymmetric in the second and third argument that means*

$$b(u,v,w) = -b(u,w,v) \qquad \forall u,v,w \in V.$$

Weak formulation

For $f \in L^2(0,T;V')$ and $u_0 \in H$ the formulation is the following, find $u \in L^2(0,T;V)$ such that:

$$\begin{cases} \int\limits_0^T < u'(t), v > \varphi(t) + \nu((u(t),v))\varphi(t) + b(u(t),u(t),v)\varphi(t)\,dt = \int\limits_0^T < f(t), v > \varphi(t)\,dt, \\ u(0,\cdot) = u_0 \end{cases} \tag{2.1}$$

for all $v \in V$ and $\varphi \in C_c^\infty((0,T))$.

Up till now we have stated everything necessary to proceed proving existence and uniqueness of a solution for NSE.

2.4 Existence and uniqueness

In this section we will prove three theorems concerning existence and uniqueness. We will start with a uniqueness theorem. If nothing else is stated we always consider three dimensions.

2.4.1 Global existence of a weak solution

Theorem 2.10 (Global existence of a weak solution)**.** *For $u_0 \in H$ and $f \in L^2(0,T;V')$ there exists a solution $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ for the weak formulation* (2.1) *with $u' \in L^{\frac{4}{3}}(0,T;V')$.*

Proof. We will present the proof in 4 steps:

1. existence of a local solution,
2. a-priori estimates,
3. identify a limit of the local solution,
4. check the initial condition.

Part 1: Existence of a local solution

For the existence of a local solution we will use a Galerkin procedure. Let $V_m \subset V$ be a finite dimensional subspace of V with dimension m and let $\{\varphi_j\}_{j=1,...m}$ be a corresponding Galerkin basis. For all $u_m \in V_m$ we denote:

$$u_m(t) = \sum_{j=1}^{m} u_{mj}(t)\varphi_j, \qquad u'_m(t) = \sum_{j=1}^{m} u'_{mj}(t)\varphi_j.$$

The Galerkin problem can be described as:
Find $u_m(t) \in V_m$ such that

$$\begin{cases} < u'_m(t), v_m > +\nu((u_m(t), v_m)) + b(u_m(t), u_m(t), v_m) =< f(t), v_m > \quad \forall v_m \in V_m, \\ u_m(0) = P_m u_0, \end{cases} \tag{2.2}$$

where P_m is an orthogonal projection onto V_m. Recall that the pressure term drops out, because we are testing with divergence free functions. We test with the basis function φ_j of V_m. For each j we get

$$\begin{aligned} &\sum_{i=1}^{m} u'_{mi}(t)(\varphi_i, \varphi_j) + \nu \sum_{i=1}^{m} u_{mi}(t)((\varphi_i, \varphi_j)) + b(u_m(t), u_m(t), \varphi_j) =< f(t), \varphi_j > \\ \Leftrightarrow \quad & MU'_m + \nu A U_m + B(U_m) = F_m, \end{aligned}$$

where $M_{j,i} = (\varphi_i, \varphi_j)$, $A_{j,i} = ((\varphi_i, \varphi_j))$ and U_m is the coefficient vector corresponding to u_m. The matrix M is invertible and therefore we can multiply with the inverse

$$U'_m + \nu M^{-1} A U_m + M^{-1} B(U_m) = M^{-1} F_m.$$

The resulting equation is an ordinary differential equation. Because all matrices are continuous operaters, we can apply Carathéodory's theorem to obtain an absolutely continuous solution $U_m \in V_m$. We found a local solution.

Part 2: A-priori estimates

To obtain the a-priori estimates we test the Galerkin Problem (2.2) with our previously obtained solution. Before we do that, we recall the folowing identity first

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 =< u'_m(t), u_m(t) > .$$

We multiply with u_m and utilize the antisymmetry of b

$$\begin{aligned} &< u'_m(t), u_m(t) > +\nu((u_m(t), u_m(t))) + b(u_m(t), u_m(t), u_m(t)) =< f(t), u_m(t) > \\ \Leftrightarrow \quad & \frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu((u_m(t), u_m(t))) =< f(t), u_m(t) > . \end{aligned}$$

We consider the right-hand side first and apply an inequality for the dual pairing and Young's inequality

$$
\begin{aligned}
< f(t), u_m(t) > &\leq ||f(t)||_{V'} \, ||u_m(t)|| \\
&\leq \frac{1}{2\nu}||f(t)||_{V'}^2 + \frac{\nu}{2}||u_m(t)||^2.
\end{aligned}
$$

Now we put all the previous calculations together and get

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu((u_m(t), u_m(t))) =< f(t), u_m(t) > \\
\Rightarrow \quad & \frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \nu||u_m(t)||^2 \leq \frac{1}{2\nu}||f(t)||_{V'}^2 + \frac{\nu}{2}||u_m(t)||^2 \\
\Leftrightarrow \quad & \frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \frac{\nu}{2}||u_m(t)||^2 \leq \frac{1}{2\nu}||f(t)||_{V'}^2 \\
\Leftrightarrow \quad & \frac{d}{dt}|u_m(t)|^2 + \nu||u_m(t)||^2 \leq \frac{1}{\nu}||f(t)||_{V'}^2.
\end{aligned}
$$

We integrate over $(0, t)$

$$
|u_m(t)|^2 + \nu\int_0^t ||u_m(s)||^2 \, ds \leq \frac{1}{\nu}\int_0^t ||f(s)||_{V'}^2 \, ds + |u_m(0)| \leq \frac{1}{\nu}||f||_{L^2(0,T;V')}^2 + |u_0|.
$$

It is important to note, that the right-hand side is independent of m. Because of that we can conclude $u_m \in L^2(0,T;V) \cap L^\infty(0,T;H)$. Therefore there exists a subsequence $u_{m_l} \in L^2(0,T;V) \cap L^\infty(0,T;H)$ and $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ such that

$$
\begin{aligned}
u_{m_l} &\overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;H), \\
u_{m_l} &\rightharpoonup \tilde{u} \quad \text{in } L^2(0,T;V).
\end{aligned}
$$

Let us recall that the dual space of $L^1(0,T;H)$ is $L^\infty(0,T;H)$ and the dual space of $L^2(0,T;V)$ is $L^2(0,T;V')$. Choosing our test space to be a subspace of the overlap of $L^1(0,T;H)$ and $L^2(0,T;V')$, for example $L^2(0,T;H)$ we conclude that $u = \tilde{u}$ almost everywhere.

Teil 3: Identify a limit of the local solution

Since we will only consider the converging subsequence we will denote u_{m_l} by u_m. We managed to show weak convergence so far, but this is not enough to show that the found limit is actually the solution we are looking for. We are looking for a limit such that all terms of (2.2) converge. It is not sufficient to control the non-linear term with only weak convergence, strong convergence is needed. Showing strong convergence is our next step. We initially consider $v, \bar{v}, w \in V$

$$
\begin{aligned}
&\Big|b(v,v,w)-b(\bar v,\bar v,w)\Big| \\
=&\Big|b(v,v,w)-b(\bar v,v,w)+b(\bar v,v,w)-b(\bar v,\bar v,w)\Big| \\
=&\Big|b(v-\bar v,v,w)-b(\bar v,\bar v-v,w)\Big| \\
\leq&\Big|b(v-\bar v,v,w)\Big|+\Big|b(\bar v,\bar v-v,w)\Big| \\
=&\Big|b(v-\bar v,w,v)\Big|+\Big|b(\bar v,w,\bar v-v)\Big|.
\end{aligned}
$$

We apply the general Hölder inequality $p_1 = p_3 = 4, p_2 = 2$ and the interpolation inequality

$$
\begin{aligned}
\leq& c||v-\bar v||_{L^4}||\nabla w||_{L^2}||v||_{L^4}+||\bar v||_{L^4}||\nabla w||_{L^2}||\bar v-v||_{L^4} \\
\leq& c||w||_{H^1}||v-\bar v||_{L^4}\Big(||v||_{L^4}+||\bar v||_{L^4}\Big) \\
\leq& c||w||_{H^1}||v-\bar v||^{\frac34}|v-\bar v|^{\frac14}\Big(||v||^{\frac34}|v|^{\frac14}+||\bar v||^{\frac34}|\bar v|^{\frac14}\Big).
\end{aligned}
$$

Restating the above inequality in operators results in

$$
||B(v)-B(\bar v)||_{V'}\leq c||v-\bar v||^{\frac34}|v-\bar v|^{\frac14}\Big(||v||^{\frac34}|v|^{\frac14}+||\bar v||^{\frac34}|\bar v|^{\frac14}\Big). \tag{2.3}
$$

We choose $v, \bar v \in V$ and therefore we can control all V-norms, we merely need to control the H-norm and make sure that $|v-\bar v|^{\frac14}$ doesn't become too large.
To motivate the next step we consider the following calculations first, consider $v, \bar v \in L^\infty(0,T;H)\cap L^2(0,T;V)$ and $1 \leq r < \frac43$

$$
||B(v)-B(\bar v)||^r_{L^r(0,T;V')} = \int_0^T ||B(v)-B(\bar v)||^r_{V'}\,dt.
$$

We insert (2.3):

$$
\leq \int_0^T \Big(c||v-\bar v||^{\frac34}|v-\bar v|^{\frac14}\Big(||v||^{\frac34}|v|^{\frac14}+||\bar v||^{\frac34}|\bar v|^{\frac14}\Big)\Big)^r\,dt.
$$

We use the fact that $(a+b)^n \leq c(a^n+b^n)$ holds for $a, b > 0, n \geq 1$:

$$
\begin{aligned}
\leq& \int_0^T \Big(c||v-\bar v||^{\frac34}|v-\bar v|^{\frac14}\Big(||v||^{\frac34}|v|^{\frac14}_{L^\infty(0,T;H)}+||\bar v||^{\frac34}|\bar v|^{\frac14}_{L^\infty(0,T;H)}\Big)\Big)^r\,dt \\
\leq& \int_0^T \Big(c||v-\bar v||^{\frac34}|v-\bar v|^{\frac14}\Big(||v||^{\frac34}+||\bar v||^{\frac34}\Big)\Big(|v|^{\frac14}_{L^\infty(0,T;H)}+|\bar v|^{\frac14}_{L^\infty(0,T;H)}\Big)\Big)^r\,dt \\
\leq& c\Big(|v|^{\frac r4}_{L^\infty(0,T;H)}+|\bar v|^{\frac r4}_{L^\infty(0,T;H)}\Big)\int_0^T ||v-\bar v||^{\frac{3r}{4}}|v-\bar v|^{\frac r4}\Big(||v||^{\frac{3r}{4}}+||\bar v||^{\frac{3r}{4}}\Big)\,dt.
\end{aligned}
$$

Using the Minkowski inequality for $||v-\bar{v}||^{\frac{3r}{4}}$ simplifies out expression.

$$
\begin{aligned}
\leq & c\Big(|v|^{\frac{r}{4}}_{L^\infty(0,T;H)}+|\bar{v}|^{\frac{r}{4}}_{L^\infty(0,T;H)}\Big)\int_0^T\Big(||v||^{\frac{3r}{4}}+||\bar{v}||^{\frac{3r}{4}}\Big)|v-\bar{v}|^{\frac{r}{4}}\Big(||v||^{\frac{3r}{4}}+||\bar{v}||^{\frac{3r}{4}}\Big)\,dt \\
\leq & c\underbrace{\Big(|v|^{\frac{r}{4}}_{L^\infty(0,T;H)}+|\bar{v}|^{\frac{r}{4}}_{L^\infty(0,T;H)}\Big)}_{\gamma}\int_0^T|v-\bar{v}|^{\frac{r}{4}}\Big(||v||^{\frac{3r}{2}}+||\bar{v}||^{\frac{3r}{2}}\Big)\,dt \\
= & c\gamma\int_0^T|v-\bar{v}|^{\frac{r}{4}}||v||^{\frac{3r}{2}}\,dt+c\gamma\int_0^T|v-\bar{v}|^{\frac{r}{4}}||\bar{v}||^{\frac{3r}{2}}\,dt.
\end{aligned}
$$

Because $u\in L^2(0,T;V)$ we can restrain all $||.||$-norms. Using Hölder's inequality we can enforce the exponent of the V-norms to be 2. Choosing $p=\frac{4}{3r}$ results in $\frac{3r}{2}\cdot p=2$. For q we get $q=\frac{4}{4-3r}$.

$$
\begin{aligned}
\leq & c\gamma\left(\int_0^T|v-\bar{v}|^{\frac{r}{4}\frac{4}{4-3r}}\,dt\right)^{\frac{4-3r}{4}}\left(\int_0^T||v||^2\,dt\right)^{\frac{3r}{4}}+c\gamma\left(\int_0^T|v-\bar{v}|^{\frac{r}{4}\frac{4}{4-3r}}\,dt\right)^{\frac{4-3r}{4}}\left(\int_0^T||\bar{v}||^2\,dt\right)^{\frac{3r}{4}} \\
\leq & c\gamma\left(\int_0^T|v-\bar{v}|^{\frac{r}{4-3r}}\,dt\right)^{\frac{4-3r}{4}}\left(\left(\int_0^T||v||^2\,dt\right)^{\frac{3r}{4}}+\left(\int_0^T||\bar{v}||^2\,dt\right)^{\frac{3r}{4}}\right) \\
\leq & c\Big(|v|^{\frac{r}{4}}_{L^\infty(0,T;H)}+|\bar{v}|^{\frac{r}{4}}_{L^\infty(0,T;H)}\Big)\Big(||v||^{\frac{3r}{2}}_{L^2(0,T;V)}+||\bar{v}||^{\frac{3r}{2}}_{L^2(0,T;V)}\Big)||v-\bar{v}||^{\frac{r}{4}}_{L^{\frac{r}{4-3r}}(0,T;H)}.
\end{aligned}
$$

If we can manage to achieve strong convergence in $L^s(0,T;H)$ for $s=\frac{r}{4-3r}$ and $r\in[1,\frac{4}{3})$, then we get strong convergence of the non-linear term in $L^r(0,T;V')$. We will use Lion-Aubin's lemma to show it. We note that it is enough to show that

$$u_m\in L^2(0,T;V)\cap H^\sigma(0,T;H)$$

for $\sigma\in(0,1)$ to ensure compactness in $L^2(0,T;H)$. We will show compactness for $s=2$, thus $r=\frac{8}{7}$.
Let us us consider $|u_m|^2_{H^\sigma(0,T;H)}$ now and estimate it from above independent of m.

$$
\begin{aligned}
|u_m|^2_{H^\sigma(0,T;H)} &= \int_0^T\int_0^T|t-s|^{-(1+2\sigma)}|u_m(t)-u_m(s)|^2\,ds\,dt \\
&= \int_0^T\int_0^T|t-s|^{-(1+2\sigma)}(u_m(t)-u_m(s),u_m(t)-u_m(s))\,ds\,dt \\
&= \int_0^T\int_0^T|t-s|^{-(1+2\sigma)}(u_m(t)-u_m(s),u_m(t))\,ds\,dt \\
&\quad -\int_0^T\int_0^T|t-s|^{-(1+2\sigma)}(u_m(t)-u_m(s),u_m(s))\,ds\,dt
\end{aligned}
$$

We apply Fubini's theorem and rename the variables

$$
\begin{aligned}
=&\int_0^T\int_0^T |t-s|^{-(1+2\sigma)}(u_m(t)-u_m(s),u_m(t))\,ds\,dt\\
&\quad -\int_0^T\int_0^T |t-s|^{-(1+2\sigma)}(u_m(s)-u_m(t),u_m(t))\,ds\,dt\\
=&\int_0^T\int_0^T |t-s|^{-(1+2\sigma)}(u_m(t)-u_m(s),u_m(t))\,ds\,dt\\
&\quad +\int_0^T\int_0^T |t-s|^{-(1+2\sigma)}(u_m(t)-u_m(s),u_m(t))\,ds\,dt\\
=&2\int_0^T\int_0^T |t-s|^{-(1+2\sigma)}(u_m(t)-u_m(s),u_m(t))\,ds\,dt.
\end{aligned}
$$

We used Carathéodory's theorem to optain a solution of the Galerkin problem. Since the solution u_m is absolutely continuous, it justifies our next step.

$$
\begin{aligned}
=&2\int_0^T\int_0^T\int_s^t |t-s|^{-(1+2\sigma)}(u_m'(\tau),u_m(t))\,d\tau\,ds\,dt\\
=&2\int_0^T\int_0^T\int_s^t |t-s|^{-(1+2\sigma)}(f(\tau)-\nu Au_m(\tau)-Bu_m(\tau),u_m(t))\,d\tau\,ds\,dt\\
=&2\int_0^T\int_0^T\int_s^t |t-s|^{-(1+2\sigma)}(f(\tau)),u_m(t))\,d\tau\,ds\,dt\\
&\quad -2\int_0^T\int_0^T\int_s^t |t-s|^{-(1+2\sigma)}(\nu Au_m(\tau),u_m(t))\,d\tau\,ds\,dt\\
&\quad -2\int_0^T\int_0^T\int_s^t |t-s|^{-(1+2\sigma)}(Bu_m(\tau),u_m(t))\,d\tau\,ds\,dt\\
=&(f-term)+(A-term)+(B-term).
\end{aligned}
$$

Considering the triple integral, we can use the substitution rule to reformulate it into a more suitable expression:

$$
\begin{aligned}
\int_0^T\int_0^T\int_s^t d\tau\,ds\,dt &= \int_0^T\int_0^t\int_s^t d\tau\,ds\,dt+\int_0^T\int_t^T\int_s^t d\tau\,ds\,dt\\
&=\int_0^T\int_{\tau=0}^t\int_{s=0}^\tau ds\,d\tau\,dt-\int_0^T\int_{\tau=t}^T\int_{s=\tau}^T ds\,d\tau\,dt.
\end{aligned}
$$

We use the reformulation above in all terms and estimate them from above, starting with the $B - term$.

$$B - Term = 2\int_0^T\int_0^T\int_s^t |t-s|^{-(1+2\sigma)}(Bu_m(\tau), u_m(t))\, d\tau\, ds\, dt$$

$$\begin{aligned}
\leq& 2\int_0^T\int_{\tau=0}^t\int_{s=0}^\tau |t-s|^{-(1+2\sigma)}||Bu_m(\tau)||_{V'}||u_m(t)||\, ds\, d\tau\, dt \\
&\quad + 2\int_0^T\int_{\tau=t}^T\int_{s=\tau}^T |s-t|^{-(1+2\sigma)}||Bu_m(\tau)||_{V'}||u_m(t)||\, ds\, d\tau\, dt \\
=& 2\int_0^T\int_{\tau=0}^t ||Bu_m(\tau)||_{V'}||u_m(t)|| \left(\int_{s=0}^\tau |t-s|^{-(1+2\sigma)}\, ds\right) d\tau\, dt \\
&\quad + 2\int_0^T\int_{\tau=t}^T ||Bu_m(\tau)||_{V'}||u_m(t)|| \left(\int_{s=\tau}^T |s-t|^{-(1+2\sigma)}\, ds\right) d\tau\, dt \\
=& 2\int_0^T\int_{\tau=0}^t ||Bu_m(\tau)||_{V'}||u_m(t)|| \left(\frac{1}{2\sigma}\Big((t-\tau)^{-2\sigma} - t^{-2\sigma}\Big)\right) d\tau\, dt \\
&\quad + 2\int_0^T\int_{\tau=t}^T ||Bu_m(\tau)||_{V'}||u_m(t)|| \left(\frac{1}{-2\sigma}\Big((T-t)^{-2\sigma} - (\tau-t)^{-2\sigma}\Big)\right) d\tau\, dt \\
=& \frac{1}{\sigma}\int_0^T\int_{\tau=0}^t ||Bu_m(\tau)||_{V'}||u_m(t)||(t-\tau)^{-2\sigma}\, d\tau\, dt - \frac{1}{\sigma}\int_0^T\int_{\tau=0}^t ||Bu_m(\tau)||_{V'}||u_m(t)||t^{-2\sigma}\, d\tau\, dt \\
&- \frac{1}{\sigma}\int_0^T\int_{\tau=t}^T ||Bu_m(\tau)||_{V'}||u_m(t)||(T-t)^{-2\sigma}\, d\tau\, dt + \frac{1}{\sigma}\int_0^T\int_{\tau=t}^T ||Bu_m(t)||_{V'}||u_m(t)||(\tau-t)^{-2\sigma}\, d\tau\, dt \\
=& \frac{1}{\sigma}(A - B - C + D).
\end{aligned}$$

We will only consider A because the other terms can be treated analogously.

$$A = \int_0^T\int_{\tau=0}^t ||Bu_m(\tau)||_{V'}||u_m(t)||(t-\tau)^{-2\sigma}\, d\tau\, dt.$$

Using Fubini's theorem and the Hölder inequality results in

$$
\begin{aligned}
&= \int_0^T \int_{t=\tau}^T ||Bu_m(\tau)||_{V'} ||u_m(t)|| (t-\tau)^{-2\sigma} \, dt \, d\tau \\
&= \int_0^T \left(\int_{t=\tau}^T ||u_m(t)|| (t-\tau)^{-2\sigma} \, dt \right) ||Bu_m(\tau)||_{V'} \, d\tau \\
&\leq c \int_0^T \underbrace{\left(\int_\tau^T ||u_m(t)||^2 \, dt \right)^{\frac{1}{2}}}_{\leq ||u_m||_{L^2(0,T,V)} \leq \infty} \underbrace{\left(\int_\tau^T (t-\tau)^{-4\sigma} \, dt \right)^{\frac{1}{2}}}_{<\infty \Leftrightarrow \sigma < \frac{1}{4}} ||Bu_m(\tau)||_{V'} \, d\tau \\
&\leq C \int_0^T ||Bu_m(\tau)||_{V'} \, d\tau \\
&= C \int_0^T \sup_{v \in V, ||v||=1} \Big| B\big(u_m(\tau), u_m(\tau), v\big) \Big| \, d\tau = C \int_0^T \sup_{v \in V, ||v||=1} \Big| -B\big(u_m(\tau), v, u_m(\tau)\big) \Big| \, d\tau \\
&\leq C \int_0^T ||u_m(\tau)||_{L^4} \underbrace{||\nabla v||_{L^2}}_{\leq 1} ||u_m(\tau)||_{L^4} \, d\tau \\
&\leq C \int_0^T ||u_m(\tau)||_{L^4}^2 \, d\tau \leq C \int_0^T \underbrace{|u_m(\tau)|^{\frac{1}{2}}}_{\leq ||u_m||_{L^\infty(H)}^{\frac{1}{2}} \leq const} ||u_m(\tau)||^{\frac{3}{2}} \, d\tau \\
&\leq C ||u_m||_{L^{\frac{3}{2}}(V)}^{\frac{3}{2}} \leq C ||u_m||_{L^2(V)}^{\frac{3}{2}} \leq const.
\end{aligned}
$$

Analogously a similar result can be shown for B, C and D and also for the $f-Term$ and the $A-Term$. Thereby we showed

$$|u_m|_{H^\sigma(0,T;H)}^2 \leq const,$$

holds for $\sigma < \frac{1}{4}$. From Lions-Aubin's theorem we can conclude that

$$L^2(0,T;V) \cap H^\sigma(0,T;H) \overset{c}{\hookrightarrow} L^2(0,T;H),$$

which means that in our case it holds that

$$Bu_m \to Bu \qquad \text{in } L^2(0,T,H).$$

So far we have showed everything we need to show to proove that the limit we found in part 2 solves the weak NSE.

Let us recall our Galerkin problem: find $u_m \in V_m$ such that

$$\int_0^T < u_m'(t), v > \varphi(t) + \nu((u_m(t), v))\varphi(t) + (b(u_m(t), u_m(t), v)\varphi(t)\,dt = \int_0^T < f(t), v > \varphi(t)\,dt$$

holds for all $v \in V_m$ and $\varphi \in C_c^\infty((0,T))$. Let us consider each term seperately.

Because $u_m \overset{*}{\rightharpoonup} u$ holds in $L^\infty(0,T;H)$ we conclude

$$\begin{aligned}\int_0^T < u_m'(t), v > \varphi(t)\,dt &= -\int_0^T < u_m(t), v > \varphi'(t)\,dt \\ &\longrightarrow -\int_0^T < u(t), v > \varphi'(t)\,dt = \int_0^T < u'(t), v > \varphi(t)\,dt.\end{aligned}$$

Because $u_m \rightharpoonup u$ holds in $L^2(0,T;V)$ we conclude

$$\int_0^T \nu((u_m(t), v))\varphi(t)\,dt \longrightarrow \int_0^T \nu((u(t), v))\varphi(t)\,dt.$$

Because of Lions-Aubin's theorem we get

$$\int_0^T (b(u_m(t), u_m(t), v)\varphi(t)\,dt \longrightarrow \int_0^T b(u(t), u(t), v)\varphi(t)\,dt.$$

We now know that:

$$\int_0^T < u'(t), v > \varphi(t)\,dt + \int_0^T \nu((u(t), v))\varphi(t)\,dt + \int_0^T b(u(t), u(t), v)\varphi(t)\,dt = \int_0^T (f, v)\varphi(t)\,dt,$$

for all $v \in V_m, m \in \mathbb{N}, \varphi(t) \in C_c^\infty((0,T))$. We get the statement we want by stating the completeness of the Galerkin approximation:

$$\bigcup_{m \in \mathbb{N}} V_m = V.$$

As a consequence every term converges, therefore the whole equation converges. If the limit of u coincides with u_0 for $t \to 0$, then we showed existence of a solution. Before we check the initial condition we consider the time derivative first.
For the limit $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ it holds using the interpolation inequality that

$$|b(u,u,w)| = |b(u,w,u)| \le ||u||^2_{L^4}||\nabla w||_{L^2} \le c||\nabla u||^{\frac{3}{2}}_{L^2}||u||^{\frac{1}{2}}_{L^2}||\nabla w||_{L^2} \le C||u||^{\frac{3}{2}}||\nabla w||_{L^2}.$$

Equivalently it also holds that

$$||B(u)||^{\frac{4}{3}}_{L^{\frac{4}{3}}(0,T;V')} \le \int_0^T (||u(t)||^{\frac{3}{2}})^{\frac{4}{3}}\, dt = \int_0^T ||u(t)||^2\, dt < \infty.$$

Therefore we can conclude that $u' \in L^{\frac{4}{3}}(0,T;V')$ because

$$u' = f - Au - Bu \in L^{\frac{4}{3}}(0,T;V') \tag{2.4}$$

Part 4: check initial condition

As the last step, we need to show that the initial condition is satisfied by our candidate u. We have to show that $u(0) = u_0$ holds. Therefore we consider an arbitrary $v \in V_m$ and do the following calculation.

$$-(u(0),v) = \Big[(u(t),v)\frac{T-t}{T}\Big]_{t=0}^{T} = \int_0^T \frac{d}{dt}\left((u(t),v)\frac{T-t}{T}\right) dt$$
$$= \int_0^T (u'(t),v)\frac{T-t}{T}\, dt - \int_0^T (u(t),v)\frac{1}{T}\, dt.$$

We insert equation (2.4) for $u'(t)$ and add a zero in form of the discrete solution $u_m(t)$

$$= \int_0^T \Big((f(t),v) - \nu((u(t),v)) - b(u(t),u(t),v)\Big)\frac{T-t}{T}\, dt - \int_0^T (u(t),v)\frac{1}{T}\, dt$$
$$+ \int_0^T \Big((u_m'(t) - f(t),v) + \nu((u_m(t),v)) + b(u_m(t),u_m(t),v)\Big)\frac{T-t}{T}\, dt$$
$$= \nu\int_0^T \Big(((u_m(t),v)) - ((u(t),v))\Big)\frac{T-t}{T}\, dt + \int_0^T \Big(b(u_m(t),u_m(t),v) - b(u(t),u(t),v)\Big)\frac{T-t}{T}\, dt$$
$$+ \int_0^T (u_m'(t),v)\frac{T-t}{T}\, dt - \int_0^T (u(t),v)\frac{1}{T}\, dt.$$

Considering $\lim\limits_{m\to\infty}$ results in the first and second term canceling each other out and the remaining terms yield:

$$
\begin{aligned}
&\lim_{m\to\infty}\int_0^T (u_m'(t),v)\frac{T-t}{T} - (u(t),v)\frac{1}{T}\,dt \\
&= \lim_{m\to\infty}\left(\left[(u_m(t),v)\frac{T-t}{T}\right]_0^T + \underbrace{\int_0^T (u_m(t),v)\frac{1}{T}\,dt - \int_0^T (u(t),v)\frac{1}{T}\,dt)}_{\longrightarrow 0}\right) \\
&= \lim_{m\to\infty} -(u_m(0),v) = -(u_0,v).
\end{aligned}
$$

We showed that the initial values coincide for $v \in V_m, m \in \mathbb{N}$, and again with the completeness of the Galerkin approximation the above statement holds for $v \in V$. We hereby showed that our limit u satisfies the initial condition and therefore is a weak solution of (2.2). □

Remark *So far we showed that there exists a weak solution $u \in L^2(0,T;V)\cap L^\infty(0,T;H)$ however we only get $u' \in L^{\frac{4}{3}}(0,T;V')$. Furthermore we have no uniqueness.*

2.4.2 Uniqueness of a solution

Theorem 2.11 (Uniqueness)**.** *There is no more then one solution $u \in L^\infty(0,T;H)\cap L^2(0,T;V)$ such that*

$$u \in L^8(0,T;L^4(\Omega)).$$

Proof. Let $u \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap L^8(0,T;L^4(\Omega))$. First of all we state the following:

$$
\begin{aligned}
||Bu||^2_{L^2(0,T;V')} &= \int_0^T ||Bu(t)||^2_{V'}\,dt \\
&= \int_0^T \sup_{v\in V,||v||=1} |\,(b(u(t),u(t),v)\,|\,dt = \int_0^T \sup_{v\in V,||v||=1} |-(b(u(t),v,u(t))\,|\,dt \\
&\leq c\int_0^T \sup_{v\in V,||v||=1} ||u(t)||^2_{L^4(\Omega)}\,||v||\,dt = c\int_0^T ||u(t)||^2_{L^4(\Omega)}\,dt \\
&\leq c\left(\int_0^T ||u(t)||^8_{L^4(\Omega)}\,dt\right)^{\frac{1}{4}}\left(\int_0^T 1^{\frac{3}{4}},dt\right)^{\frac{4}{3}} = c||u||^2_{L^8(0,T;L^4)} < \infty.
\end{aligned}
$$

This means that $Bu \in L^2(0,T;V')$. Let us consider our equation again and reorder it:

$$u' = f - Au - Bu \in L^2(0,T;V') \Longrightarrow u \in L^2(0,T;V').$$

Now we can start with the proof, let us consider two solutions $u, \bar{u}$ of

$$\begin{aligned} < u'(t), v > + \nu((u,v)) + b(u,u,v) &= < f, v > \qquad \forall v \in V, \\ u(0) &= u_0. \end{aligned}$$

We subtract both solutions from each other and utilize the linearity of $((\cdot,\cdot))$.

$$< u'(t) - \bar{u}'(t), v > + \nu((u(t) - \bar{u}(t), v)) + b\Big(u(t), u(t), v\Big) - b\Big(\bar{u}(t), \bar{u}(t), v\Big) = 0$$

Because $u(t), \bar{u}(t) \in V$, we can use $v = u(t) - \bar{u}(t)$ as a testfunction. For the first term we get

$$< u'(t) - \bar{u}'(t), u(t) - \bar{u}(t) > = \frac{1}{2}\frac{d}{dt}|u(t) - \bar{u}(t)|^2$$

and for the second we get

$$((u(t) - \bar{u}(t), u(t) - \bar{u}(t))) = ||\nabla(u(t) - \bar{u}(t))||_{L^2}^2.$$

For the whole equation we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||u(t) - \bar{u}(t)||^2 + \nu||\nabla(u(t) - \bar{u}(t))||_{L^2}^2 = & b\Big(\bar{u}(t), \bar{u}(t), u(t) - \bar{u}(t)\Big) \\ & - b\Big(u(t), u(t), u(t) - \bar{u}(t)\Big) \end{aligned} \tag{2.5}$$

Let us now consider the right-hand side of the equation and by adding a fitting zero we can simplify it:

$$\begin{aligned} & b\Big(\bar{u}(t), \bar{u}(t), u(t) - \bar{u}(t)\Big) - b\Big(u(t), \bar{u}(t), u(t) - \bar{u}(t)\Big) + b\Big(u(t), \bar{u}(t), u(t) - \bar{u}(t)\Big) \\ & \qquad - b\Big(u(t), u(t), u(t) - \bar{u}(t)\Big) \\ = & b\Big(\bar{u}(t) - u(t), \bar{u}(t), u(t) - \bar{u}(t)\Big) - \underbrace{b\Big(u(t), u(t) - \bar{u}(t), u(t) - \bar{u}(t)\Big)}_{=0} \\ = & b\Big(u(t) - \bar{u}(t), u(t) - \bar{u}(t), \bar{u}(t)\Big) \\ \leq & c||u(t) - \bar{u}(t)||_{L^4} \underbrace{||\nabla(u(t) - \bar{u}(t))||_{L^2}}_{=||u(t) - \bar{u}(t)||} ||\bar{u}(t)||_{L^4} \\ \leq & c||u(t) - \bar{u}(t)||_{L^2}^{\frac{1}{4}} ||u(t) - \bar{u}(t)||^{\frac{3}{4}} ||u(t) - \bar{u}(t)|| \; ||\bar{u}(t)||_{L^4} \\ \leq & \underbrace{||u(t) - \bar{u}(t)||^{\frac{7}{4}}}_{a} \underbrace{c||u(t) - \bar{u}(t)||_{L^2}^{\frac{1}{4}} \; ||\bar{u}(t)||_{L^4}}_{b} \end{aligned}$$

Using the scaled Young inequality with $p = \frac{8}{7}$, $q = 8$ und $\varepsilon = \frac{\nu}{2}$ reaches a more suitable form

$$\leq c\frac{\nu}{2}||u(t) - \bar{u}(t)||^2 + cC_\nu|u(t) - \bar{u}(t)|^2||\bar{u}(t)||_{L^4}^8.$$

Inserting what we calculated so far for the right-hand side into (2.5) almost finishes the proof.

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|u(t) - \bar{u}(t)|^2 + \nu||\nabla(u(t) - \bar{u}(t))||_{L^2}^2 \quad &\leq \frac{\nu}{2}||u(t) - \bar{u}(t)||^2 \\
&\quad + cC_\nu|u(t) - \bar{u}(t)|^2||\bar{u}(t)||_{L^4}^8 \\
\Rightarrow \frac{1}{2}\frac{d}{dt}|u(t) - \bar{u}(t)|^2 + \underbrace{\frac{\nu}{2}||u(t) - \bar{u}(t)||^2}_{\geq 0} \quad &\leq cC_\nu|u(t) - \bar{u}(t)|^2||\bar{u}(t)||_{L^4}^8.
\end{aligned}$$

We apply Gronwalls inequality.

$$\begin{aligned}
\Rightarrow \frac{d}{dt}|u(t) - \bar{u}(t)|^2 \quad &\leq \underbrace{|u(t) - \bar{u}(t)|^2}_{a} \underbrace{\tilde{c}C_\nu||\bar{u}(t)||_{L^4}^8}_{\lambda(t)} \\
\Rightarrow |u(t) - \bar{u}(t)|^2 \quad &\leq \underbrace{|u(0) - \bar{u}(0)|}_{=0} e^{\Lambda(t)}.
\end{aligned}$$

As a consequence of the last inequality we have shown uniqueness. □

2.4.3 Existence and uniqueness for small initial data

Before we can state the theorem and prove it, we need to review a Sobolev's embedding theorem and another interpolation inequality.

Theorem 2.12 (Soboloev theorem)**.** *Let $k_1, k_2 \in \mathbb{N}, k_2 \leq k_2, 1 \leq p_1, p_2 < \infty$ satisfying*

$$k_1 - \frac{d}{p_1} \geq k_2 - \frac{d}{p_2}$$

where d is the dimension. Then it holds that:

$$W^{k_1,p_1} \hookrightarrow W^{k_2,p_2}$$

which means that $\forall v \in W^{k_1,p_1}$ it holds

$$||v||_{W^{k_2,p_2}} \leq c||v||_{W^{k_1,p_1}}.$$

Remark *Later we will apply the theorem above on the spaces $L^6(\Omega)$ and $L^3(\Omega)$ to get the following embeddings for three dimensions*

$$L^6 \hookrightarrow H^1, \text{ and } L^3 \hookrightarrow H^{\frac{1}{2}}.$$

Theorem 2.13 (Sobolev embedding)**.** *Let Ω be a domain with Lipschitz boundary, $m \geq 1, 1 \leq p < \infty, k \geq 0, 0 < \alpha < 1$ satisfying*

$$k - \frac{d}{p} = k + \alpha,$$

where d is the dimension. Then it holds that

$$W^{k,p}(\Omega) \hookrightarrow C^{k,\alpha}(\bar{\Omega}).$$

Remark *This theorem will later be the main argument for the uniqueness.*

Theorem 2.14 (Interpolation in Sobolev spaces)**.** *Let $0 \leq s_1 < s_2$ and $\theta \in (0,1)$, then it holds $\forall \varphi \in H^{s_2}$ satisfying $s = \theta s_1 + (1-\theta)s_2$*

$$||\varphi||_{H^s} \leq c||\varphi||^{\theta}_{H^{s_1}}||\varphi||^{1-\theta}_{H^{s_2}}.$$

Theorem 2.15 (Existence and uniqueness for small initial data)**.** *Let Ω be an open and bounded domain with a C^2 boundary, $u_0 \in V, f \in L^\infty(0,T;H)$, then there exists a $T^* = T^*(u_0, f, \nu)$ such that there exists a unique solution for* (2.1) *satisfying*

$$\begin{aligned} u &\in L^\infty(0,T^*;V) \cap L^2(0,T^*;H^2(\Omega)), \\ u' &\in L^2(0,T^*;H). \end{aligned}$$

Proof. The idea of this proof is based on using eigenfunctions of the Stokes operator as testfunctions. We define

$$A : V \to V', \qquad A(v) = a(v,\cdot) = \langle Av, \cdot \rangle = ((v,\cdot)).$$

Now we consider the Friedrichs' extension to the operator A

$$A_F : \mathcal{D}(A) \to H,$$

where

$$\mathcal{D}(A) = \{v \in V : Av \in H\} = (H^2(\Omega))^3 \cap V.$$

A_F can be considered as an operator acting on H, $A_F : H \to H$ and is a linear, bounded and symmetric operator. Consequently A_F is selfadjoint. Using Lax-Milgram and the fact that $V \overset{c}{\hookrightarrow} H$ we conclude that A_F^{-1} is compact. For a more detailed analysis and the proved statement of the Friedrichs' extension, refer to [Tri97], chapter 4.1.9. The Hilbert-Schmidt theorem, see [RR04], Theorem 8.94, provides us with a set of eigenfunctions $\{\varphi_k\} \subset \mathcal{D}(A)$ and their corresponding eigenvalues $\{\lambda_k\}$ such that the eigenfunctions form a complete set in H. Since the eigenfunctions are a Galerkin basis in V, we can define $V_m = span\{\varphi_1, \varphi_2, \cdots \varphi_m\}$. Our Galerkin problem can be formulated in the following way: find $u_m(t) \in V_m$ such that

$$\begin{cases} < u'_m(t), v_m > +\nu((u_m(t), v_m)) + b(u_m(t), u_m(t), v_m) =< f(t), v_m > \quad \forall v_m \in V_m, \\ u_m(0) = P_m u_0 \end{cases}$$

holds.
As our first step we will test the equation with $v_m = Au_m$. Before we go into detail we will do some auxiliary calculations first.

$$((u_m(t), Au_m(t))) = (Au_m(t), Au_m(t)) = ||Au_m(t)||^2_{L^2} = |Au_m(t)|^2$$

and

$$< u'_m(t), Au_m(t) > = \frac{1}{2}\frac{d}{dt}||u_m(t)||^2.$$

These results are fundamental for the proof.
We use $v_m = Au_m(t)$ as a testfunction and get

$$\frac{1}{2}\frac{d}{dt}||u_m(t)||^2 + \nu|Au_m(t)|^2 + b(u_m(t), u_m(t), Au_m(t)) =< f(t), Au_m(t) > .$$

Existence of a local solution and a-priori estimates

Local existence is again a result of Carethéodory's theorem. For the a-priori estimates we consider

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||u_m(t)||^2 + \nu|Au_m(t)|^2 &\leq |< f(t), Au_m(t) >| + |b(u_m(t), u_m(t), Au_m(t))|. \\ &\leq ||f||_{V'}|Au_m(t)| + c||u_m(t)||_{L^6}||\nabla u_m(t)||_{L^3}||Au_m(t)||_{L^2} \end{aligned}$$

Using Sobolev's theorem we can estimate the L^6 norms by the H^1 Norms. The L^3 norm can be estimated by the $H^{\frac{1}{2}}$ norm and using the interpolation inequality with $S_1 = 0, s_2 = 1$ we can estimate the $H^{\frac{1}{2}}$ norm by the product of the L^2 and H^1 norms.

$$\begin{aligned} &\leq ||f||_{V'}|Au_m(t)| + c||u_m(t)||\ ||\nabla u_m(t)||^{\frac{1}{2}}_{L^2}||\nabla u_m(t)||^{\frac{1}{2}}_{H^1}||Au_m(t)||_{L^2} \\ &\leq ||f||_{V'}|Au_m(t)| + c||u_m(t)||\ ||u_m(t)||^{\frac{1}{2}}||u_m(t)||^{\frac{1}{2}}_{H^2}||Au_m(t)||_{L^2} \\ &= ||f||_{V'}|Au_m(t)| + c||u_m(t)||^{\frac{3}{2}}||u_m(t)||^{\frac{1}{2}}_{H^2}||Au_m(t)||_{L^2}. \end{aligned}$$

Using Cattabriga's theorem, stating that the Laplacian controls the H^2 norm, that means $||u_m(t)||_{H^2} \leq c||Au_m(t)||_{L^2}$, we get the following:

$$= ||f||_{V'}|Au_m(t)| + c||u_m(t)||^{\frac{3}{2}}|Au_m(t)|^{\frac{3}{2}}.$$

Using Young's inequality for the first summand with $p = q = 2, \varepsilon = \frac{\nu}{4}$ und for the second summand with $p = \frac{4}{3}, q = 4, \varepsilon = \frac{\nu}{4}$

$$\begin{aligned}
&\leq \frac{1}{\nu}||f||_{V'}^2 + \frac{\nu}{4}|Au_m(t)|^2 + \frac{\nu}{4}|Au_m(t)|^2 + cC_\varepsilon||u_m(t)||^{\frac{3}{2}\cdot 4} \\
&\leq \frac{1}{\nu}||f||_{V'}^2 + \frac{\nu}{2}|Au_m(t)|^2 + cC_\varepsilon||u_m(t)||^6.
\end{aligned}$$

After absorbing $|Au_m(t)|^2$ by the left-hand side, our initial inequality becomes

$$\frac{d}{dt}||u_m(t)||^2 + \nu|Au_m(t)|^2 \leq \frac{2}{\nu}||f||_{V'}^2 + C||u_m(t)||^6.$$

Because $|Au_m(t)|^2$ is positive we can drop this term on the left-hand side and get

$$\frac{d}{dt}||u_m(t)||^2 \leq \frac{2}{\nu}||f||_{V'}^2 + C||u_m(t)||^6. \tag{2.6}$$

To get the needed a-priori estimates we define an auxiliary function

$$y(t) = 1 + ||u_m(t)||^2.$$

Obviously it holds that $y(t) \geq 1$ and $y(t)^3 \geq 1$. Using $y(t)$ and defining

$$\lambda = 2 \cdot \max\left\{\frac{2}{\nu}||f||_{V'^2}, C\right\}$$

we can rewrite (2.6)

$$\begin{aligned}
& & y'(t) &\leq \lambda y^3(t) \\
&\Leftrightarrow & \frac{y'(t)}{y^3(t)} &\leq \lambda \\
&\Leftrightarrow & \frac{d}{dt}\left(\frac{-1}{2y^2(t)}\right) &\leq \lambda
\end{aligned}$$

Integrating over $[0, t]$ leads to:

$$\begin{aligned}
&\Rightarrow & \frac{1}{2y^2(0)} - \frac{1}{2y^2(t)} &\leq t\lambda \\
&\Leftrightarrow & y(t) &\leq \frac{2y(0)}{\sqrt{1 - 2t\lambda y^2(0)}}.
\end{aligned}$$

We need to garantee that $1 - 2t\lambda y^2(0) > 0$ is always true. Choosing $t \in (0, T^*), T^* < \frac{1}{2\lambda y^2(0)}$ and applying the inequality $||u_m(0)|| \leq c||u_0||$ we get:

$$\Rightarrow \qquad ||u_m(t)||^2 \leq \frac{(1 + ||u_m(0)||^2)^2}{\sqrt{1 - 2t\lambda(1 + ||u_m(0)||^2)^2}} \leq \frac{(1 + c||u_0||^2)^2}{\sqrt{1 - 2t\lambda(1 + c||u_0||^2)^2}}.$$

We managed to show that $u_m(t)$ can be bounded in $L^2(0, T^*; V)$ and in $L^\infty(0, T^*; V)$ and $Au_m(t)$ can be bounded in $L^2(0, T^*; H)$ and $L^\infty(0, T^*; H)$. Furthermore we get $u_m'(t) \in L^2(0, T^*; H)$ by rearranging the equation.
We searched for a solution in $H^2 \cap V$ and achieved additional regularity for our found solution $u_m(t)$. Hence the following holds for our local solution.

$$u_m \in H^1(0, T^*; H^2(\Omega) \cap V)$$

Going to the limit and intial condition

As we did in the global existence theorem, we need to specify out limit function and show that it satisfies the equation in some sense. In this case this part becomes very simple. Using Sobolev's embedding theorem we state

$$H^2(\Omega) \cap V \hookrightarrow C^{0,\frac{1}{2}}(\bar{\Omega})$$

und

$$H^1([0, T^*]) \hookrightarrow C^{0,\frac{1}{2}}([0, T^*]).$$

Thus we know that our local solution is continuous up to the boundary. The transition to the limit is done without any additional effort. We recieve a continuous solution for the NSE. $\square$

3 The Navier-Stokes-α Model

This section will discuss a turbulence model, namely the Navier-Stokes-α model. We will briefly discuss why turbulence models are necessary by taking a look at direct numerical simulations and observing their limits. After doing so, we will state the NS-α equations and show existence and uniqueness of a solution. In the last part we will show that the solution of the NS-α converges to a solution of NSE if α tends to zero, which is a very important property of a model. This chapter is based on [FHT02], this paper also focuses on estimating the dimension of the global attractor and link it with the degrees of freedom in turbulent flows.

3.1 Direct numerical simulations

Direct numerical simulations (DNS) are a fundamental approach in solving PDEs. There is no further modeling of turbulence necessary. A standard finite element discretization satisfying the inf-sup-condition can be used. For rather small Reynolds numbers one gets very good results. But once we increase the Reynolds number, we will run into problems. Recall that we have to choose our grid width according to the Kolmogorov's scales,

$$h \sim Re^{-\frac{3}{4}}.$$

We will do an example calculation relating to real life values of the Reynolds number and only consider the efforts on storing all the data that we produce instead of the accumulated costs.

Example: Let us consider a flow around an airplane wing during flight, then the Reynolds numbers can easily reach magnitudes of 10^6 and higher. Assume we have three dimensions, a uniform grid and a rectangular domain ($\Omega = 1m^3$).

Assuming we have $Re = 10^6$, the grid width has to be approximately $10^{6\cdot(-\frac{3}{4})}m = 10^{-\frac{9}{2}}m$ or smaller. So in each time step we compute values on $(10^{\frac{9}{2}})^3 = 10^{\frac{27}{2}}$ grid points because of the dimension. The velocity field contains three components, one for each direction and the pressure contains only one component, that means that on each grid point we need to compute 4 values, this results in evaluating $4 \cdot 10^{\frac{27}{2}}$ points. Each of these points requires at least 8 Bytes to be stored on the hard disc. So the total needed storage space is

$$10^{\frac{27}{2}} \cdot 4 \cdot 8 = 3.2 \cdot 10^{\frac{29}{2}} \text{ Bytes} \approx 3.2 \cdot 10^{\frac{5}{2}} \text{ Terrabytes} \approx 1\text{Petabyte}.$$

These immense storage costs are unbearable, because they occur every time step. We ignored the computational time that is needed to evaluate all the data and still the outcome is unsatisfactory.

As a conclusion, we can see that DNS can be used for relatively small Reynolds numbers but as soon as it becomes higher DNS becomes impractical.

3.2 The NS-α equations

For the numerical application of the NSE it is not practical to use DNS to obtain solutions with a fine resolution. A common approach of turbulence modeling is to model the influence of the smaller scales on the larger scales. As a consequence we can compute a solution on a coarser grid. In general additional errors occur and we have to find a balance between computational cost and precision. One ansatz out of many is the Navier-Stokes-α approach, where the convection term $(\nabla \times u) \times u$ in the rotational form of NSE is replaced by $(\nabla \times u) \times \bar{u}$. The variable $\bar{u}$ is a filtered velocity field with filter $(I - \alpha^2 \Delta)^{-1}$. In addition to filtering the velocity, we want to ensure that the filtered solution is also divergence free, since u was divergence free, so to enforce this condition we introduce a Lagrange function λ. The resulting system of equations is the following:

$$\begin{cases} \partial_t u - \nu \Delta u + (\nabla \times u) \times \bar{u} + \nabla P = f \\ \bar{u} - \alpha^2 \Delta \bar{u} + \nabla \lambda = u \\ \nabla \cdot u = \nabla \cdot \bar{u} = 0 \\ \text{Initial and boundary condition.} \end{cases} \tag{3.1}$$

In literature, the NS-α equations are often referred to as the viscous Camassa-Holm equations. For a more detailed derivation of the equations see [CFH+98, FHT02, CH93]

3.3 Different representations of the NS-α approach

In the literature one can find several different definitions of the Navier-Stokes-α approach, but they can be transformed into each other. Sometimes the formulation changes slightly depending on the boundary condition. Basically, the main difference is in the non-linear part of the NSE. As we derived in the first chapter the NSE have the following form:

$$\begin{aligned} \partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p =& f, \\ \nabla \cdot u =& 0. \end{aligned}$$

Our first step will be to transform the standard NSE into the NSE in rotational form, for which we use an identity connecting the cross product and the scalar product of a velocity field u:

$$(u \cdot \nabla) u = (\nabla \times u) \times u + \frac{1}{2} \nabla u^2.$$

Inserting the identity above in the NSE results in the rotational form of the NSE.

$$\begin{aligned} & \partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p =f \\ \Leftrightarrow \quad & \partial_t u - \nu \Delta u + (\nabla \times u) \times u + \nabla P =f, \end{aligned}$$

where $P = p + \frac{1}{2} u^2$ is the modified pressure. It is important for the theoretical treatment that we have the the term $(\nabla \times u) \times u$. It is crucial because it has the following property:

$$((\nabla \times u) \times v, u) = 0.$$

As we saw in the previous section, the NS-α approach is basically replacing the convection direction by a filtered version of u. The filter in this case is the inverse of the Helmholtz operator $(I - \alpha^2 \Delta)$ with an additional Lagrange function λ to ensure the divergence free constraint on the filtered solution.
In case of periodic BC the Lagrange function λ vanishes. Intuitively this makes sense, because the divergence free constraint comes from mass conservation. Since the right-hand side u of our auxiliary problem is divergence free, it doesn't have any sources or sinks, which means that there is no mass loss inside the domain. Due to the periodic BC there is no mass loss across the boundary either, which explains why λ vanishes in that case.

3.4 Preliminaries

In this Chapter we will mainly follow [FHT02]. We consider the following problem on $\Omega = [0, L]^3$ with $\alpha \geq 0$:

$$\begin{cases} \partial_t(u - \alpha^2 \Delta u) - \nu\Delta(u - \alpha^2\Delta u) + (\nabla \times (u - \alpha^2 \Delta u)) \times u + \nabla P = f, \\ \nabla \cdot u = \nabla \cdot (u - \alpha^2 \Delta u) = 0, \\ u(0, x) = u_0(x). \end{cases} \tag{3.2}$$

Since we have a periodic BC, the Lagrange function λ vanishes. For simplicity reasons we assume that the right-hand side is time-independent, $f(t, x) = f(x)$. Before we define our spaces H and V, we will conclude useful properties of the velocity field u first. Let us integrate equation (3.2) over the whole domain Ω and integrate by parts. We can see that all terms besides the time derivative vanish:

$$\frac{d}{dt} \int_\Omega (u - \alpha^2 \Delta u)\, dx = \int_\Omega f\, dx \tag{3.3}$$

Since we have periodic BC, the Laplacian becomes spatially periodic and therefore vanishes in the integral mean:

$$\int_\Omega \Delta u\, dx = 0.$$

Inserting the previous equation into (3.3) results in:

$$\frac{d}{dt} \int_\Omega u\, dx = \int_\Omega f\, dx.$$

We will only consider forcing terms f that have a zero mean, $\int_\Omega f\, dx = 0$. Therefore the mean of the solution u is invariant. In addition we assume that our initial condition u_0 has zero mean as well, this results in a vanishing mean of u. These calculations and assumptions motivate the definition of the spaces H and V:

$$H = \left\{ v \in (L^2(\Omega))^3 : \nabla \cdot v = 0, \int_\Omega v\, dx = 0 \right\}$$

and

$$V = \left\{ v \in (H_0^1(\Omega))^3 : \nabla \cdot v = 0, \int_\Omega v\, dx = 0 \right\},$$

where the derivatives are meant in a weak or distributional sense respectively.

Remark *It holds that V is densely embedded in H and H' is densely embedded in V'. Hence they form a Gelfand triple*

$$V \subset H \cong H' \subset V'.$$

Remark *We will abbreviate the norms in V with $||\cdot||$ and the norms in H with $|\cdot|$.*

Remark *The space*

$$\mathcal{V} = \left\{ \psi : \psi \text{ is a trigonometric function defined on } \Omega, \nabla \cdot \psi = 0, \int_{\Omega} \psi = 0 \right\}$$

is dense in the spaces H and V.

We define the orthogonal L^2 projection:

$$P_\sigma : (\dot{L}^2(\Omega))^3 \to H,$$

where

$$(\dot{L}^2(\Omega))^3 := \left\{ v \in (L^2(\Omega))^3 : \int_{\Omega} v\, dx = 0 \right\}.$$

In addition we define the Stokes operator A and its domain $\mathcal{D}(A)$:

$$A = -P_\sigma \Delta, \qquad \mathcal{D}(A) = (H^2(\Omega))^3 \cap V.$$

Remark *In case of periodic BC the restriction of A on $\mathcal{D}(A)$ is a selfadjoint operator with compact inverse. We can apply the theorem of Hilbert-Schmidt and get a set of eigenfunctions of the Stokes operator $\{\varphi_k\}_{k\in\mathbb{N}}$ with its corresponding eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$. The eigenfunctions form an orthonormal basis of H and the eigenvalues are positive and are sorted, $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k \to \infty$.*

Next we will show a connection between the Stokes operator and the H^2-norm, more precisely they are equivalent.

$|Aw| \leq c||w||_{H^2(\Omega)}$ holds for all $w \in \mathcal{D}(A)$ because the $H^2(\Omega)$ norm has more terms.
$||w||_{H^2(\Omega)} \leq C|Aw|$ holds for all $w \in \mathcal{D}(A)$ because of the Poincaré inequality.

With a similar argument we can conclude the equivalence of the $H^1(\Omega)$ norm and the $|A^{\frac{1}{2}}\cdot|$ norm.

$$c\left|A^{\frac{1}{2}}w\right| \leq ||w||_{H^1(\Omega)} \leq C\left|A^{\frac{1}{2}}w\right| \qquad \text{holds for all } w \in V.$$

In particular we can show that $V = \mathcal{D}(A^{\frac{1}{2}})$ and therefore $||\cdot|| = |A^{\frac{1}{2}}\cdot|$. For a more detailed analysis refer to [Tem87, CF89]. So far we finished everything concerning the spaces, we need to describe some notation and some very crucial estimates before we can start to prove existence and uniqueness.
Let $u, v \in V$, we denote $B(u,v) = P_\sigma((u\cdot\nabla)v)$ and $\tilde{B}(u,v) = P_\sigma((\nabla\times v)\times u)$. We notice that $(B(u,v),v) = 0$ and recall the identity

$$(u \cdot \nabla)v + \sum_{j=1}^{3} v_j \nabla u_j = (\nabla \times u) \times v + \nabla(u \cdot v) \tag{3.4}$$

and the fact that $\{\nabla p : p \in H^1(\Omega)\}$ is orthogonal to H in $L^2(\Omega)$. As a consequence the term $\nabla(u \cdot v)$ will vanish if we test the equation with $w \in V$. Let us focus on testing $\sum_{j=1}^{3} v_j \nabla u_j$ with $w \in V$:

$$\int_\Omega \left(\sum_{j=1}^{3} v_j \nabla u_j \right) w \, dx = \int_\Omega \sum_{j=1}^{3} \sum_{i=1}^{3} v_j \nabla_i u_j w_i \, dx.$$

We use Einstein notation to loose the sums and keep the overview.

$$\begin{aligned}
&= \int_\Omega v_j \nabla_i u_j w_i \, dx = \int_\Omega v_j w_i \nabla_i u_j \, dx \\
&= -\int_\Omega \nabla_i (v_j w_i) u_j \, dx \\
&= -\int_\Omega \nabla_i v_j w_i u_j \, dx - \underbrace{\int_\Omega \nabla_i \underbrace{w_i}_{\in H^1(\Omega)} \underbrace{v_j u_j}_{\in H} \, dx}_{=0} \\
&= -\int_\Omega w_i \nabla_i v_j u_j \, dx \\
&= -\int_\Omega ((w \cdot \nabla) v) \, u \, dx.
\end{aligned}$$

We use previous calculations and (3.4) to conclude:

$$(\tilde{B}(u, v), w) = (B(u, v), w) - (B(w, v), u)$$

and

$$(\tilde{B}(u, v), w) = -(\tilde{B}(w, v), u).$$

As a consequence $(\tilde{B}(u, v), u) = 0$ holds.
Our next step is going to be the discussion of two lemmata that sum up all the necessary relations and inequalities for our theorems.

Lemma 3.1. *1. The operator A can be extended continuously on $V = \mathcal{D}(A^{\frac{1}{2}})$ with values in $V' = \mathcal{D}(A^{-\frac{1}{2}})$ such that:*

$$\langle Au, v \rangle_{V' \times V} = (A^{\frac{1}{2}} u, A^{\frac{1}{2}} v) \qquad \textit{for every } u, v \in V.$$

2. *The operator A^2 can be extended continuously on $\mathcal{D}(A)$ with values in $\mathcal{D}(A)'$ such that:*

$$\langle A^2 u, v\rangle_{\mathcal{D}(A)'\times\mathcal{D}(A)} = (Au, Av) \qquad \textit{for every } u, v \in \mathcal{D}(A).$$

Proof. The proof can be found in [Tem87]. □

Lemma 3.2. *1. The operator $\tilde{B}$ can be extended continuously from $V \times V$ with values in V' and in addition it satisfies the following inequalities for all $u, v, w \in V$:*

$$\left|(\tilde{B}(u,v), w)_{V'\times V}\right| \leq c|u|^{\frac{1}{2}}\,||u||^{\frac{1}{2}}\,||v||\,||w||,$$
$$\left|(\tilde{B}(u,v), w)_{V'\times V}\right| \leq c||u||\,||v||\,|w|^{\frac{1}{2}}\,||w||^{\frac{1}{2}}.$$

2. For all $u \in H, v \in V, w \in \mathcal{D}(A)$ it holds:

$$\left|(\tilde{B}(u,v), w)_{\mathcal{D}(A)'\times\mathcal{D}(A)}\right| \leq c|u|\,||v||\,||w||^{\frac{1}{2}}\,|Aw|^{\frac{1}{2}},$$
$$\left|(\tilde{B}(w,v), u)_{V'\times V}\right| \leq c||w||^{\frac{1}{2}}\,|Aw|^{\frac{1}{2}}\,||v||\,|u|.$$

3. For all $u \in V, v \in H, w \in \mathcal{D}(A)$ it holds:

$$\left|(\tilde{B}(u,v), w)_{\mathcal{D}(A)'\times\mathcal{D}(A)}\right| \leq c\left(|u|^{\frac{1}{2}}\,||u||^{\frac{1}{2}}\,|v|\,|Aw| + |v|\,||u||\,||w||^{\frac{1}{2}}|Aw|^{\frac{1}{2}}\right),$$
$$\left|(\tilde{B}(w,v), u)_{V'\times V}\right| \leq c\left(||w||^{\frac{1}{2}}\,||Aw||^{\frac{1}{2}}\,|v|\,||u|| + |Aw|\,|v|\,|u|^{\frac{1}{2}}\,||u||^{\frac{1}{2}}\right).$$

Proof. All inequalities will be proven in a similar way. First we show all statements using functions from $\mathcal{V}$ and then using a density argument to get the above inequalities. The second part of each estimates will be done analogously to the first part, so only the first part will be discussed.
Let $u, v, w \in \mathcal{V}$.

1. We apply the general Hölder inequality to $(\tilde{B}(u,v), w)_{V'\times V}$

$$\begin{aligned}\left|(\tilde{B}(u,v), w)_{V'\times V}\right| &= |(B(u,v), w) - (B(w,v), u)| \\ &\leq c||u||_{L^3(\Omega)}\,||\nabla v||_{L^2(\Omega)}\,||w||_{L^6(\Omega)}.\end{aligned}$$

We apply the Sobolev embedding theorem and interpolate in Sobolev spaces.

$$\leq c|u|^{\frac{1}{2}}\,||u||^{\frac{1}{2}}\,||v||\,||w||.$$

Since $\mathcal{V}$ is dense in V we conclude the proof.

2. Let us recall Agmon's inequality in $\mathbb{R}^3$: $||u||_{L^\infty} \leq C||u||^{\frac{1}{2}}_{H^1(\Omega)}||u||^{\frac{1}{2}}_{H^2(\Omega)}$. It follows that:

$$\left|(\tilde{B}(u,v),w)_{\mathcal{D}(A)'\times\mathcal{D}(A)}\right| \leq c|u|\,||v||\,||w||_{L^\infty} \\ \leq c|u|\,||v||\,||w||^{\frac{1}{2}}|Aw|^{\frac{1}{2}}.$$

And again a densitiy argument concludes the proof.

3. We consider:

$$\left|(\tilde{B}(u,v),w)_{V'\times V}\right| = |(B(u,v),w)| + |(B(w,v),u)| \\ \leq c||u||_{L^3(\Omega)}\,||\nabla w||_{L^6(\Omega)}\,|v| + c|v|\,||u||\,||w||_{L^\infty} \\ \leq c\left(|u|^{\frac{1}{2}}\,||u||^{\frac{1}{2}}\,|v|\,|Aw| + |v|\,||u||\,||w||^{\frac{1}{2}}|Aw|^{\frac{1}{2}}\right).$$

The density of $\mathcal{V}$ in V concludes the proof.

□

Definition (Solution of (3.2))**.** Let $f \in H$, $u_0 \in V$ and $T > 0$. A function $u \in C([0,T);V) \cap L^2([0,T);\mathcal{D}(A))$ with $u' \in L^2([0,T);H)$ is called **solution of** (3.2) if for all $w \in \mathcal{D}(A)$ and all $t_0 < t \in [0,T)$

$$(u(t)+\alpha^2 Au(t),w) - (u(t_0)+\alpha^2 Au(t_0),w) + \nu\int_{t_0}^{t}(u(s)+\alpha^2 Au(s),w)\,ds \\ + \int_{t_0}^{t}\langle\tilde{B}(u(s),u(s)+\alpha^2 Au(s),w\rangle_{\mathcal{D}(A)'\times\mathcal{D}(A)}\,ds = \int_{t_0}^{t}(f(s),w)\,ds$$

holds.

Remark *A different way of writing down the above condition, is by using dual pairing:*

$$\left\langle\frac{d}{dt}(u+\alpha^2 Au),w\right\rangle_{\mathcal{D}(A)'\times\mathcal{D}(A)} + \nu\left\langle A(u+\alpha^2 Au),w\right\rangle_{\mathcal{D}(A)'\times\mathcal{D}(A)} \\ + \left\langle\tilde{B}(u,u+\alpha^2 Au),w\right\rangle_{\mathcal{D}(A)'\times\mathcal{D}(A)} = (f,w),$$

for all $w \in \mathcal{D}(A)$ and almost every $t \in [0,T]$. We sometimes use the operator equation that can be formulated as: The equation

$$\frac{d}{dt}(u+\alpha^2 Au) + \nu A(u+\alpha^2 Au) + \tilde{B}(u,u+\alpha^2 Au) = f$$

holds in $L^2([0,T];\mathcal{D}(A))$.

3.5 Existence and uniqueness

Theorem 3.3 (Global existence and uniqueness)**.** *There exists a uniquely defined solution u of* (3.2)*. In addition the solution u also satisfies*

$$u \in L^\infty_{loc}((0,T]; H^3(\Omega)).$$

Proof. The proof can be divided essentially in these parts:

1. Existence of a local solution.
2. Estimates of the $H^1(\Omega)$-, $H^2(\Omega)$- and $H^3(\Omega)$-norms of u.
3. Passing to the limit.
4. Uniqueness.

Part 1: Existence of a local solution

Similar to the previous proofs we will use a Galerkin approximation to obtain a local description of the problem. Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of H consisting of eigenfunctions of the Stokes operator A with corresponding eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$. Let $H_m = span(\varphi_1, \varphi_2, ..., \varphi_m)$ and P_m the orthogonal projection of H onto H_m. The resulting Galerkin problem corresponding to (3.2) is:

$$\begin{cases} \dfrac{d}{dt}(u_m(t) + \alpha^2 A u_m(t)) + \nu A(u_m(t) + \alpha^2 A u_m(t)) \\ \qquad\qquad\qquad + \tilde{B}(u_m(t), u_m(t) + \alpha^2 A u_m(t)) = P_m f, \\ \qquad\qquad\qquad\qquad\qquad\qquad u_m(0,x) = P_m u_0(x). \end{cases} \tag{3.5}$$

We notice that the non-linear part is only quadratic, so we conclude with Carethéodory's theorem the short time existence and uniqueness of a absolutely continuous solution u_m on $(-\tau_m, T_m)$. To extend the solution to the whole time interval we need to show that the our solution u_m can be bounded for larger T_m.

Part 2: Estimates of the $H^1(\Omega)$-, the $H^2(\Omega)$- and the $H^3(\Omega)$-norms of u

$H^1(\Omega)$ **norm:**
We multiply equation (3.5) with $u_m(t)$ and integrate over Ω:

$$\begin{aligned} &\big\langle \partial_t(u_m(t) + \alpha^2 A u_m(t)), u_m(t)\big\rangle \\ &\quad + \nu \underbrace{\big(A(u_m(t) + \alpha^2 A u_m(t)), u_m(t)\big)}_{(u_m(t),Au_m(t))+\alpha^2(Au_m(t),Au_m(t))} \\ &\quad + \underbrace{\Big(\tilde{B}(u_m(t), u_m(t) + \alpha^2 A u_m(t)), u_m(t)\Big)}_{=0} &&= (f, u_m(t)) \\ \Leftrightarrow &\frac{1}{2}\frac{d}{dt}\left(|u_m(t)|^2 + \alpha^2 ||u_m(t)||^2\right) + \nu\left(||u_m(t)||^2 + \alpha^2 |Au_m(t)|^2\right) &&= (P_m f, u_m(t)). \end{aligned} \tag{3.6}$$

We will estimate the right-hand side from above such that all terms containing u_m can be absorbed by the left-hand side. All other resulting terms should be given. We multiply an identity in form of $(A^{-1}A)$ and $(A^{-\frac{1}{2}}A^{\frac{1}{2}})$ to the right-hand side and obtain through application of Hölder's and Young's inequality the following estimations:

$$|(P_m f, u_m(t))| \leq \begin{cases} |A^{-1}f|\,|Au_m(t)| \\ |A^{-\frac{1}{2}}f|\,||u_m(t)|| \end{cases} \leq \begin{cases} \dfrac{|A^{-1}f|^2}{2\nu\alpha^2} + \dfrac{\nu}{2}\alpha^2|Au_m(t)|^2, \\ \dfrac{|A^{-\frac{1}{2}}f|^2}{2\nu} + \dfrac{\nu}{2}||u_m(t)||^2. \end{cases} \tag{3.7}$$

We define

$$K_1 = \min\left(\frac{|A^{-1}f|^2}{\nu\alpha^2}, \frac{|A^{-\frac{1}{2}}f|^2}{\nu}\right)$$

and use K_1 in the estimates (3.7):

$$|(P_m f, u_m(t))| \leq \frac{1}{2}K_1 + \frac{\nu}{2}||u_m(t)||^2 + \frac{\nu}{2}\alpha^2|Au_m(t)|^2.$$

We insert the above inequality into (3.6), absorb the last two terms and multiply by 2:

$$\frac{d}{dt}(|u_m(t)|^2 + \alpha^2||u_m(t)||^2) + \nu(||u_m(t)||^2 + \alpha^2|Au_m(t)|^2) \leq K_1. \tag{3.8}$$

We want to apply the Gronwall's inequality to obtain an upper bound that is independent of m. Let us consider some auxiliary calculations first:

$$\begin{aligned} ||u_m(t)||^2 &= \int_\Omega \nabla u_m(t) : \nabla u_m(t) = \int_\Omega u_m(t)\, Au_m(t)\, dx = \int_\Omega u_m(t) \sum_{i=1}^m c_i\lambda_i\varphi_i\, dx \\ &\geq \lambda_1 \int_\Omega u_m(t) \sum_{i=1}^m c_i\varphi_i\, dx = \lambda_1 \int_\Omega u_m(t)\cdot u_m(t)\, dx = \lambda_1|u_m(t)|^2. \end{aligned} \tag{3.9}$$

We know that λ_1 is the smallest eigenvalue. A similar calculation can be done for $|Au_m|^2$:

$$|Au_m(t)|^2 \geq \lambda_1||u_m(t)||^2. \tag{3.10}$$

Inserting both inequalities into (3.8) results in:

$$\begin{aligned} &\frac{d}{dt}(|u_m(t)|^2 + \alpha^2||u_m(t)||^2) + \nu\lambda_1\left(|u_m(t)|^2 + \alpha^2||u_m(t)||^2\right) \leq K_1 \\ \Leftrightarrow\quad &\frac{d}{dt}(|u_m(t)|^2 + \alpha^2||u_m(t)||^2) \leq K_1 - \nu\lambda_1\left(|u_m(t)|^2 + \alpha^2||u_m(t)||^2\right). \end{aligned}$$

We apply Gronwall's inequality with $\lambda(s) = -\nu\lambda_1$ and $g = K_1$.

$$\begin{aligned} |u_m(t)|^2 + \alpha^2||u_m(t)||^2 &\leq \underbrace{e^{-\nu\lambda_1 t}}_{\leq 1}(|u_m(0)|^2 + \alpha^2||u_m(0)||^2) + \frac{K_1}{\nu\lambda_1}\underbrace{(1 - e^{-\nu\lambda_1 t})}_{\leq 1} \\ &\leq |u_m(0)|^2 + \alpha^2||u_m(0)||^2 + \frac{K_1}{\nu\lambda_1} \\ &\leq |u_0|^2 + \alpha^2||u_0||^2 + \frac{K_1}{\nu\lambda_1}. \end{aligned}$$

Defining $k_1 := |u_0|^2 + \alpha^2||u_0||^2 + \frac{K_1}{\nu\lambda_1}$, simplifies the inequality and we get:

$$|u_m(t)|^2 + \alpha^2||u_m(t)||^2 \leq k_1. \tag{3.11}$$

Integrating t over $[0,T]$ results in the boundedness of the $H^1(\Omega)$ norm.

$H^2(\Omega)$**-Norm:**
We consider equation (3.8) and integrate over $[t, t+\tau]$ to obtain a useful result first:

$$\begin{aligned}
&\int_t^{t+\tau} \frac{d}{dt}(|u_m(s)|^2 + \alpha^2||u_m(s)||^2) + \nu(||u_m(s)||^2 + \alpha^2|Au_m(s)|^2)\,ds \leq \int_t^{t+\tau} K_1\,ds \\
\Leftrightarrow &\underbrace{(|u_m(t+\tau)|^2 + \alpha^2||u_m(t+\tau)||^2)}_{\geq 0} - (|u_m(t)|^2 + \alpha^2||u_m(t)||^2) \\
&\qquad + \int_t^{t+\tau} \nu(||u_m(s)||^2 + \alpha^2|Au_m(s)|^2)\,ds \leq \tau K_1 \\
\Rightarrow &\int_t^{t+\tau} \nu(||u_m(s)||^2 + \alpha^2|Au_m(s)|^2)\,ds \leq \tau K_1 + (|u_m(t)|^2 + \alpha^2||u_m(t)||^2) \\
\Rightarrow &\int_t^{t+\tau} \nu(||u_m(s)||^2 + \alpha^2|Au_m(s)|^2)\,ds \leq \tau K_1 + k_1.
\end{aligned} \tag{3.12}$$

The resulting inequality (3.12) will be useful throughout this thesis. Now we consider the Galerkin equation (3.5) and test it with Au_m:

$$\begin{aligned}
&\left\langle \frac{d}{dt}(u_m(t) + \alpha^2 Au_m(t)), Au_m(t) \right\rangle \\
&\quad + \nu\Big(A\Big(u_m(t) + \alpha^2 Au_m(t)\Big), Au_m(t)\Big) \\
&\qquad + \Big(\tilde{B}(u_m(t), u_m(t) + \alpha^2 Au_m(t)), Au_m(t)\Big) \quad = (P_m f, Au_m(t)) \\
\Leftrightarrow &\frac{1}{2}\frac{d}{dt}(||u_m(t)||^2 \\
&\quad + \alpha^2|Au_m(t)|^2) + \nu(|Au_m(t)|^2 + \alpha^2|A^{\frac{3}{2}}u_m(t)|^2) \\
&\qquad + \Big(\tilde{B}(u_m(t), u_m(t) + \alpha^2 Au_m(t)), Au_m(t)\Big) \quad = (P_m f, Au_m(t).)
\end{aligned} \tag{3.13}$$

As we did before we will estimate the right-hand side such that we can absorb all non given terms by the left-hand side. We will use a similar inequality by multiplying an identity in form of $(A^{-\frac{1}{2}}A^{\frac{1}{2}})$ to the right-hand side and then apply Hölder's and Young's inequality:

$$|(P_m f, Au_m(t))| \leq \begin{cases} |A^{-\frac{1}{2}}f|\,|A^{\frac{3}{2}}u_m(t)| \\ |f|\,|Au_m(t)| \end{cases} \leq \begin{cases} \dfrac{|A^{-\frac{1}{2}}f|^2}{\nu\alpha^2} + \dfrac{\nu}{4}\alpha^2|A^{\frac{3}{2}}u_m(t)|^2, \\ \dfrac{|f|^2}{\nu} + \dfrac{\nu}{4}|Au_m(t)|^2. \end{cases} \tag{3.14}$$

We define

$$K_2 = \min\left(\frac{|A^{-\frac{1}{2}}f|^2}{\nu\alpha^2}, \frac{|f|^2}{\nu}\right).$$

Using K_2 and (3.14) in equation (3.13) we can absorb $|A^{\frac{3}{2}}u_m(t)|^2$ and $|Au_m(t)|^2$ by the left-hand side:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||u_m(t)||^2 + \alpha^2|Au_m(t)|^2) + \frac{3}{4}\nu(|Au_m(t)|^2 + \alpha^2|A^{\frac{3}{2}}u_m(t)|^2) \\
&\qquad + \left(\tilde{B}(u_m(t), u_m(t) + \alpha^2 Au_m(t)), Au_m(t)\right) \le K_2 \\
\Rightarrow\quad &\frac{1}{2}\frac{d}{dt}(||u_m(t)||^2 + \alpha^2|Au_m(t)|^2) + \frac{3}{4}\nu(|Au_m(t)|^2 + \alpha^2|A^{\frac{3}{2}}u_m(t)|^2) \\
&\qquad \le K_2 + \left|\left(\tilde{B}(u_m(t), u_m(t) + \alpha^2 Au_m(t)), Au_m(t)\right)\right|.
\end{aligned}$$

We will now apply the estimates for $\tilde{B}$ from Lemma 3.2 and also consider the following calculations:

$$||u_m(t)|| = (A^2\underbrace{A^{-2}u_m(t)}_{\le\lambda_1^{-2}u_m(t)}, Au_m(t))^{\frac{1}{2}} \le \lambda_1^{-1}(A^2u_m(t), Au_m(t))^{\frac{1}{2}} = \lambda_1^{-1}|A^{\frac{3}{2}}u_m(t)|.$$

Putting these thoughts together results in:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||u_m(t)||^2 + \alpha^2|Au_m(t)|^2) + \frac{3}{4}\nu(|Au_m(t)|^2 + \alpha^2|A^{\frac{3}{2}}u_m(t)|^2) \\
&\le K_2 + \left|\left(\tilde{B}(u_m(t), u_m(t) + \alpha^2 Au_m(t)), Au_m(t)\right)\right| \\
&\le K_2 + ||u_m(t)||\left(\underbrace{||u_m(t)||}_{\le\lambda_1^{-1}|A^{\frac{3}{2}}u_m(t)|} + \alpha^2\underbrace{||Au_m(t)||}_{=|A^{\frac{3}{2}}u_m(t)|}\right)|Au_m(t)|^{\frac{1}{2}}\underbrace{||Au_m(t)||^{\frac{1}{2}}}_{|A^{\frac{3}{2}}u_m(t)|^{\frac{1}{2}}} \\
&\le K_2 + ||u_m(t)||(\lambda_1^{-1} + \alpha^2)|A^{\frac{3}{2}}u_m(t)|^{\frac{3}{2}}|Au_m(t)|^{\frac{1}{2}}.
\end{aligned}$$

We use Young's inequality in such ways that we can absorb $|A^{\frac{3}{2}}u_m(t)|^{\frac{3}{2}}$ by the left-hand side:

$$\begin{aligned}
&\le K_2 + \underbrace{||u_m(t)||(\lambda_1^{-1} + \alpha^2)|Au_m(t)|^{\frac{1}{2}}}_{q=4}\underbrace{|A^{\frac{3}{2}}u_m(t)|^{\frac{3}{2}}}_{p=\frac{4}{3}} \\
&\le K_2 + \frac{1}{36}(\nu\alpha^2)^{-3}||u_m(t)||^4(\lambda_1^{-1} + \alpha^2)^4|Au_m(t)|^2 + \frac{\nu\alpha^2}{4}|A^{\frac{3}{2}}u_m(t)|^2.
\end{aligned}$$

The whole inequality has the following form now:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||u_m(t)||^2+\alpha^2|Au_m(t)|^2)+\frac{3}{4}\nu\left(|Au_m(t)|^2+\alpha^2|A^{\frac{3}{2}}u_m(t)|^2\right)\\
&\qquad\leq K_2+\frac{1}{36}(\nu\alpha^2)^{-3}||u_m(t)||^4(\lambda_1^{-1}+\alpha^2)^4|Au_m(t)|^2+\frac{\nu\alpha^2}{4}|A^{\frac{3}{2}}u_m(t)|^2\\
\Leftrightarrow\ &\frac{1}{2}\frac{d}{dt}(||u_m(t)||^2+\alpha^2|Au_m(t)|^2)+\frac{3}{4}\nu|Au_m(t)|^2+\nu\frac{1}{2}\alpha^2|A^{\frac{3}{2}}u_m(t)|^2\\
&\qquad\leq K_2+\frac{1}{36}(\nu\alpha^2)^{-3}||u_m(t)||^4(\lambda_1^{-1}+\alpha^2)^4|Au_m(t)|^2.
\end{aligned}
$$

We decrease the left-hand side even more and multiply by 2:

$$
\begin{aligned}
\frac{d}{dt}(||u_m(t)||^2+\alpha^2|Au_m(t)|^2)+\nu\left(|Au_m(t)|^2+\alpha^2|A^{\frac{3}{2}}u_m(t)|^2\right)&\\
\leq 2K_2+2c(\nu\alpha^2)^{-3}||u_m(t)||^4(\lambda_1^{-1}+\alpha^2)^4|Au_m(t)|^2.&
\end{aligned}
\tag{3.15}
$$

Before we continue to estimate any further, let us consider equation (3.11) and (3.12) first and draw two conclusions.

From equation (3.11) we can conclude

$$||u_m(t)||^4\leq\left(\frac{k_1}{\alpha^2}\right)^2$$

and from equation (3.12) we get

$$\int_s^t |Au_m(r)|^2\,dr\leq(t-s)K_1+k_1.$$

We will use these estimates in the next calculation. Let us integrate equation (3.15) over $[s,t]$. For the left-hand side we get:

$$
\begin{aligned}
&\int_s^t \frac{d}{dr}(||u_m(r)||^2+\alpha^2|Au_m(r)|^2)+\underbrace{\nu(|Au_m(r)|^2+\alpha^2|A^{\frac{3}{2}}u_m(r)|^2)}_{\geq 0}\,dr\\
&\qquad\geq\quad ||u_m(t)||^2+\alpha^2|Au_m(t)|^2-(||u_m(s)||^2+\alpha^2|Au_m(s)|^2),
\end{aligned}
$$

and for the right-hand side we get

$$
\begin{aligned}
&\int_s^t 2K_2 + 2c(\nu\alpha^2)^{-3}||u_m(r)||^4(\lambda_1^{-1}+\alpha^2)^4|Au_m(r)|^2\,dr \\
\leq& 2(t-s)K_2 + \int_s^t 2c(\nu\alpha^2)^{-3}\left(\frac{k_1}{\alpha^2}\right)^2(\lambda_1^{-1}+\alpha^2)^4|Au_m(r)|^2\,dr \\
\leq& 2(t-s)K_2 + \left(\frac{2ck_1^2}{(\nu\alpha^2)^3\alpha^4}\right)(\lambda_1^{-1}+\alpha^2)^4\int_s^t |Au_m(r)|^2\,dr \\
\leq& 2(t-s)K_2 + \left(\frac{2ck_1^2}{(\nu\alpha^2)^3\alpha^4}\right)(\lambda_1^{-1}+\alpha^2)^4\left((t-s)K_1+k_1\right).
\end{aligned}
$$

Putting both calculations together results in the following inequality:

$$
\left.
\begin{aligned}
&||u_m(t)||^2+\alpha^2|Au_m(t)|^2 \\
&\quad \leq 2(t-s)K_2 + \left(\frac{2ck_1^2}{(\nu\alpha^2)^3\alpha^4}\right)(\lambda_1^{-1}+\alpha^2)^4\left((t-s)K_1+k_1\right) \\
&\qquad + (||u_m(s)||^2+\alpha^2|Au_m(s)|^2)
\end{aligned}
\right\} \tag{3.16}
$$

We now integrate s over [0,t]:

$$
\begin{aligned}
&t(||u_m(t)||^2+\alpha^2|Au_m(t)|^2) \\
&\quad\leq \int_0^t 2(t-s)K_2 + \left(\frac{2ck_1^2}{(\nu\alpha^2)^3\alpha^4}\right)(\lambda_1^{-1}+\alpha^2)^4\left((t-s)K_1+k_1\right) \\
&\qquad + (||u_m(s)||^2+\alpha^2|Au_m(s)|^2)ds \\
&\quad= K_2t^2 + \left(\frac{2ck_1^2}{(\nu\alpha^2)^3\alpha^4}\right)(\lambda_1^{-1}+\alpha^2)^4\left(\frac{t^2}{2}K_1+tk_1\right) + \int_0^t (||u_m(s)||^2+\alpha^2|Au_m(s)|^2)ds \\
&\quad\leq K_2t^2 + \left(\frac{2ck_1^2}{(\nu\alpha^2)^3\alpha^4}\right)(\lambda_1^{-1}+\alpha^2)^4\left(\frac{t^2}{2}K_1+tk_1\right) + \frac{1}{\nu}(tK_1+k_1).
\end{aligned}
$$

This estimate holds for $t>0$ but is unbounded for large t, we need another estimate to control larger times. If $t \geq \frac{1}{\nu\lambda_1}$ then we can integrate equation (3.16) over $[t-\frac{1}{\nu\lambda_1},t]$ instead:

$$
\begin{aligned}
&\frac{1}{\nu\lambda_1}(||u_m(t)||^2+\alpha^2|Au_m(t)|^2) \\
&\quad\leq K_2\left(\frac{1}{\nu\lambda_1}\right)^2 + \left(\frac{2ck_1^2}{(\nu\alpha^2)^3\alpha^4}\right)(\lambda_1^{-1}+\alpha^2)^4\left(\frac{1}{2(\nu\lambda_1)^2}K_1+\frac{1}{\nu\lambda_1}k_1\right) \\
&\qquad + \frac{1}{\nu}(\frac{1}{\nu\lambda_1}K_1+k_1).
\end{aligned}
$$

We showed that $||u_m(t)||^2 + \alpha^2|Au_m(t)|^2$ is finite for all $t > 0$ which is equivalent to the boundedness of the $H^2(\Omega)$ norm. We can say that there exists a bounded function $k_2(t)$ that satisfies

$$||u_m(t)||^2 + \alpha^2|Au_m(t)|^2 \leq k_2(t). \tag{3.17}$$

$H^3(\Omega)$**-Norm:**
To obtain a constraint on the H^3 norm of our velocity field u we will find an upper bound for $||v_m(t)||$ that is independent of m by testing the Galerkin equation (3.5) with $Av_m(t)$, where $v_m(t) = u_m(t) + \alpha^2 Au_m(t)$:

$$\begin{aligned}
&\langle \partial_t v_m(t), Av_m(t)\rangle + \nu(Av_m(t), Av_m(t)) + (\tilde{B}(u_m(t), v_m(t)), Av_m(t)) = (f, Av_m(t))\\
\Leftrightarrow\quad & \frac{1}{2}\frac{\partial}{\partial t}||v_m(t)||^2 + \nu|Av_m(t)|^2 + (\tilde{B}(u_m(t), v_m(t)), Av_m(t)) = (f, Av_m(t))\\
\Rightarrow\quad & \frac{1}{2}\frac{\partial}{\partial t}||v_m(t)||^2 + \nu|Av_m(t)|^2 \leq |(f, Av_m(t))| + |(\tilde{B}(u_m(t), v_m(t)), Av_m(t))|.
\end{aligned}$$

We use part 3 of Lemma 3.2 on the non-linear part and apply Hölder's and Young's inequality to the forcing term:

$$\begin{aligned}
\Rightarrow\quad & \frac{1}{2}\frac{\partial}{\partial t}||v_m(t)||^2 + \nu|Av_m(t)|^2 \leq \frac{1}{\nu}|f|^2_{V'} + \frac{\nu}{4}|Av_m(t)|^2 + |(\tilde{B}(u_m(t), v_m(t)), Av_m(t))|\\
\Rightarrow\quad & \frac{1}{2}\frac{\partial}{\partial t}||v_m(t)||^2 + \frac{3\nu}{4}|Av_m(t)|^2 \leq \frac{1}{\nu}|f|^2_{V'} + c||u_m(t)||^{\frac{1}{2}}|Au_m(t)|^{\frac{1}{2}}||v_m(t)||\,|Av_m(t)|.
\end{aligned}$$

We know that the H^1 norm and the H^2 norm are bounded from above and therefore the $||u_m(t)||$ norm and the $|Au_m(t)|$ norm are bounded as well:

$$\begin{aligned}
\Rightarrow\quad & \frac{1}{2}\frac{\partial}{\partial t}||v_m(t)||^2 + \frac{3\nu}{4}|Av_m(t)|^2 \leq \frac{1}{\nu}|f|^2_{V'} + ck_1^{\frac{1}{2}}k_2^{\frac{1}{2}}||v_m(t)||\,|Av_m(t)|\\
\Rightarrow\quad & \frac{1}{2}\frac{\partial}{\partial t}||v_m(t)||^2 + \frac{3\nu}{4}|Av_m(t)|^2 \leq \frac{1}{\nu}|f|^2_{V'} + Ck_1^{\frac{1}{2}}k_2^{\frac{1}{2}}\frac{1}{\nu}||v_m(t)||^2 + \frac{\nu}{4}|Av_m(t)|^2\\
\Rightarrow\quad & \frac{\partial}{\partial t}||v_m(t)||^2 + \nu|Av_m(t)|^2 \leq \frac{2}{\nu}|f|^2_{V'} + Ck_1^{\frac{1}{2}}k_2^{\frac{1}{2}}\frac{1}{\nu}||v_m(t)||^2\\
\Rightarrow\quad & \frac{\partial}{\partial t}||v_m(t)||^2 \leq \frac{2}{\nu}|f|^2_{V'} + Ck_1^{\frac{1}{2}}k_2^{\frac{1}{2}}\frac{1}{\nu}||v_m(t)||^2.
\end{aligned}$$

Applying the Gronwall lemma will result in an upper bound of $||v_m(t)||$ that only depends on the initial condition and the forcing term but not on m.

So far we showed that the Galerkin equations have a unique solution. Before we continue to show that they converge nicely let us gather what estimates we have. Considering equations (3.11), (3.12) and (3.17) we can conclude the following upper bounds that are independent of m:

$$\begin{aligned}
||u_m||^2_{L^\infty([0,T];V)} &\leq \frac{k_1}{\alpha^2}, & ||v_m||^2_{L^\infty([0,T];V')} &\leq k_1,\\
||u_m||^2_{L^2([0,T];\mathcal{D}(A))} &\leq \frac{TK_1 + k_1}{\nu\alpha^2}, & ||v_m||^2_{L^2([0,T];H)} &\leq \frac{TK_1 + k_1}{\nu},\\
||u_m||^2_{L^\infty([0,T];\mathcal{D}(A))} &\leq \frac{k_2}{\alpha^2}, & ||v_m||^2_{L^\infty([0,T];H)} &\leq k_2.
\end{aligned}$$

We only considered the solution $u_m(t)$ so far but we need to consider the first time derivative as well. This is our next step and we will achieve that by estimating the $L^2([0,T];\mathcal{D}(A)')$ norm of Av_m and $\tilde{B}(u_m, v_m)$:

$$\begin{aligned}
||Av_m||^2_{L^2([0,T];\mathcal{D}(A)')} &= \int_0^T ||Av_m||^2_{\mathcal{D}(A)'}dt = \sup_{w\in\mathcal{D}(A)} \int_0^T \left\langle A(u_m + \alpha^2 Au_m), w\right\rangle^2 dt \\
&= \sup_{w\in\mathcal{D}(A)} \int_0^T \langle u_m + \alpha^2 Au_m, \underbrace{Aw}_{\in H}\rangle^2 dt \\
&\leq \sup_{\mu\in H} \int_0^T \left\langle u_m + \alpha^2 Au_m, \mu\right\rangle^2 dt \\
&= ||v_m||_{L^2([0,T];H')} = ||v_m||_{L^2([0,T];H)} \\
&\leq \frac{TK_1 + k_2}{\nu}
\end{aligned}$$

Using the inequality of Lemma 3.2 we can estimate $\tilde{B}(u_m, v_m)$ in a similar way:

$$\begin{aligned}
||\tilde{B}(u_m, v_m)||^2_{L^2([0,T];\mathcal{D}(A)')} &= \int_0^T ||\tilde{B}(u_m, v_m)||^2_{\mathcal{D}(A)'}dt \\
&= \sup_{w\in\mathcal{D}(A),\,||w||=1} \int_0^T \left|\left\langle \tilde{B}(u_m, v_m), w\right\rangle_{\mathcal{D}(A)'\times\mathcal{D}(A)}\right|^2 dt \\
&\leq \sup_{w\in\mathcal{D}(A),\,||w||=1} \int_0^T \left(c||u_m||^{\frac{1}{2}}\,|Au_m|^{\frac{1}{2}}||v_m||\,|w|\right)^2 dt.
\end{aligned}$$

We estimate $|w| \leq (\lambda_1)^{-\frac{1}{2}}||w|| = (\lambda_1)^{-\frac{1}{2}}$ and $||v_m||^2_{L^\infty([0,T];V')} \leq k_1$.

$$\begin{aligned}
&\leq ck_1^2(\lambda_1)^{-1} \int_0^T ||u_m||\,|Au_m|\,dt \\
&\leq ck_1^2(\lambda_1)^{-1} \int_0^T C||u_m||^2 + \alpha^2|Au_m|^2\,dt \\
&\leq c\frac{k_1^2(TK_1 + k_2)}{\nu\lambda_1\alpha^2}.
\end{aligned}$$

Now we can estimate the first time derivative by reordering the terms in the operator equation.

$$\frac{dv_m}{dt} = f - Av_m - \tilde{B}(u_m, v_m).$$

We take the $L^2([0,T];\mathcal{D}(A)')$ norm of the whole equation and insert the previous estimates for all the terms of the right-hand side. As a consequence there exists a constant $\tilde{k}$ depending only on $\nu, \lambda_1, f, \alpha$ and T such that the time derivative can be estimated by it.

$$\left|\left|\frac{dv_m}{dt}\right|\right|^2_{L^2([0,T];\mathcal{D}(A)')} \leq \tilde{k}$$

and in particular

$$\left|\left|\frac{du_m}{dt}\right|\right|^2_{L^2([0,T];H)} \leq \frac{\tilde{k}}{\alpha^2}.$$

Part 3: Passing to the limit

We will use Lion-Aubin's compactness theorem (Theorem 2.9) to obtain our candidate for the limit. We conclude that there exists a subsequence u_{m_k} of u_m satisfying

$$\begin{aligned} u_{m_k} &\longrightarrow u && \text{weakly in } L^2([0,T];\mathcal{D}(A)), \\ u_{m_k} &\longrightarrow u && \text{strongly in } L^2([0,T];V), \\ u_{m_k} &\longrightarrow u && \text{in } C([0,T];H), \end{aligned}$$

and equivalently

$$\begin{aligned} v_{m_k} &\longrightarrow v && \text{weakly in } L^2([0,T];H), \\ v_{m_k} &\longrightarrow v && \text{strongly in } L^2([0,T];V'), \\ v_{m_k} &\longrightarrow v && \text{in } C([0,T];\mathcal{D}(A)'). \end{aligned}$$

From here on we will always consider the converging subsequence u_{m_k} and label it with u_m and accordingly v_{m_k} with v_m. We insert the converging subsequence into the Galerkin equation (3.5) and observe what happens when we pass to the limit. We want the whole equation to converge to an equation of u instead of an equation of u_m, so we need to make sure that all terms converge properly.
For $t_0, t \in [0,T]$ and $w \in \mathcal{D}(A)$ it holds

$$\begin{aligned} &\int_{t_0}^{t} \left(\frac{dv_m(s)}{ds}, w\right) ds + \nu \int_{t_0}^{t} (Av_m(s), w)\, ds \\ &\qquad + \int_{t_0}^{t} (\tilde{B}(u_m, v_m), P_m w)\, ds &&= \int_{t_0}^{t} (P_m f, w)\, ds \\ \Leftrightarrow &(v_m(t), w)\, ds + \nu \int_{t_0}^{t} (Av_m(s), w)\, ds \\ &\qquad + \int_{t_0}^{t} (\tilde{B}(u_m, v_m), P_m w)\, ds &&= (P_m f, w)(t - t_0)\, ds + (v(t_0), w). \end{aligned}$$

We will consider each term by itself and show convergence.

Because v_m converges weakly in $L^2([0,T];H)$ we conclude

$$(v_m(t),w) \longrightarrow (v(t),w).$$

Because v_m converges in $C([0,T];\mathcal{D}(A)')$ we conclude

$$\nu\int_{t_0}^{t}(Av_m(s),w)\,ds = \nu\int_{t_0}^{t}(v_m(s),Aw)\,ds \longrightarrow \nu\int_{t_0}^{t}(v(s),Aw)\,ds = \nu\int_{t_0}^{t}(Av(s),w)\,ds.$$

It is obvious that the orthogonal projection P_m also converges without any troubles.

$$(P_mf,w) \longrightarrow (f,w).$$

The only term left is the non-linear term. We subtract what we want from what we have and try to show that the difference tends to zero with increasing m:

$$\begin{aligned}
&\left|\int_{t_0}^{t}(\tilde{B}(u_m(s),v_m(s)),P_mw)\,ds - \int_{t_0}^{t}\left\langle \tilde{B}(u(s),v(s)),w\right\rangle\,ds\right| \\
\leq&\left|\int_{t_0}^{t}(\tilde{B}(u_m(s),v_m(s)),P_mw) - \left\langle \tilde{B}(u_m(s),v_m(s)),w\right\rangle\,ds\right| \\
&\quad+\left|\int_{t_0}^{t}\left\langle \tilde{B}(u_m(s),v_m(s)),w\right\rangle - \left\langle \tilde{B}(u(s),v_m(s)),w\right\rangle\,ds\right| \\
&\qquad+\left|\int_{t_0}^{t}\left\langle \tilde{B}(u(s),v_m(s)),w\right\rangle - \left\langle \tilde{B}(u(s),v(s)),w\right\rangle\,ds\right| \\
=&\left|\int_{t_0}^{t}\left\langle \tilde{B}(u_m(s),v_m(s)),P_mw-w\right\rangle\,ds\right| + \left|\int_{t_0}^{t}\left\langle \tilde{B}(u_m(s)-u(s),v_m(s)),w\right\rangle\,ds\right| \\
&\quad+\left|\int_{t_0}^{t}\left\langle \tilde{B}(u(s),v_m(s)-v(s)),w\right\rangle\,ds\right| \\
=&I_1+I_2+I_3.
\end{aligned}$$

We will consider each term separately and show that it vanishes. For I_1 we will use Lemma 3.2 and (3.10), namely the inequality $\lambda_1||w||^2 \leq |Aw|^2$, $\forall w \in \mathcal{D}(A)$:

$$
\begin{aligned}
I_1 &\leq \left| \int_{t_0}^{t} c|u_m(s)|\,||v_m(s)|| \underbrace{||P_m w - w||^{\frac{1}{2}}}_{\leq \lambda_1^{-\frac{1}{4}} A(P_m w - w)|^{\frac{1}{2}}} |A(P_m w - w)|^{\frac{1}{2}}\, ds \right| \\
&\leq \left| \int_{t_0}^{t} \frac{c}{\lambda_1^{\frac{1}{4}}} |u_m(s)|\,||v_m(s)||\,|A(P_m w - w)|\, ds \right| \\
&= \frac{c}{\lambda_1^{\frac{1}{4}}} |A(P_m w - w)| \left| \int_{t_0}^{t} |u_m(s)|\,||v_m(s)||\, ds \right|.
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality and increasing the integration range results in:

$$
\begin{aligned}
&\leq \frac{c}{\lambda_1^{\frac{1}{4}}} |A(P_m w - w)| \left(\int_{t_0}^{t} |u_m(s)|^2\, ds \right)^{\frac{1}{2}} \left(\int_{t_0}^{t} ||v_m(s)||^2\, ds \right)^{\frac{1}{2}} \\
&\leq \frac{c}{\lambda_1^{\frac{1}{4}}} \underbrace{|A(P_m w - w)|}_{\longrightarrow 0} \underbrace{\left(\int_{0}^{T} |u_m(s)|^2\, ds \right)^{\frac{1}{2}} \left(\int_{0}^{T} ||v_m(s)||^2\, ds \right)^{\frac{1}{2}}}_{bounded} \\
&\longrightarrow 0.
\end{aligned}
$$

The Terms I_2 and I_3 can be done in a similar fashion. As a consequence of our calculations we conclude that the non-linear term converges too. We found a regular solution of the equations.

Uniqueness

The last part of the proof is showing uniqueness and we will do it by assuming there are two solutions u_1, u_2, with initial values u_{10}, u_{20}. We denote $v_1 = (u_1 + \alpha^2 A u_1), v_2 = (u_2 + \alpha^2 A u_2)$ and the differences $\Delta u = u_1 - u_2, \Delta v = v_1 - v_2$. The solutions u_1, u_2 satisfy

$$
\begin{aligned}
\frac{d}{dt} v_1 + \nu A v_1 + \tilde{B}(u_1, v_1) = f, \\
\frac{d}{dt} v_2 + \nu A v_2 + \tilde{B}(u_2, v_2) = f.
\end{aligned}
$$

Subtracting one from the other yields:

$$
\begin{aligned}
& 0 = \frac{d}{dt}\Delta v + \nu A \Delta v + \tilde{B}(u_1, v_1) - \tilde{B}(u_2, v_2), \\
\Leftrightarrow \quad & 0 = \frac{d}{dt}\Delta v + \nu A \Delta v + \tilde{B}(u_1, v_1) - \tilde{B}(u_2, v_1) + \tilde{B}(u_2, v_1) - \tilde{B}(u_2, v_2), \\
\Leftrightarrow \quad & 0 = \frac{d}{dt}\Delta v + \nu A \Delta v + \tilde{B}(\Delta u, v_1) + \tilde{B}(u_2, \Delta v) \qquad (3.18)
\end{aligned}
$$

Our next step is to test above equation with Δu, but first we calculate:

$$\begin{aligned}
\left\langle \frac{d}{dt}\Delta v(t), \Delta u(t) \right\rangle &= \left\langle \frac{d}{dt}\left(\Delta u(t) + \alpha^2 A\Delta u(t)\right), \Delta u(t) \right\rangle \\
&= \frac{1}{2}\frac{d}{dt}\left(|\Delta u(t)|^2 + \alpha^2||\Delta u(t)||^2\right) \\
\nu \langle A\Delta v(t), \Delta u(t)\rangle &= \nu\left(||\Delta u(t)||^2 + \alpha^2|A\Delta u(t)|^2\right), \\
\left\langle \tilde{B}(\Delta u(t), v_1(t)), \Delta u(t) \right\rangle &= 0.
\end{aligned}$$

After testing with Δu and simplifying equation (3.18), we get:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|\Delta u(t)|^2 + \alpha^2||\Delta u(t)||^2) \\
&\quad + \nu(||\Delta u(t)||^2 + \alpha^2|A\Delta u(t)|^2) \\
&\qquad + \left\langle \tilde{B}(u_2(t), \Delta v(t)), \Delta u(t) \right\rangle && = 0 \\
\Rightarrow \quad &\frac{1}{2}\frac{d}{dt}(|\Delta u(t)|^2 + \alpha^2||\Delta u(t)||^2) \\
&\quad + \nu(||\Delta u(t)||^2 + \alpha^2|A\Delta u(t)|^2) && \leq \left|\left\langle \tilde{B}(u_2(t), \Delta v(t)), \Delta u(t) \right\rangle\right|.
\end{aligned}$$

Our next step is to estimate the right-hand side from above such that we get the initial condition and some terms that can be absorbed by the left-hand side. Therefore we use part 3 of Lemma 3.2 for the non-linear term:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|\Delta u(t)|^2 + \alpha^2||\Delta u(t)||^2) + \nu(||\Delta u(t)||^2 + \alpha^2|A\Delta u(t)|^2) \\
\leq \quad & c\left(||u_2(t)||^{\frac{1}{2}}|Au_2(t)|^{\frac{1}{2}}|\Delta v(t)|\,||\Delta u(t)|| + |Au_2(t)|\,|\Delta v(t)|\,|\Delta u(t)|^{\frac{1}{2}}||\Delta u(t)||^{\frac{1}{2}}\right) \\
= \quad & c|\Delta v(t)|\left(||u_2(t)||^{\frac{1}{2}}|Au_2(t)|^{\frac{1}{2}}\,||\Delta u(t)|| + |Au_2(t)|\,|\Delta u(t)|^{\frac{1}{2}}||\Delta u(t)||^{\frac{1}{2}}\right).
\end{aligned}$$

We will use the scaled Young's inequality to seperate the product with $p = q = 2$, $\varepsilon = \frac{\nu}{2\eta}$, where $\eta = (2\alpha^2 + \frac{1}{\lambda_1})$:

$$\leq \quad cC_\varepsilon\left(||u_2(t)||^{\frac{1}{2}}|Au_2(t)|^{\frac{1}{2}}\,||\Delta u(t)|| + |Au_2(t)|\,|\Delta u(t)|^{\frac{1}{2}}||\Delta u(t)||^{\frac{1}{2}}\right)^2 + \frac{\nu}{2\eta}|\Delta v(t)|^2. \tag{3.19}$$

Let us consider only the term $\frac{\nu}{2\eta}|\Delta v(t)|^2$ first:

$$\begin{aligned}
\frac{\nu}{2\eta}|\Delta v(t)|^2 &= \frac{\nu}{2\eta}\left|\Delta u(t) + \alpha^2 A\Delta u(t)\right|^2 \\
&= \frac{\nu}{2\eta}\Big(\underbrace{|\Delta u(t)|^2}_{\leq \frac{1}{\lambda_1}||\Delta u(t)||^2} + \alpha^4|A\Delta u(t)|^2 + 2\alpha^2\underbrace{(\Delta u(t), A\Delta u(t))}_{=||\Delta u(t)||^2}\Big) \\
&\leq \frac{\nu}{2\eta}\left(\frac{1}{\lambda_1}||\Delta u(t)||^2 + \alpha^4|A\Delta u(t)|^2 + 2\alpha^2||\Delta u(t)||^2\right) \\
&\leq \frac{\nu}{2\eta}\left(\left(2\alpha^2 + \frac{1}{\lambda_1}\right)||\Delta u(t)||^2 + \alpha^4|A\Delta u(t)|^2\right).
\end{aligned}$$

We can estimate $\alpha^4 \leq \alpha^2(2\alpha^2) \leq \alpha^2\left(2\alpha^2 + \frac{1}{\lambda_1}\right)$ and simplify further:

$$\leq \frac{\nu}{2}\left(||\Delta u(t)|| + \alpha^2|A\Delta u(t)|^2\right).$$

Inserting this calculation into (3.19) will result in:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|\Delta u(t)|^2 + \alpha^2||\Delta u(t)||^2) + \nu(||\Delta u(t)||^2 + \alpha^2|A\Delta u(t)|^2) \\
&\leq \quad cC_\varepsilon\left(||u_2(t)||^{\frac{1}{2}}|Au_2(t)|^{\frac{1}{2}}\,||\Delta u(t)|| + |Au_2(t)|\,|\Delta u(t)|^{\frac{1}{2}}||\Delta u(t)||^{\frac{1}{2}}\right)^2 \\
&\quad + \frac{\nu}{2}\left(||\Delta u(t)|| + \alpha^2|A\Delta u(t)|^2\right).
\end{aligned}$$

We will absorb $\frac{\nu}{2}\left(||\Delta u|| + \alpha^2|A\Delta u|^2\right)$ by the left-hand side. Since $\frac{\nu}{2}\left(||\Delta u|| + \alpha^2|A\Delta u|^2\right)$ is positive, we can decrease the left-hand side by dropping this term. Recall that all u_i are bounded in the $H^1(\Omega)$- and $H^2(\Omega)$-norms. We multiply by two and get:

$$\begin{aligned}
&\frac{d}{dt}(|\Delta u(t)|^2 + \alpha^2||\Delta u(t)||^2) \\
&\leq \quad 2cC_\varepsilon\Big(||u_2(t)||^{\frac{1}{2}}|Au_2(t)|^{\frac{1}{2}}\,||\Delta u(t)|| + |Au_2(t)|\,|\Delta u(t)|^{\frac{1}{2}}||\Delta u(t)||^{\frac{1}{2}}\Big)^2 \\
&\leq \quad 2cC_\varepsilon\Big(\underbrace{||u_2(t)||\,|Au_2(t)|}_{\leq const}||\Delta u(t)||^2 + \underbrace{|Au_2(t)|^2}_{\leq const}\ \underbrace{|\Delta u(t)|}_{\leq\lambda^{-1}||\Delta u(t)||}\ \ ||\Delta u(t)||\Big).
\end{aligned}$$

We apply (3.9) and add $|\Delta u(t)|^2$ to the right-hand side and since it is positive, the inequality still holds.

$$\begin{aligned}
&\leq \quad 2CC_\varepsilon\lambda_1^{-1}\alpha^{-2}||\alpha^2\Delta u(t)||^2 \\
&\leq \quad 2CC_\varepsilon\lambda_1^{-1}\alpha^{-2}\left(|\Delta u(t)|^2 + \alpha^2||\Delta u(t)||^2\right).
\end{aligned}$$

The last step is to apply Gronwall's lemma and estimate $|\Delta u(t)|^2 + \alpha^2||\Delta u(t)||^2$ by the initial values and assuming the initial values for u_1 and u_2 coincide, we get uniqueness.

□

3.6 Convergence of the NS-α solution to a solution of NSE

Up till now we showed that there always exists a unique solution for the NS-α equations. Since we modeled turbulence to achieve an approximation of the true solution, we are interested in how good this approximation is and of course if it is converging at all. In this section we will study the limit of α going to zero and show that our sequence of NS-α solutions converges to a solution of the NSE.

Theorem 3.4. *Let $f \in H, u_0 \in V$ and let u_α be the solution of the NS-α problem for an arbitrary $\alpha \geq 0$. We abbreviate $v_\alpha = u_\alpha + \alpha^2 Au_\alpha$. Then there exist subsequences $u_{\alpha_j}, v_{\alpha_j}$ and a function u such that the following statements hold for $\alpha \to 0^+$:*

1. $u_{\alpha_j} \to u$, *strongly in* $L^2_{Loc}([0,\infty); H)$,
2. $u_{\alpha_j} \to u$, *weakly in* $L^2_{Loc}([0,\infty); V)$,
3. $v_{\alpha_j} \to u$, *strongly in* $L^2_{Loc}([0,\infty); H)$,
4. $v_{\alpha_j} \to u$, *strongly in* $L^2_{Loc}([0,\infty); V')$,
5. *For every* $T \in (0,\infty)$ *and every* $w \in H$: $(u_{\alpha_j}, w) \to (u, w)$ *uniformly on* $[0,T]$.

This means u *is a weak solution of the NSE.*

Proof. Let $T \geq 0$ arbitrary but fixed. Since (3.11) and (3.12) are also true for u_α and v_α we conclude that there are subsequences $u_{\alpha_j}, v_{\alpha_j}$ and u, v such that:

$$\begin{aligned} u_{\alpha_j} &\to u \qquad \text{weakly in } L^2((0,T];V), \\ v_{\alpha_j} &\to v \qquad \text{weakly in } L^2((0,T];H). \end{aligned}$$

Our next step will be to show the boundedness of the time derivative. Before we go further, let us do some auxiliary calculations first. We know that for an eigenpair (λ,φ) of an operator A it holds:

$$A^{-1}\varphi = \lambda^{-1}\varphi.$$

In our case all eigenvalues λ_i of A are positive and therefore the eigenvalues of $(I+\alpha A)$ are larger than one, which means that the eigenvalues of $(I+\alpha A)^{-1}$ are smaller than one and it holds for arbitrary $w \in H$:

$$\left|(I+\alpha A)^{-1}w\right| = \left|\sum_i c_i (I+\alpha\lambda_i)^{-1}\varphi_i\right| \leq \left|\sum_i c_i\varphi_i\right| = |w|.$$

Let us consider the NS-α equation now and find an upper bound for the time derivative:

$$\begin{aligned}
& \frac{dv_\alpha}{dt} + \nu A v_\alpha + \tilde{B}(u_\alpha, v_\alpha) = f \\
\Leftrightarrow\quad & \frac{du_\alpha}{dt} + \nu A u_\alpha + (I+\alpha^2 A)^{-1}\tilde{B}(u_\alpha, v_\alpha) = (I+\alpha^2 A)^{-1} f \\
\Leftrightarrow\quad & A^{-1}\frac{du_\alpha}{dt} + \nu u_\alpha + A^{-1}(I+\alpha^2 A)^{-1}\tilde{B}(u_\alpha, v_\alpha) = A^{-1}(I+\alpha^2 A)^{-1} f \\
\Rightarrow\quad & \left|\left|A^{-1}\frac{du_\alpha}{dt}\right|\right|_{L^2((0,T],H')} \leq \left|\left|A^{-1}(I+\alpha^2 A)^{-1} f\right|\right|_{L^2((0,T],H')} + \nu \left|\left|u_\alpha\right|\right|_{L^2((0,T],H')} \\
& \qquad\qquad\qquad\qquad + \left|A^{-1}(I+\alpha^2 A)^{-1}\tilde{B}(u_\alpha, v_\alpha)\right|_{L^2((0,T],H')} \\
\Rightarrow\quad & \left|\left|A^{-1}\frac{du_\alpha}{dt}\right|\right|_{L^2((0,T],H')} \leq \left|\left|A^{-1} f\right|\right|_{L^2((0,T],H')} + \nu \left|\left|u_\alpha\right|\right|_{L^2((0,T],H} \\
& \qquad\qquad\qquad\qquad + \left|\left|A^{-1}\tilde{B}(u_\alpha, v_\alpha)\right|\right|_{L^2((0,T],H')}.
\end{aligned}$$

Since the nonlinear part is the only term that is not bounded yet we will focus on it. Applying Lemma 3.2 and the inequality $|u_\alpha|^{\frac{1}{2}} = (|u_\alpha|^2)^{\frac{1}{4}} \leq (\frac{1}{\lambda_1}||u_\alpha||^2)^{\frac{1}{4}} = \left(\frac{1}{\lambda_1}\right)^{\frac{1}{4}} ||u_\alpha||^{\frac{1}{2}}$:

$$
\begin{aligned}
\left|A^{-1}\tilde{B}(u_\alpha(t), v_\alpha(t))\right| &= \sup_{w\in H, |w|=1} \left|(A^{-1}\tilde{B}(u_\alpha(t), v_\alpha(t)), w)\right| \\
&= \sup_{w\in H, |w|=1} \left|(\tilde{B}(u_\alpha(t), v_\alpha(t)), A^{-1}w)\right| \\
&\leq c \sup_{w\in H, |w|=1} \Big(|u_\alpha(t)|^{\frac{1}{2}} ||u_\alpha(t)||^{\frac{1}{2}} |v_\alpha(t)| \underbrace{|AA^{-1}w|}_{=1} \\
&\qquad + |v_\alpha(t)|\, ||u_\alpha(t)||\, \underbrace{||A^{-1}w||^{\frac{1}{2}}}_{\leq \lambda^{-\frac{1}{4}} |w|^{\frac{1}{2}}} \Big) \\
&\leq 2c\lambda^{-\frac{1}{4}} |v_\alpha(t)|\, ||u_\alpha(t)|| \\
&\leq 2c\lambda^{-\frac{1}{4}} \left(|u_\alpha(t)|\, ||u_\alpha(t)|| + \underbrace{\alpha^2 |Au_\alpha(t)|\, ||u_\alpha(t)||}_{\geq 0} \right).
\end{aligned}
$$

We square the whole inequality and use the inequality: $(a+b)^2 \leq 2(a^2+b^2)$:

$$
\begin{aligned}
\left|A^{-1}\tilde{B}(u_\alpha(t), v_\alpha(t))\right|^2 &\leq 4c^2\lambda^{-\frac{1}{2}} \left(|u_\alpha(t)|\, ||u_\alpha(t)|| + \alpha^2 |Au_\alpha(t)|\, ||u_\alpha(t)|| \right)^2 \\
&\leq 4c^2\lambda^{-\frac{1}{2}} \left(|u_\alpha(t)|^2\, ||u_\alpha(t)||^2 + \alpha^2 |Au_\alpha(t)|^2 \alpha^2 ||u_\alpha(t)||^2 \right).
\end{aligned}
$$

We apply (3.11):

$$
\leq 4c^2\lambda^{-\frac{1}{2}} k_1 \left(||u_\alpha(t)||^2 + \alpha^2 |Au_\alpha(t)|^2 \right).
$$

We integrate over $[0, T]$ and use inequality (3.12):

$$
\begin{aligned}
\int_0^T \left|A^{-1}\tilde{B}(u_\alpha(t), v_\alpha(t))\right|^2 dx &\leq \int_0^T 4c^2\lambda^{-\frac{1}{2}} k_1 \left(||u_\alpha(t)||^2 + \alpha^2 |Au_\alpha(t)|^2 \right) dx \\
&\leq 4c^2\lambda^{-\frac{1}{2}} k_1 \frac{1}{\nu} (TK_1 + k_1).
\end{aligned}
$$

Using previous calculations we conclude a bound for the time-derivative:

$$
\begin{aligned}
\int_0^T \left\| \frac{du_\alpha(t)}{dt} \right\|_{\mathcal{D}(A)'}^2 dt &= \int_0^T \sup_{w \in \mathcal{D}(A)} \left\langle \frac{du_\alpha(t)}{dt}, w \right\rangle_{\mathcal{D}(A)' \times \mathcal{D}(A)} dt \\
&= \int_0^T \sup_{\tilde{w} \in H} \left\langle \frac{du_\alpha(t)}{dt}, A^{-1}\tilde{w} \right\rangle_{\mathcal{D}(A)' \times \mathcal{D}(A)} dt \\
&= \int_0^T \sup_{\tilde{w} \in H} \left\langle A^{-1}\frac{du_\alpha(t)}{dt}, \tilde{w} \right\rangle_{H \times H} dt \\
&= \int_0^T \left| A^{-1}\frac{du_\alpha(t)}{dt} \right|^2 dt < \infty.
\end{aligned}
$$

Finally we found an upper bound for the time derivative in the $L^2((0,T], \mathcal{D}(A)')$ norm.

We have shown the following upper bounds so far:

- u_α in $L^2((0,T], V)$,
- v_α in $L^2((0,T], H)$,
- $\frac{du_\alpha(t)}{dt}$ in $L^2((0,T], \mathcal{D}(A)')$.

These bounds are sufficient to apply Aubin's compactness theorem and obtain a subsequence that we will also denote by $\{u_{\alpha_j}\}_j$ that converges to u strongly in $L^2((0,T], H)$. Furthermore we have:

$$
\begin{aligned}
\int_0^T \left| A^{-\frac{1}{2}}(v_{\alpha_j}(t) - u_{\alpha_j}(t)) \right|^2 dt &= \int_0^T \left| A^{-\frac{1}{2}}(u_{\alpha_j}(t) + \alpha_j^2 A u_{\alpha_j}(t) - u_{\alpha_j}(t)) \right|^2 dt \\
&= \int_0^T \left| \alpha_j^2 A^{\frac{1}{2}} u_{\alpha_j}(t) \right|^2 dt \\
&= \alpha_j^2 ||u_\alpha||_{L^2((0,T],V)}^2 \\
&\leq \alpha_j^2 (TK_1 + k_1).
\end{aligned}
$$

This means that v_{α_j} converges to u_{α_j} strongly in $L^2((0,T], V')$ and that $v(t) = u(t)$ almost everywhere in $[0,T]$.

We now managed to find a limit u of the subsequence u_{α_j}, we need to briefly argue that the limit satisfies the NSE. Since we have strong convergence in $L^2((0,T], V')$ all the terms in NS-α equations converge nicely.

$\square$

4 Numerical analysis

In the previous section, we showed the existence of a solution by verifying the convergence of a sequence of solutions in a continuously growing space H_m. This result does not contain any information about the convergence speed of these solutions, which is relevant for numerical computations. In this section, we will deal with this question and deduce an upper bound for the error in terms of the best approximation in the chosen FEM space under the assumption of no-slip and periodic BC.

4.1 Preliminaries

The NS-α problem states:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (\nabla \times u) \times \bar{u} + \nabla P = f, \\ \bar{u} - \alpha^2 \Delta \bar{u} + \nabla \lambda = u, \\ \nabla \cdot u = \nabla \cdot \bar{u} = 0, \\ \text{Initial and boundary condition.} \end{cases} \tag{4.1}$$

Let us define our spaces and the corresponding discrete spaces. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain then we define:

$$\begin{aligned} X \equiv H_0^1(\Omega) &= \left\{ v \in H^1(\Omega) : v|_{\partial\Omega} = 0 \right\}, \\ Q \equiv L_0^2(\Omega) &= \left\{ q \in L^2(\Omega) : \int_\Omega q \, dx = 0 \right\}, \\ V &= \left\{ v \in X : \int_\Omega q \nabla \cdot v \, dx = 0, \quad \forall q \in Q \right\}. \end{aligned}$$

Remark *Analogously to the previous chapter, we will abbreviate the L^2 norm by $|\cdot|$ and the H^1 seminorm by $||\cdot||$, every other seminorm in $H^k(\Omega)$ will be denoted as $|\cdot|_k$.*

For the discrete spaces, we will assume conforming finite element spaces $X_h \subset X, V_h \subset V, Q_h \subset Q$, satisfying the discrete uniform Ladyschenskaja-Babuška-Brezzi condition, also known as the discrete inf-sup condition. The condition reads: there exists $\beta > 0$ such that for all mesh sizes $h > 0$

$$\inf_{q_h \in Q_h} \sup_{v_h \in X_h} \frac{(q_h, \nabla \cdot v_h)}{||q_h|| \, ||\nabla v_h||} \geq \beta > 0$$

holds. Now we define the finite element discretisation of (4.1):
For $f \in L^2(0, T; X'(\Omega))$, find $u_h, \overline{u_h} \in X_h$ and $p_h, \lambda_h \in Q_h$ satisfying

$$\begin{cases} \left\langle \dfrac{\partial u_h}{\partial t}, v_h \right\rangle + \nu(\nabla u_h, \nabla v_h) + ((\nabla \times u_h) \times \overline{u_h}, v_h) + (\nabla p_h, v_h) = (f, v_h) & \forall v_h \in X_h, \\ (\overline{u_h}, v_h) + \alpha^2(\nabla \overline{u_h}, \nabla v_h) + (\nabla \lambda_h, v_h) = (u_h, v_h) & \forall v_h \in X_h, \\ (\nabla \cdot u_h, q_h) = (\nabla \cdot \overline{u_h}, q_h) = 0 & \forall q_h \in Q_h. \end{cases} \tag{4.2}$$

An equivalent problem is formulated over discretely divergence free functions V_h:
For $f \in L^2(0,T;X'(\Omega))$ find $u_h, \overline{u_h} \in V_h$ satisfying

$$\begin{cases} \left\langle \dfrac{\partial u_h}{\partial t}, v_h \right\rangle + \nu(\nabla u_h, \nabla v_h) + ((\nabla \times u_h) \times \overline{u_h}, v_h) = (f, v_h) & \forall v_h \in V_h, \\ (\overline{u_h}, v_h) + \alpha^2(\nabla \overline{u_h}, \nabla v_h) = (u_h, v_h) & \forall v_h \in V_h, \\ (\nabla \cdot u_h, q_h) = (\nabla \cdot \overline{u_h}, q_h) = 0 & \forall q_h \in Q_h. \end{cases} \tag{4.3}$$

Definition (Discrete differential filter)**.** Let $\psi \in L^2(\Omega)$ and $\alpha > 0$. Then, he filtered version of ψ is denoted by $\overline{\psi}^h \in V_h$ and is the unique solution of

$$(\psi, v_h) = (\overline{\psi}^h, v_h) + \alpha^2(\nabla\overline{\psi}^h, \nabla v_h) \qquad \forall v_h \in V_h. \tag{4.4}$$

Definition (Discrete Laplacian)**.** Let $\Psi \in X$ then the discrete Laplacian $\Delta_h : X \to V_h$ is defined by

$$\Delta_h \Psi = \Xi_h,$$

where Ξ_h is the unique solution of

$$(\Xi_h, v_h) = -(\nabla\Psi, \nabla v_h) \qquad \forall v_h \in V_h. \tag{4.5}$$

Using these definitions we can draw some useful conclusions. For $\psi_h \in V_h$ we use (4.4) with $v_h = \overline{\psi_h}^h$ to get:

$$(\psi_h, \overline{\psi_h}^h) = |\overline{\psi_h}^h|^2 + \alpha^2 |\nabla \overline{\psi_h}^h|^2. \tag{4.6}$$

Additionally, we can reorder (4.4) and since $\Delta_h \overline{\psi_h}^h \in V_h$ we can test with $v_h = \psi_h - (\overline{\psi_h}^h - \alpha^2 \Delta_h \overline{\psi_h}^h)$ to get:

$$\left|\psi_h - (\overline{\psi_h}^h - \alpha^2 \Delta_h \overline{\psi_h}^h)\right|^2 = 0$$

and therefore

$$\psi_h = \overline{\psi_h}^h - \alpha^2 \Delta_h \overline{\psi_h}^h \tag{4.7}$$

holds almost everywhere. Since we assumed $\psi_h, \overline{\psi_h}^h \in V_h$, we can apply the gradient to ψ_h which results in the following identity:

$$(\nabla \psi_h, \nabla v_h) = (\nabla \overline{\psi_h}^h, \nabla v_h) - \alpha^2 (\nabla \Delta_h \overline{\psi_h}^h, \nabla v_h) \qquad \forall v_h \in V_h.$$

Choosing $v = \overline{\psi_h}^h$ and applying (4.5) results in:

$$\begin{aligned}
\Rightarrow \quad & (\nabla \psi_h, \nabla \overline{\psi_h}^h) = (\nabla \overline{\psi_h}^h, \nabla \overline{\psi_h}^h) - \alpha^2 (\nabla \Delta_h \overline{\psi_h}^h, \nabla \overline{\psi_h}^h) \\
\Leftrightarrow \quad & (\nabla \psi_h, \nabla \overline{\psi_h}^h) = |\nabla \overline{\psi_h}^h|^2 + \alpha^2 |\Delta_h \overline{\psi_h}^h|^2. \qquad (4.8)
\end{aligned}$$

These are important identities that will be used throughout this chapter.

4.2 Stability and convergence

Theorem 4.1 (Stability). *Let $u_h \in V_h$ satisfy (4.3) and assume the FEM space X_h has no-slip or periodic BC. Then $\exists M(\Omega) > 0$, independent of α, g, ν such that if $0 < \alpha \leq M h \nu^{\frac{1}{4}}$ is true, u_h satisfies:*

$$|\overline{u_h(t)}^h|^2 + \alpha^2 |\nabla \overline{u_h(t)}^h|^2 + \nu \int_0^T |\nabla \overline{u_h(t)}^h|^2 + \alpha^2 |\Delta \overline{u_h(t)}^h|^2 \, dt \ \leq \ K,$$

$$|u_h(t)|^2 + \nu \int_0^T |\nabla u_h(t)|^2 \, dt \ \leq \ K.$$

Proof. Since u_h is a solution of (4.3) we can use the test function $v_h = \overline{u_h}^h$:

$$\begin{aligned}
& \underbrace{\left(\frac{\partial u_h(t)}{\partial t}, \overline{u_h(t)}^h\right)}_{= \frac{1}{2}\frac{\partial}{\partial t}\left(|\overline{u_h(t)}^h|^2 + \alpha^2 |\nabla \overline{u_h(t)}^h|^2\right)} + \nu \underbrace{(\nabla u_h(t), \nabla \overline{u_h(t)}^h)}_{= |\nabla \overline{u_h(t)}^h|^2 + \alpha^2 |\Delta \overline{u_h(t)}^h|^2} \\
& \qquad + \underbrace{((\nabla \times u_h(t)) \times \overline{u_h(t)}, \overline{u_h(t)}^h)}_{=0} \qquad = (f, \overline{u_h(t)}^h) \\
\Leftrightarrow \quad & \frac{1}{2}\frac{\partial}{\partial t}\left(|\overline{u_h(t)}^h|^2 + \alpha^2 |\nabla \overline{u_h(t)}^h|^2\right) \\
& \qquad + \nu \left(|\nabla \overline{u_h(t)}^h|^2 + \alpha^2 |\Delta \overline{u_h(t)}^h|^2\right) \qquad = (f, \overline{u_h(t)}^h).
\end{aligned}$$

Considering only the right-hand side we apply Hölder's and Young's inequality to manipulate $\overline{u_h}^h$ such that it can be absorbed by the left-hand side.

$$(f, \overline{u_h(t)}^h) \;\leq\; ||f||_{X'}\, |\nabla \overline{u_h(t)}^h| \;\leq\; \frac{1}{2\nu}||f||^2_{X'} + \frac{\nu}{2}|\nabla \overline{u_h(t)}^h|^2$$

This results in:

$$\frac{\partial}{\partial t}\left(|\overline{u_h(t)}^h|^2 + \alpha^2|\nabla \overline{u_h(t)}^h|^2\right) + \nu\left(|\nabla \overline{u_h(t)}^h|^2 + \alpha^2|\Delta \overline{u_h(t)}^h|^2\right) \;\leq\; \frac{1}{\nu}\nu||f||^2_{X'}$$

and after integration over $[0, T]$ we get:

$$\begin{aligned}
|\overline{u_h(T)}^h|^2 + \alpha^2|\nabla \overline{u_h(T)}^h|^2 + \nu \int_0^T \left(|\nabla \overline{u_h(t)}^h|^2 + \alpha^2|\Delta \overline{u_h(t)}^h|^2\right) dt \\
\leq \quad \underbrace{\frac{1}{\nu}\int_0^T ||f||^2_{X'}\, dt + |\overline{u_h(0)}^h|^2 + \alpha^2|\nabla \overline{u_h(0)}^h|^2}_{constant}.
\end{aligned}$$

This proves the first part. For the second part we recall the inverse inequality for no-slip or periodic BC:

$$|\nabla \psi_h| \leq \frac{C}{h}|\psi_h| \qquad \forall \psi_h \in X_h. \tag{4.9}$$

Assuming $0 < \alpha \leq Mh\nu^{\frac{1}{4}} = Ch < \infty$ we can apply the inverse inequality on (4.6) and (4.8) to obtain:

$$c(\psi_h, \overline{\psi_h}^h) \leq |\psi_h|^2 \leq C(\psi_h, \overline{\psi_h}^h) \tag{4.10}$$

and

$$c(\nabla\psi_h, \overline{\nabla\psi_h}^h) \leq |\nabla\psi_h|^2 \leq C(\nabla\psi_h, \overline{\nabla\psi_h}^h). \tag{4.11}$$

This is sufficient to show the second part.

□

Theorem 4.2 (Existence of a solution)**.** *Consider* (4.3) *then the mapping*

$$u_h : [0, T] \to V_h$$

exists. In particular every solution u_h *satisfies:*

$$u_h \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)).$$

Proof. This proof is straight-forward and analog to the existence of the local solution proofs from the last sections. We select a basis of V_h expand (4.3) in terms of these basis functions, consequently this results in a system of ordinary differential equations and its solvability is shown by using Carathéodory's theorem and the stability estimates. □

Theorem 4.3 (FEM approximations). *The following FEM estimates hold for $k = 0, 1$:*

$$\begin{aligned} |\overline{u} - \overline{u}^h| &\le c\left(\alpha h^k + h^{k+1}\right)|\overline{u}|_{k+1}, \\ |\nabla(u - \overline{u})| &\le c|\nabla u|, \end{aligned}$$

and as a consequence:

$$|\nabla(u - \overline{u}^h)| \le C\left(|\nabla u| + h^k|\overline{u}|_{k+1}\right).$$

Proof. These estimates can be proven by using standard FEM approximation inequalities. For a more detailed analysis see [LMNR08] Chapter two.
The third inequality results from the triangle inequality and the inverse inequality:

$$\begin{aligned} |\nabla(u - \overline{u}^h)| &\le |\nabla(u - \overline{u})| + \underbrace{|\nabla(\overline{u} - \overline{u}^h)|}_{\le \frac{c}{h}|\overline{u} - \overline{u}^h|} \\ &\le c|\nabla u| + \frac{c}{h}|\overline{u} - \overline{u}^h| \\ &\le c|\nabla u| + \frac{c}{h}\underbrace{\left(\alpha h^k + h^{k+1}\right)}_{=Ch^{k+1},\text{ because }\alpha = Ch}|\overline{u}|_{k+1} \\ &\le C\left(|\nabla u| + h^k|\overline{u}|_{k+1}\right). \end{aligned}$$

□

Remark *If we assume $\alpha \le Mh\nu^{\frac{1}{4}}$ then:*

$$\begin{aligned} |(u - \overline{u}^h)| &\le |u - \overline{u}| + |\overline{u} - \overline{u}^h| \\ &\le \alpha^2|\Delta\overline{u}| + |\overline{u} - \overline{u}^h| \\ &\le \alpha^2|\Delta\overline{u}| + c\left(\alpha h^k + h^{k+1}\right)|\overline{u}|_{k+1} \\ &\le \alpha^2|\Delta\overline{u}| + c\left(M\nu^{\frac{1}{4}}h^{k+1} + h^{k+1}\right)|\overline{u}|_{k+1} \\ &\le \alpha^2|\Delta\overline{u}| + c(1 + \nu^{\frac{1}{4}})h^{k+1}|\overline{u}|_{k+1}. \end{aligned}$$

So we get the following estimate that we are going to use during the proof:

$$|u - \overline{u}^h| \le c(1 + \nu^{\frac{1}{4}})(\alpha^2|\Delta\overline{u}| + h^{k+1}|\overline{u}|_{k+1}). \tag{4.12}$$

Definition (Strong solution of NSE)**.** A solution u of NSE is called a strong solution if for $u_0 \in V$ and

$$u \in C([0,T);V) \cap L^2([0,T);H^2(\Omega)) \quad \text{and} \quad \frac{\partial}{\partial t} u(t,\cdot) \in L^2([0,T];H)$$

is satisfied.

Theorem 4.4 (Convergence)**.** *Let $X_h \subset X$ be a conforming FEM space satisfying the inverse inequality and therefore is equipped with no-slip or periodic BC. Assume $u(t,\cdot) \in V$ is a strong solution of NSE, satisfying*

$$\nabla \times u \in L^2(0,T;L^\infty(\Omega)), \qquad \nabla u \in L^4(0,T;L^2(\Omega)) \tag{4.13}$$

and the filtered solution $\overline{u}$ satisfies:

$$\overline{u} \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \tag{4.14}$$

independent of α. Let $0 < \alpha < 1$, then there exists a constant $M > 0$ such that if for some fixed $\tilde{M} > 0$

$$0 < \tilde{M} h \le \alpha \le M h \nu^{\frac{1}{4}} \tag{4.15}$$

the solution $u_h \in V_h$ of (4.3) satisfies:

$$\begin{aligned}
&\sup_{0\le t\le T} |u(t) - u_h(t)|^2 + \int_0^T |u(s) - u_h(s)|^2 \, ds \\
&\le \quad \sup_{0\le t\le T} \inf_{\tilde{u}(t)\in V_h} |u(t) - \tilde{u}(t)|^2 + |u(0) - \tilde{u}(0)|^2 + \inf_{\tilde{u}(t)\in V_h} \int_0^T |u(s)) - \tilde{u}(s)|^2 \, ds \\
&\quad + C \inf_{\tilde{u}(t)\in V_h} \int_0^T \left(\alpha^4 |\Delta \overline{u(s)}|^2 + h^{2k+2} |\overline{u(s)}|^2_{k+1} \right) ds \\
&\quad + C \inf_{\tilde{u}(t)\in V_h} \int_0^T \left| \frac{\partial}{\partial s} (u(s)) - \tilde{u}(s)) \right|^2 ds \\
&\quad + C \inf_{\tilde{u}(t)\in V_h} \left(\int_0^T |\nabla (u(s)) - \tilde{u}(s))|^4 \, ds \right)^{\frac{1}{2}} + C \inf_{\tilde{u}(t)\in V_h} \int_0^T |P - q_h|^2 \, ds
\end{aligned}$$

Proof. To derive the error bound, we will divide the proof into different parts to help us keep the overview. It will have the following structure:

1. Obtain the error equation,
2. Split the error term into $u - u_h = \eta + \phi_h$,
3. Estimating the trilinear term,
   3.1 Estimating $((\nabla \times \eta) \times u, \overline{\phi_h}^h)$,
   3.2 Estimating $((\nabla \times \phi_h) \times u, \overline{\phi_h}^h)$,
   3.3 Estimating $((\nabla \times u_h) \times (\overline{u_h}^h - u_h), \overline{\phi_h}^h)$,
4. Continue to estimate the inequality from part 2, using results from part 3,
5. Putting everything together.

We will sometimes drop the explicit time dependency for the sake of notation. In general u, u_h and all their filtered versions are time dependent but all test functions $v_h \in V_h$ are not.

Part 1: Obtain the error equation
We subtract (4.3) from the weak formulation of (4.1) and add a zero in form of $(-q_h, \nabla \cdot v_h)$, for arbitrary $q_h \in Q_h$. Then for all $v_h \in V_h$ it holds:

$$\begin{aligned}\left(\frac{\partial}{\partial t}(u - u_h), v_h\right) + \nu(\nabla(u - u_h), \nabla v_h) = (P - q_h, \nabla \cdot v_h) + ((\nabla \times u_h) \times \overline{u_h}^h, v_h)\\ - ((\nabla \times u) \times u, v_h)\end{aligned}$$

The above equation is called error equation.

Part 2: Split the error
For arbitrary $\tilde{u} \in V_h$ we can split the error:

$$u(t) - u_h(t) = \underbrace{u(t) - \tilde{u}}_{=\eta(t)} + \underbrace{\tilde{u} - u_h(t)}_{=\phi_h(t)} = \eta(t) + \phi_h(t).$$

We insert it in the error equation and rearrange the terms such that only terms containing ϕ_h are on the left-hand side:

$$\begin{aligned}\left(\frac{\partial}{\partial t}\phi_h, v_h\right) + \nu(\nabla\phi_h, \nabla v_h) =& (P - q_h, \nabla \cdot v_h) + ((\nabla \times u_h) \times \overline{u_h}^h, v_h)\\ & - ((\nabla \times u) \times u, v_h) - \left(\frac{\partial}{\partial t}\eta, v_h\right) - \nu(\nabla\eta, v_h).\end{aligned} \tag{4.16}$$

We will test (4.16) with $v_h = \overline{\phi_h}^h$ and apply (4.6) and (4.8).

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\left(|\overline{\phi_h}^h|^2+\alpha^2|\nabla\overline{\phi_h}^h|^2\right)+\nu\left(|\nabla\overline{\phi_h}^h|^2+\alpha^2|\Delta_h\overline{\phi_h}^h|^2\right)\\
&\qquad=(P-q_h,\nabla\cdot\overline{\phi_h}^h)+((\nabla\times u_h)\times\overline{u_h}^h,\overline{\phi_h}^h)-((\nabla\times u)\times u,\overline{\phi_h}^h)\\
&\qquad\quad-\left(\frac{\partial}{\partial t}\eta,\overline{\phi_h}^h\right)-\nu(\nabla\eta,\nabla\overline{\phi_h}^h)\\
&\qquad\le|P-q_h|\left|\nabla\cdot\overline{\phi_h}^h\right|+\left|\frac{\partial}{\partial t}\eta\right|\left|\overline{\phi_h}^h\right|+\nu\,|\nabla\eta|\left|\nabla\overline{\phi_h}^h\right|\\
&\qquad\quad+((\nabla\times u_h)\times\overline{u_h}^h,\overline{\phi_h}^h)-((\nabla\times u)\times u,\overline{\phi_h}^h).
\end{aligned}$$

Part 3: Estimating the trilinear term

We will only consider the trilinear part of the estimate and show upper bounds in terms of η and expressions we can absorb by the left-hand side. Firstly we will add a zero:

$$\nabla\times u=\nabla\times u_h-\nabla\times(u_h-u)$$

and insert it into the second part of the trilinear term:

$$\begin{aligned}
&((\nabla\times u_h)\times\overline{u_h}^h,\overline{\phi_h}^h)-((\nabla\times u)\times u,\overline{\phi_h}^h)\\
&\quad=((\nabla\times u_h)\times\overline{u_h}^h,\overline{\phi_h}^h)-((\nabla\times u_h)\times u,\overline{\phi_h}^h)+((\nabla\times(u_h-u))\times u,\overline{\phi_h}^h).
\end{aligned}$$

We simplify and split $u_h-u=-(u-u_h)=-\eta-\phi_h$.

$$\begin{aligned}
&=((\nabla\times u_h)\times(\overline{u_h}^h-u),\overline{\phi_h}^h)+((\nabla\times(u_h-u))\times u,\overline{\phi_h}^h)\\
&=((\nabla\times u_h)\times(\overline{u_h}^h-u),\overline{\phi_h}^h)-((\nabla\times\eta)\times u,\overline{\phi_h}^h)-((\nabla\times\phi_h)\times u,\overline{\phi_h}^h).
\end{aligned}$$

Before we start showing an upper bound for all three terms let us consider some auxiliary calculations regarding the constant C_ε in Young's inequality, since we are going to use Young's inequality multiple times in one calculation. The following statement hold for C_ε:

$$C_\varepsilon=\frac{(p\,\varepsilon)^{-\frac{q}{p}}}{p}=C\varepsilon^{-\frac{q}{p}}.$$

Part 3.1: Estimating $((\nabla\times\eta)\times u,\overline{\phi_h}^h)$

We will apply the general Hölder inequality and Young's inequality twice, once with $p=4,q=\frac{4}{3}$ and once with $p=\frac{3}{2},q=3$:

$$\begin{aligned}
((\nabla\times\eta)\times u,\overline{\phi_h}^h)&\le C|\nabla\eta|\,|\nabla u|\,|\overline{\phi_h}^h|^{\frac{1}{2}}\,|\nabla\overline{\phi_h}^h|^{\frac{1}{2}}\\
&\le C\nu^{-\frac{1}{3}}|\nabla\eta|^{\frac{4}{3}}\,|\nabla u|^{\frac{4}{3}}\,|\overline{\phi_h}^h|^{\frac{2}{3}}+\frac{\nu}{24}|\nabla\overline{\phi_h}^h|^2\\
&\le\frac{2}{3}(C\nu^{-\frac{1}{3}})^{\frac{3}{2}}|\nabla\eta|^2+\frac{1}{3}|\nabla u|^4\,|\overline{\phi_h}^h|^2+\frac{\nu}{24}|\nabla\overline{\phi_h}^h|^2\\
&\le C\nu^{-\frac{1}{2}}|\nabla\eta|^2+|\nabla u|^4\,|\overline{\phi_h}^h|^2+\frac{\nu}{24}|\nabla\overline{\phi_h}^h|^2.
\end{aligned}\tag{4.17}$$

Part 3.2: Estimating $((\nabla \times \phi_h) \times u, \overline{\phi_h}^h)$
We will use identity (4.7) and Hölder's inequality:

$$\begin{aligned}
\left((\nabla \times \phi_h) \times u, \overline{\phi_h}^h\right) &= \left(\left(\nabla \times (\overline{\phi_h}^h - \Delta_h \overline{\phi_h}^h)\right) \times u, \overline{\phi_h}^h\right) \\
&= \left((\nabla \times \overline{\phi_h}^h) \times u, \overline{\phi_h}^h\right) - \alpha^2 \left((\nabla \times \Delta_h \overline{\phi_h}^h) \times u, \overline{\phi_h}^h\right) \\
&\leq C \left(|\nabla \overline{\phi_h}^h| \, |\nabla u| \, |\overline{\phi_h}^h|^{\frac{1}{2}} \, |\nabla \overline{\phi_h}^h|^{\frac{1}{2}} \right. \\
&\qquad \left. + \alpha^2 |\nabla \Delta_h \overline{\phi_h}^h| \, |\nabla u| \, |\overline{\phi_h}^h|^{\frac{1}{2}} \, |\nabla \overline{\phi_h}^h|^{\frac{1}{2}} \right) \\
&\leq C \left(|\nabla \overline{\phi_h}^h|^{\frac{3}{2}} \, |\overline{\phi_h}^h|^{\frac{1}{2}} \, |\nabla u| + \alpha^2 |\nabla \Delta_h \overline{\phi_h}^h| \, |\nabla u| \, |\overline{\phi_h}^h|^{\frac{1}{2}} \, |\nabla \overline{\phi_h}^h|^{\frac{1}{2}} \right).
\end{aligned}$$

Since we can absorb $|\nabla \overline{\phi_h}^h|^2$ terms with the left-hand side, we will use Young's inequality to get the right exponent, we also force the constant to be $\frac{\nu}{48}$. For the first term we will use $p = \frac{4}{3}$ and for the second term $p = 4$.

$$\begin{aligned}
&\leq \frac{\nu}{48} |\nabla \overline{\phi_h}^h|^2 + C\nu^{-3} |\overline{\phi_h}^h|^2 \, |\nabla u|^4 \\
&\qquad + \frac{\nu}{48} |\nabla \overline{\phi_h}^h|^2 + \alpha^{\frac{8}{3}} C \nu^{-\frac{1}{3}} |\nabla \Delta_h \overline{\phi_h}^h|^{\frac{4}{3}} \, |\nabla u|^{\frac{4}{3}} \, |\overline{\phi_h}^h|^{\frac{2}{3}} \\
&\leq \frac{\nu}{24} |\nabla \overline{\phi_h}^h|^2 + C\nu^{-3} |\overline{\phi_h}^h|^2 \, |\nabla u|^4 \\
&\qquad + \alpha^{\frac{8}{3}} C \nu^{-\frac{1}{3}} |\nabla \Delta_h \overline{\phi_h}^h|^{\frac{4}{3}} \, |\nabla u|^{\frac{4}{3}} \, |\overline{\phi_h}^h|^{\frac{2}{3}}.
\end{aligned}$$

We use Young's inequality with $p = 3, q = \frac{3}{2}$ and force the constant to be $c\frac{\mu}{\nu^2}$ where μ is a variable, that will be chosen later.

$$\begin{aligned}
&\leq \frac{\nu}{24} |\nabla \overline{\phi_h}^h|^2 + C\nu^{-3} |\overline{\phi_h}^h|^2 \, |\nabla u|^4 \\
&\qquad + \frac{1}{4} \alpha^4 C \nu^{\frac{1}{2}} \mu^{-\frac{1}{2}} |\nabla \Delta_h \overline{\phi_h}^h|^2 + C \frac{\mu}{\nu^2} |\nabla u|^4 \, |\overline{\phi_h}^h|^2 \\
&\leq \frac{\nu}{24} |\nabla \overline{\phi_h}^h|^2 + C \left(\nu^{-3} + \mu\nu^{-2}\right) |\overline{\phi_h}^h|^2 \, |\nabla u|^4 \\
&\qquad + \frac{1}{4} \alpha^4 C \nu^{\frac{1}{2}} \mu^{-\frac{1}{2}} |\nabla \Delta_h \overline{\phi_h}^h|^2
\end{aligned}$$

Every term on the right-hand side besides $|\nabla \Delta_h \overline{\phi_h}^h|^2$ can be either absorbed or controlled. We use the inverse inequality (4.9) and the existence of the given constant M to obtain an upper bound for $|\nabla \Delta_h \overline{\phi_h}^h|^2$ that can be absorbed by the left-hand side:

$$\alpha^2 |\nabla \Delta_h \overline{\phi_h}^h|^2 \leq \alpha^2 \frac{\tilde{C}}{h^2} |\Delta_h \overline{\phi_h}^h|^2 \leq \tilde{C} M^2 \nu^{\frac{1}{2}} |\Delta_h \overline{\phi_h}^h|^2.$$

Let us insert this in the above inequality and choose $\mu = (C\tilde{C}M^2)^2$:

$$((\nabla \times \phi_h) \times u, \overline{\phi_h}^h) \leq \frac{\nu}{24} |\nabla \overline{\phi_h}^h|^2 + C \left(\nu^{-3} + \mu\nu^{-2}\right) |\overline{\phi_h}^h|^2 \, |\nabla u|^4 + \frac{\nu}{4} \alpha^2 |\Delta_h \overline{\phi_h}^h|^2. \quad (4.18)$$

Part 3.3: Estimating $((\nabla \times u_h) \times (\overline{u_h}^h - u), \overline{\phi_h}^h)$
For the last term of the trilinear part, we split $u - \overline{u_h}^h$ and use the linearity of the discrete filtering operation:

$$\begin{aligned} u - \overline{u_h}^h &= (u - \overline{u}^h) + (\overline{u}^h - \overline{u_h}^h) \\ &= (u - \overline{u}^h) + \overline{(u - \tilde{u})}^h + \overline{(\tilde{u} - u_h)}^h \\ &= (u - \overline{u}^h) + \overline{\eta}^h + \overline{\phi_h}^h . \end{aligned}$$

Inserting this identity into $((\nabla \times u_h) \times (\overline{u_h}^h - u_h), \overline{\phi_h}^h)$ results in:

$$\begin{aligned} ((\nabla \times u_h) \times (\overline{u_h}^h - u), \overline{\phi_h}^h) &= ((\nabla \times u_h) \times (\overline{u}^h - u), \overline{\phi_h}^h) - ((\nabla \times u_h) \times \overline{\eta}^h, \overline{\phi_h}^h) \\ &\quad - \underbrace{((\nabla \times u_h) \times \overline{\phi_h}^h, \overline{\phi_h}^h)}_{=0} . \end{aligned}$$

We rewrite $u_h = u - (u - u_h)$ in the first term and use $u - u_h = \eta + \phi_h$.

$$\begin{aligned} &= ((\nabla \times u) \times (\overline{u}^h - u), \overline{\phi_h}^h) - ((\nabla \times (u - u_h)) \times (\overline{u}^h - u), \overline{\phi_h}^h) \\ &\quad - ((\nabla \times u_h) \times \overline{\eta}^h, \overline{\phi_h}^h) \\ &= ((\nabla \times u) \times (\overline{u}^h - u), \overline{\phi_h}^h) - ((\nabla \times \eta) \times (\overline{u}^h - u), \overline{\phi_h}^h) \\ &\quad - ((\nabla \times \phi_h) \times (\overline{u}^h - u), \overline{\phi_h}^h) - ((\nabla \times u_h) \times \overline{\eta}^h, \overline{\phi_h}^h) \\ &= I - II - III - IV. \end{aligned}$$

We have to bound all the terms on the right-hand side. For **term I** we will use (4.12), (4.13) and *Theorem* 4.3.

$$\begin{aligned} I = ((\nabla \times u) \times (\overline{u}^h - u), \overline{\phi_h}^h) &\leq C||\nabla \times u||_{L^\infty(\Omega)} |u - \overline{u}^h| \, |\overline{\phi_h}^h| \\ &\leq C||\nabla \times u||^2_{L^\infty(\Omega)} |\overline{\phi_h}^h|^2 + \frac{1}{2}|u - \overline{u}^h|^2 \\ &\leq C||\nabla \times u||^2_{L^\infty(\Omega)} |\overline{\phi_h}^h|^2 + C(1 + \nu^{\frac{1}{4}})^2 \left(\alpha^4 |\Delta \overline{u}|^2 + h^{2k+2} |\overline{u}|^2_{k+1}\right) \end{aligned}$$

The index k can be 0 or 1.
For **term II** we use Young's inequality with $p = 4$ and a second time with $p = \frac{3}{2}$ in addition to *Theorem* 4.3:

$$\begin{aligned} II = ((\nabla \times \eta) \times (\overline{u}^h - u), \overline{\phi_h}^h) &\leq C|\nabla \eta| \, |\nabla(\overline{u}^h - u)| \, |\overline{\phi_h}^h|^{\frac{1}{2}} \, |\nabla \overline{\phi_h}^h|^{\frac{1}{2}} \\ &\leq \frac{\nu}{24} |\nabla \overline{\phi_h}^h|^2 + C\nu^{-\frac{1}{3}} \left(|\nabla \eta|^{\frac{4}{3}} \, |\nabla(\overline{u}^h - u)|^{\frac{4}{3}} \, |\overline{\phi_h}^h|^{\frac{3}{2}} \right) \\ &\leq \frac{\nu}{24} |\nabla \overline{\phi_h}^h|^2 + C\nu^{-\frac{1}{3}} \left(|\nabla \eta|^2 + |\nabla(\overline{u}^h - u)|^4 \, |\overline{\phi_h}^h|^2 \right) \\ &\leq \frac{\nu}{24} |\nabla \overline{\phi_h}^h|^2 + C\nu^{-\frac{1}{3}} \left(|\nabla \eta|^2 + (|\nabla u| + h^k |\overline{u}|_{k+1})^4 \, |\overline{\phi_h}^h|^2 \right) \end{aligned}$$

Because (4.14) holds, we can choose $k = 1$.

$$\leq \frac{\nu}{24}|\nabla\overline{\phi_h}^h|^2 + C\nu^{-\frac{1}{3}}\left(|\nabla\eta|^2 + \left(|\nabla u|^4 + h^4|\nabla\overline{u}|^4\right)|\overline{\phi_h}^h|^2\right)$$

In order to bound **term III**, we rewrite $\phi_h = \overline{\phi_h}^h - \alpha^2\Delta_h\overline{\phi_h}^h$ and use Young's inequality.

$$\begin{aligned}
III &= ((\nabla\times\phi_h)\times(\overline{u}^h - u), \overline{\phi_h}^h) \\
&= ((\nabla\times\overline{\phi_h}^h)\times(\overline{u}^h - u), \overline{\phi_h}^h) - ((\nabla\times\alpha^2\Delta_h\overline{\phi_h}^h)\times(\overline{u}^h - u), \overline{\phi_h}^h) \\
&\leq C|\nabla\overline{\phi_h}^h|\,|\nabla(\overline{u}^h - u)|\,|\overline{\phi_h}^h|^{\frac{1}{2}}\,|\nabla\overline{\phi_h}^h|^{\frac{1}{2}} + \alpha^2 C|\nabla\Delta_h\overline{\phi_h}^h|\,|\nabla(\overline{u}^h - u)|\,|\overline{\phi_h}^h|^{\frac{1}{2}}\,|\nabla\overline{\phi_h}^h|^{\frac{1}{2}} \\
&= C|\nabla\overline{\phi_h}^h|^{\frac{3}{2}}\,|\nabla(\overline{u}^h - u)|\,|\overline{\phi_h}^h|^{\frac{1}{2}} + \alpha^2 C|\nabla\Delta_h\overline{\phi_h}^h|\,|\nabla(\overline{u}^h - u)|\,|\overline{\phi_h}^h|^{\frac{1}{2}}\,|\nabla\overline{\phi_h}^h|^{\frac{1}{2}} \\
&\leq \frac{\nu}{48}|\nabla\overline{\phi_h}^h|^2 + C\nu^{-3}\underbrace{|\nabla(\overline{u}^h - u)|^4}_{\leq(|\nabla u|^4 + h^4|\nabla\overline{u}|^4)}|\overline{\phi_h}^h|^2 + \frac{\nu}{48}|\nabla\overline{\phi_h}^h|^2 \\
&\quad + \alpha^{\frac{8}{3}}C\nu^{-\frac{1}{3}}|\nabla\Delta_h\overline{\phi_h}^h|^{\frac{4}{3}}\,|\nabla(\overline{u}^h - u)|^{\frac{4}{3}}\,|\overline{\phi_h}^h|^{\frac{2}{3}}
\end{aligned}$$

Analogously to part 3.2 we insert a new variable μ to loose all the constants in front of the descrete Laplacian.

$$\begin{aligned}
&\leq \frac{\nu}{24}|\nabla\overline{\phi_h}^h|^2 + C\nu^{-3}(|\nabla u|^4 + h^4|\nabla\overline{u}|^4)|\overline{\phi_h}^h|^2 + \frac{1}{4}\alpha^4 C\nu^{\frac{1}{2}}\mu^{-\frac{1}{2}}|\nabla\Delta_h\overline{\phi_h}^h|^2 \\
&\quad + C\frac{\mu}{\nu^2}\underbrace{|\nabla(\overline{u}^h - u)|^4}_{\leq(|\nabla u|^4 + h^4|\nabla\overline{u}|^4)}|\overline{\phi_h}^h|^2 \\
&\leq \frac{\nu}{24}|\nabla\overline{\phi_h}^h|^2 + C(\nu^{-3} + \nu^{-2})(|\nabla u|^4 + h^4|\nabla\overline{u}|^4)|\overline{\phi_h}^h|^2 + \frac{\nu}{4}\alpha^4|\Delta_h\overline{\phi_h}^h|^2.
\end{aligned}$$

There is one more term to bound to finish bounding the trilinear term. To obtain an upper bound for **term IV**, we will use the following identity:

$$(\nabla\times a)\times b = a\times(\nabla\times b) + a\cdot\nabla b + b\cdot\nabla a - \nabla(a\cdot b). \tag{4.19}$$

Furthermore we will use integration by parts and the fact that we have periodic or no-slip BC:

$$(\overline{\eta}^h\cdot\nabla u_h, \overline{\phi_h}^h) = -(\overline{\eta}^h\cdot\nabla\overline{\phi_h}^h, u_h) - (\nabla\cdot\overline{\eta}^h, u_h\cdot\overline{\phi_h}^h) \tag{4.20}$$

and

$$(\nabla(\overline{\eta}^h\cdot u_h), \overline{\phi_h}^h) = -(\overline{\eta}^h\cdot u_h, \nabla\cdot\overline{\phi_h}^h). \tag{4.21}$$

We insert (4.19),(4.20) and (4.21) into IV:

$$\begin{aligned}
IV &= ((\nabla\times u_h)\times\overline{\eta}^h, \overline{\phi_h}^h) \\
&= (u_h\times(\nabla\times\overline{\eta}^h), \overline{\phi_h}^h) + (u_h\cdot\nabla\overline{\eta}^h, \overline{\phi_h}^h) + (\overline{\eta}^h\cdot\nabla u_h, \overline{\phi_h}^h) - (\nabla(u_h\cdot\overline{\eta}^h), \overline{\phi_h}^h) \\
&= (u_h\times(\nabla\times\overline{\eta}^h), \overline{\phi_h}^h) + (u_h\cdot\nabla\overline{\eta}^h, \overline{\phi_h}^h) - (\overline{\eta}^h\cdot\nabla\overline{\phi_h}^h, u_h) - (\nabla\cdot\overline{\eta}^h, u_h\cdot\overline{\phi_h}^h) \\
&\quad + (\overline{\eta}^h\cdot u_h, \nabla\cdot\overline{\phi_h}^h).
\end{aligned}$$

All terms can be estimated in the same way.

$$
\begin{aligned}
&\leq C|u_h|^{\frac{1}{2}}\,|\nabla u_h|^{\frac{1}{2}}\,|\nabla\overline{\eta}^h|\,|\nabla\overline{\phi_h}^h| \\
&\leq C\nu^{-1}|u_h|\,|\nabla u_h|\,|\nabla\overline{\eta}^h|^2 + \frac{\nu}{24}|\nabla\overline{\phi_h}^h|^2.
\end{aligned}
$$

Summary of triliniar term
Let us gather what we have shown so far for the non-linear term

$$
\begin{aligned}
&((\nabla\times u_h)\times\overline{u_h}^h,\overline{\phi_h}^h) - ((\nabla\times u)\times u,\overline{\phi_h}^h) \\
&\leq C\nu^{-\frac{1}{2}}|\nabla\eta|^2 + |\nabla u|^4\,|\overline{\phi_h}^h|^2 + \frac{\nu}{24}|\nabla\overline{\phi_h}^h|^2 && \text{Part 3.1} \\
&\quad + \frac{\nu}{24}|\nabla\overline{\phi_h}^h|^2 + C\left(\nu^{-3}+\mu\nu^{-2}\right)|\overline{\phi_h}^h|^2\,|\nabla u|^4 + \frac{\nu}{4}\alpha^2|\Delta_h\overline{\phi_h}^h|^2 && \text{Part 3.2} \\
&\quad + C||\nabla\times u||^2_{L^\infty(\Omega)}|\overline{\phi_h}^h|^2 + C(1+\nu^{\frac{1}{4}})^2\left(\alpha^4|\Delta\overline{u}|^2 + h^{2k+2}|\overline{u}|^2_{k+1}\right) && \text{Part 3.3 I} \\
&\quad + \frac{\nu}{24}|\nabla\overline{\phi_h}^h|^2 + C\nu^{-\frac{1}{3}}\left(|\nabla\eta|^2 + \left(|\nabla u|^4 + h^4|\nabla\overline{u}|^4\right)|\overline{\phi_h}^h|^2\right) && \text{Part 3.3 II} \\
&\quad + \frac{\nu}{24}|\nabla\overline{\phi_h}^h|^2 + C(\nu^{-3}+\nu^{-2})(|\nabla u|^4 + h^4|\nabla\overline{u}|^4)|\overline{\phi_h}^h|^2 + \frac{\nu}{4}\alpha^4|\Delta_h\overline{\phi_h}^h|^2 && \text{Part 3.3 III} \\
&\quad + C\nu^{-1}|u_h|\,|\nabla u_h|\,|\nabla\overline{\eta}^h|^2 + \frac{\nu}{3}|\nabla\overline{\phi_h}^h|^2 && \text{Part 3.3 IV} \\
&\leq \frac{5\nu}{24}|\nabla\overline{\phi_h}^h|^2 + \frac{\nu}{2}\alpha^4|\Delta_h\overline{\phi_h}^h|^2 + C\max\left(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{3}},\nu^{-1}\right)(1+|u_h|\,|\nabla u_h|)|\nabla\eta|^2 \\
&\quad + C\max\left(1,\nu^{-\frac{1}{3}},\nu^{-3},\nu^{-2}\right)\left(|\nabla u|^4 + ||\nabla\times u||^2_{L^\infty(\Omega)}\right)|\overline{\phi_h}^h|^2 \\
&\quad + C(1+\nu^{\frac{1}{4}})^2\left(\alpha^4|\Delta\overline{u}|^2 + h^{2k+2}|\overline{u}|^2_{k+1}\right).
\end{aligned}
$$

We can shorten the expression slightly by stating:

$$
\begin{aligned}
\max\left(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{3}},\nu^{-1}\right) &= \max\left(\nu^{-\frac{1}{3}},\nu^{-1}\right), \\
\max\left(1,\nu^{-\frac{1}{3}},\nu^{-3},\nu^{-2}\right) &= \max\left(1,\nu^{-3}\right).
\end{aligned}
$$

This concludes the final estimate for the trilinear part.

$$
\begin{aligned}
&((\nabla\times u_h)\times\overline{u_h}^h,\overline{\phi_h}^h) - ((\nabla\times u)\times u,\overline{\phi_h}^h) \\
&\quad\leq \frac{5\nu}{24}|\nabla\overline{\phi_h}^h|^2 + \frac{\nu}{2}\alpha^2|\Delta_h\overline{\phi_h}^h|^2 + C\max\left(\nu^{-\frac{1}{3}},\nu^{-1}\right)(1+|u_h|\,|\nabla u_h|)|\nabla\eta|^2 \\
&\qquad + C\max\left(1,\nu^{-3}\right)\left(|\nabla u|^4 + ||\nabla\times u||^2_{L^\infty(\Omega)}\right)|\overline{\phi_h}^h|^2 \\
&\qquad + C(1+\nu^{\frac{1}{4}})^2\left(\alpha^4|\Delta\overline{u}|^2 + h^{2k+2}|\overline{u}|^2_{k+1}\right).
\end{aligned} \tag{4.22}
$$

Part 4: Continue to estimate the inequality from part 2
We consider the inequality of part 2 and apply (4.22), in addition we apply Young's

inequality to all the terms besides the trilinear part, to be able to partly absorb it by left-hand side.

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\left(|\overline{\phi_h}^h|^2+\alpha^2|\nabla\overline{\phi_h}^h|^2\right)+\nu\left(|\nabla\overline{\phi_h}^h|^2+\alpha^2|\Delta_h\overline{\phi_h}^h|^2\right)\\
&\quad\le |P-q_h|\left|\nabla\cdot\overline{\phi_h}^h\right|-\left|\frac{\partial}{\partial t}\eta\right|\left|\overline{\phi_h}^h\right|-\nu\,|\nabla\eta|\left|\nabla\overline{\phi_h}^h\right|+((\nabla\times u_h)\times\overline{u_h}^h,\overline{\phi_h}^h)\\
&\qquad-((\nabla\times u)\times u,\overline{\phi_h}^h)\\
&\quad\le C\nu^{-1}\,|P-q_h|^2+\frac{3\nu}{24}\left|\nabla\overline{\phi_h}^h\right|^2+C\left|\frac{\partial}{\partial t}\eta\right|^2+C\left|\overline{\phi_h}^h\right|^2+C\nu|\nabla\eta|^2\\
&\qquad+\frac{4\nu}{24}\left|\nabla\overline{\phi_h}^h\right|+\frac{5\nu}{24}|\nabla\overline{\phi_h}^h|^2\\
&\qquad+C\max\left(\nu^{-\frac{1}{3}},\nu^{-1}\right)(1+|u_h|\,|\nabla u_h|)|\nabla\eta|^2\\
&\qquad+C\max\left(1,\nu^{-3}\right)\left(|\nabla u|^4+||\nabla\times u||^2_{L^\infty(\Omega)}\right)|\overline{\phi_h}^h|^2\\
&\qquad+\frac{\nu}{2}\alpha^2|\Delta_h\overline{\phi_h}^h|^2+C(1+\nu^{\frac{1}{4}})\left(\alpha^4|\Delta\overline{u}|^2+h^{2k+2}|\overline{u}|^2_{k+1}\right)\\
&\quad\le C\nu^{-1}\,|P-q_h|^2+C\left|\frac{\partial}{\partial t}\eta\right|^2+\frac{\nu}{2}|\nabla\overline{\phi_h}^h|^2+\frac{\nu}{2}\alpha^4|\Delta_h\overline{\phi_h}^h|^2\\
&\qquad+C\max\left(\nu,\nu^{-1}\right)(1+|u_h|\,|\nabla u_h|)|\nabla\eta|^2\\
&\qquad+C\max\left(1,\nu^{-3}\right)\left(1+|\nabla u|^4+||\nabla\times u||^2_{L^\infty(\Omega)}\right)|\overline{\phi_h}^h|^2\\
&\qquad+C(1+\nu^{\frac{1}{4}})\left(\alpha^4|\Delta\overline{u}|^2+h^{2k+2}|\overline{u}|^2_{k+1}\right)
\end{aligned}
$$

We absorb $\frac{\nu}{2}(|\nabla\overline{\phi_h}^h|^2+\alpha^2|\Delta_h\overline{\phi_h}^h|^2)$ by the left-hand side and multiply through by 2:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(|\overline{\phi_h}^h|^2+\alpha^2|\nabla\overline{\phi_h}^h|^2\right)+\nu\left(|\nabla\overline{\phi_h}^h|^2+\alpha^2|\Delta_h\overline{\phi_h}^h|^2\right)\\
&\quad\le C\nu^{-1}\,|P-q_h|^2+C\left|\frac{\partial}{\partial t}\eta\right|^2+C(1+\nu^{\frac{1}{4}})\left(\alpha^4|\Delta\overline{u}|^2+h^{2k+2}|\overline{u}|^2_{k+1}\right)\\
&\qquad+C\max\left(\nu,\nu^{-1}\right)(1+|u_h|\,|\nabla u_h|)|\nabla\eta|^2\\
&\qquad+C\max\left(1,\nu^{-3}\right)\left(1+|\nabla u|^4+||\nabla\times u||^2_{L^\infty(\Omega)}\right)|\overline{\phi_h}^h|^2.
\end{aligned}
\tag{4.23}
$$

Before we integrate above inequality we consider an auxiliary calculation for $|\nabla\eta|^2$:

$$
\begin{aligned}
\int_0^t |u_h|\,|\nabla u_h|\,|\nabla\eta|^2\,ds &\le ||u_h||_{L^\infty([0,t])}\int_0^t|\nabla u_h|\,|\nabla\eta|^2\,ds\\
&\le \underbrace{||u_h||_{L^\infty([0,t])}\left(\int_0^t|\nabla u_h|^2\,ds\right)^{\frac{1}{2}}}_{\le C\text{ because }u_h\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))}\left(\int_0^t|\nabla\eta|^4\,ds\right)^{\frac{1}{2}}\\
&\le C\left(\int_0^t|\nabla\eta|^4\,ds\right)^{\frac{1}{2}}
\end{aligned}
$$

and for $|\nabla u|^4$:

$$|\nabla u|^4 \leq C|u|^4 \leq const,$$

because $u \in C(0,T;V)$. Now we integrate (4.23) over $[0,t]$:

$$\begin{aligned}
&\left(|\overline{\phi_h(t)}^h|^2 + \alpha^2|\nabla\overline{\phi_h(t)}^h|^2\right) - \left(|\overline{\phi_h(0)}^h|^2 + \alpha^2|\nabla\overline{\phi_h(0)}^h|^2\right)\\
&\qquad + \nu\int_0^t \left(|\nabla\overline{\phi_h(s)}^h|^2 + \alpha^2|\Delta_h\overline{\phi_h(s)}^h|^2\right) ds\\
&\leq C\nu^{-1}\int_0^t |P-q_h|^2\,ds + C\int_0^t \left|\frac{\partial}{\partial t}\eta(s)\right|^2 ds\\
&\qquad + C(1+\nu^{\frac{1}{4}})^2\int_0^t \left(\alpha^4|\Delta\overline{u(s)}|^2 + h^{2k+2}|\overline{u(s)}|_{k+1}^2\right) ds\\
&\qquad + C\max\left(\nu,\nu^{-1}\right)\left(\int_0^t |\nabla\eta|^4\,ds\right)^{\frac{1}{2}}\\
&\qquad + C\max\left(1,\nu^{-3}\right)\int_0^t \underbrace{\left(1+|\nabla u(s)|^4 + ||\nabla\times u(s)||^2_{L^\infty(\Omega)}\right)}_{\leq const}|\overline{\phi_h(s)}^h|^2\,ds.
\end{aligned}$$

Our next step is to apply a version of Gronwall's lemma, the Gronwall's inequality in integral form: If

$$w(\theta) \leq a(\theta) + \int_0^\theta b(\zeta)w(\zeta)d\zeta \tag{4.24}$$

holds, then

$$w(\theta) \leq a(\theta) + \int_0^\theta a(\zeta)b(\zeta)e^{\left(\int_\zeta^\theta b(\xi)d\xi\right)}d\zeta \tag{4.25}$$

is true.
Before we use Gronwall's inequality, we have to extend the right-hand side, such that all terms on the left-hand side appear on the right-hand side. Since $\alpha^2|\nabla\overline{\phi_h(s)}^h|^2 + \nu\int_0^s \left(|\nabla\overline{\phi_h(r)}^h|^2 + \alpha^2|\Delta_h\overline{\phi_h(r)}^h|^2\right) dr$ is positive we can add this term to the right-hand side and the inequality is still valid:

$$
\begin{aligned}
&\overbrace{\left(|\overline{\phi_h(t)}^h|^2 + \alpha^2|\nabla\overline{\phi_h(t)}^h|^2\right) + \nu\int_0^t \left(|\nabla\overline{\phi_h(s)}^h|^2 + \alpha^2|\Delta_h\overline{\phi_h(s)}^h|^2\right)\, ds}^{w(t)} \\
&\left.\begin{aligned}
&\leq C\nu^{-1}\int_0^t |P - q_h|^2\, ds + C\int_0^t \left|\frac{\partial}{\partial t}\eta(s)\right|^2 ds \\
&\quad + C(1+\nu^{\frac{1}{4}})^2\int_0^t \left(\alpha^4|\Delta\overline{u(s)}|^2 + h^{2k+2}|\overline{u(s)}|_{k+1}^2\right)\, ds \\
&\quad + C\max\left(\nu,\nu^{-1}\right)\left(\int_0^t |\nabla\eta|^4\, ds\right)^{\frac{1}{2}} + \left(|\overline{\phi_h(0)}^h|^2 + \alpha^2|\nabla\overline{\phi_h}^h(0)|^2\right)
\end{aligned}\right\} a(t) \\
&\quad + \underbrace{C\max\left(1,\nu^{-3}\right)}_{b}\int_0^t \underbrace{|\overline{\phi_h(s)}^h|^2 + \underbrace{\alpha^2|\nabla\overline{\phi_h(s)}^h|^2 + \nu\int_0^s \left(|\nabla\overline{\phi_h(r)}^h|^2 + \alpha^2|\Delta_h\overline{\phi_h(r)}^h|^2\right)\, dr}_{\geq 0}}_{w(s)}\, ds.
\end{aligned}
$$

We notice that $a(t)$ is an increasing function and $b(t) \equiv b$. Let us also denote $\tilde{C} = 1 + \max(\nu, \nu^{-1})$ as the largest constant used in the above equation. For the sake of notation we will briefly adopt the notation used in (4.24) since we will use Gronwall's lemma right away:

$$
\begin{aligned}
w(t) &\leq a(t) + \tilde{C}e^{\tilde{C}t}\int_0^t a(s)ds \\
&\leq a(t) + \tilde{C}e^{\tilde{C}t}\int_0^t a(t)ds \\
&\leq \underbrace{\left(1 + t\tilde{C}e^{\tilde{C}t}\right)}_{\leq(1+T\tilde{C}e^{\tilde{C}T})=C} a(t) \leq Ca(t).
\end{aligned}
$$

Going back to our usual notation will give us:

$$\left.\begin{aligned}
&\left(|\overline{\phi_h(t)}^h|^2 + \alpha^2|\nabla\overline{\phi_h(t)}^h|^2\right) + \nu\int_0^t \left(|\nabla\overline{\phi_h(s)}^h|^2 + \alpha^2|\Delta_h\overline{\phi_h(s)}^h|^2\right) ds \\
\leq\ & C\int_0^t |P-q_h|^2\,ds + C\int_0^t \left|\frac{\partial}{\partial t}\eta(s)\right|^2 ds \\
&+ C\int_0^t \left(\alpha^4|\Delta\overline{u(s)}|^2 + h^{2k+2}|\overline{u(s)}|^2_{k+1}\right) ds \\
&+ C\left(\int_0^t |\nabla\eta|^4\,ds\right)^{\frac{1}{2}} + \left(|\overline{\phi_h(0)}^h|^2 + \alpha^2|\nabla\overline{\phi_h}^h(0)|^2\right).
\end{aligned}\right\} \tag{4.26}$$

Part 5: Putting everything together
To receive the final bound, we start from the beginning and take a look at the error $|u-u_h|$ and apply (4.10) and (4.26):

$$\begin{aligned}
&|u(t)-u_h(t)|^2 + \int_0^t |u(s)-u_h(s)|^2\,ds \\
\leq\ & |\eta(t)|^2 + |\phi_h(t)|^2 + \int_0^t |\eta(s)|^2\,ds + \int_0^t |\phi_h(s)|^2\,ds \\
\leq\ & |\eta(t)|^2 + \int_0^t |\eta(s)|^2\,ds + |\overline{\phi_h(t)}^h|^2 + \alpha^2|\nabla\overline{\phi_h(t)}^h|^2 + \nu\int_0^t \left(|\nabla\overline{\phi_h(s)}^h|^2 + \alpha^2|\Delta_h\overline{\phi_h(s)}^h|^2\right) ds \\
\leq\ & |\eta(t)|^2 + \int_0^t |\eta(s)|^2\,ds + C\int_0^t |P-q_h|^2\,ds + C\int_0^t \left|\frac{\partial}{\partial t}\eta(s)\right|^2 ds \\
&+ C\int_0^t \left(\alpha^4|\Delta\overline{u(s)}|^2 + h^{2k+2}|\overline{u(s)}|^2_{k+1}\right) ds \\
&+ C\left(\int_0^t |\nabla\eta|^4\,ds\right)^{\frac{1}{2}} + \underbrace{\left(|\overline{\phi_h(0)}^h|^2 + \alpha^2|\nabla\overline{\phi_h}^h(0)|^2\right)}_{\leq|\phi_h(0)|^2\leq|\eta(0)|^2+\underbrace{|\phi_h(0)|^2}_{=0}=|\eta(0)|^2}.
\end{aligned}$$

As a final step, we take the supremum over $t\in[0,T]$ and the infimum over $\tilde{u}\in V_h, q_h\in Q_h$ and insert the definition of $\eta(t)$.

□

After proving convergence we ask ourselves what we can expect from actual computation, therefore we consider Taylor-Hood FE basis functions, where we know that they converge with optimal rate, i.e., quadratic in terms of the gridsize h, see [QV08]. This motivates the next theorem as a final result for this chapter.

Theorem 4.5 (Order of convergence with Taylor-Hood finite elements). *Let (X_h, Q_h) be finite elements spaces corresponding to Taylor-Hoof elements $\alpha = h$ for each mesh. Assuming $u(t,\cdot)$ is a solution for NSE then the corresponding NS-α approximations converge at a rate of $\mathcal{O}(h^2)$.*

Proof. Every term on the right-hand side of $Theorem$ 4.4 can be estimated by $\mathcal{O}(h^4)$ from above using the optimality estimate of Taylor-Hood elements. Therefore this estimate holds:

$$\sup_{t\in[0,T]} |u(t) - u_h(t)|^2 + \int_0^T |u(s) - u_h(s)|^2 \, ds \le \mathcal{O}(h^4)$$
$$\Rightarrow ||u - u_h||_{C(0,T)} + ||u - u_h||_{L^2(0,T;H)} \le \mathcal{O}(h^2)$$

□

5 Numerical simulations

In our simulations, we consider the flow though a channel using an anisotropic grid in the direction perpendicular to the walls. The simulations were performed on two grids, a coarse and a finer one. The filter width parameter α should be proportional to the meshsize h, $\alpha = Ch$, where C is the filter width constant that we can choose arbitrary. In our numerical experiments we will vary the filter width constant on the same grid and compare the results. Note that unlike in the numerical analysis, α is not constant on the grid, as the grid is anisotropic. As there are different ways of measuring the width of a mesh cell, we performed our experiments for three different measures, namely the geometric mean, the diameter of the cell and the shortest edge of the cell.

5.1 Channel flow

We consider the flow through a rectangular duct Ω. The bottom wall is at $y = 0$, the top wall is at $y = 2H$ and the center line is at $y = H, z = 0$. This makes Domain Ω to be:

$$\Omega = (-2\pi, 2\pi) \times (0, 2H) \times (-\frac{2}{3}\pi, \frac{2}{3}\pi).$$

The mean flow is predominantly in the x-direction, the velocity varies mainly in the y-direction. Assuming the height in z-direction is large compared to $2H$, allows us to assume that the flow is statistically independent of z, except of course at and near the walls. We use homogeneous Dirichlet BC for the top and bottom walls $(y = 0, y = 2H)$ and periodic BC in x-direction. This choice of BCs makes the channel infinitely long and infinitely wide, so we do not have to deal with walls and the hereby arising issues. The x-, y- and z-direction are called stream wise, cross stream and span wise direction respectively. As a consequence of an infinitely long duct, the statistics no longer vary in the stream wise direction, making the flow statistics only dependent on the cross stream direction, resulting in a statistically symmetric flow around the $y = H$. For more details see [Pop00] and [JR07].

5.2 The discretization

The NS-α model is discretized in time by a Crank-Nicolson method. It is well known to be an accurate and efficient temporal discretization of the incompressible NSE.

The Crank-Nicolson scheme for (3.1) is the following:

$$\left.\begin{aligned}
u_{n+1} + \frac{1}{2}\Delta t_n\Big(&- \nu\Delta u_{n+1} + (\overline{u_{n+1}} \cdot \nabla)u_{n+1} + \nabla P_{n+1}\Big) \\
&= u_n + \frac{1}{2}\Delta t_n\Big(- \nu\Delta u_n + (\overline{u_n} \cdot \nabla)u_n + \nabla P_n\Big) + \frac{1}{2}\Delta t_n(f_{n+1} + f_n), \\
\nabla \cdot u_{n+1} &= \nabla \cdot \overline{u_{n+1}} = 0, \\
u_{n+1} &= \overline{u_{n+1}} - \alpha^2\Delta\overline{u_{n+1}} + \nabla\lambda.
\end{aligned}\right\} \qquad (5.1)$$

This is transformed into variational form and discretized by a finite element method using the Q_2/P_1^{disc} finite elements. We also replace $(\nabla u, \nabla v)$ by $2(\mathbb{D}(u), \mathbb{D}(v))$, where $\mathbb{D}(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the deformation tensor or the symmetric gradient.

$$\left.\begin{aligned}
(u_{n+1},v)+\frac{1}{2}\Delta t_n\Big(2\nu(\mathbb{D}(u_{n+1}),\mathbb{D}(v))+(\overline{u_{n+1}}\cdot\nabla)u_{n+1},v)+(P_{n+1},\nabla\cdot v)\Big)&\\
=(u_n,v)+\frac{1}{2}\Delta t_n\Big(2\nu(\mathbb{D}(u_n),\mathbb{D}(v))+(\overline{u_n}\cdot\nabla)u_n,v)+(P_n,\nabla\cdot v)\Big)&\\
+\frac{1}{2}\Delta t_n(f_{n+1}+f_n,v),&\\
(\nabla\cdot u_{n+1},q)=(\nabla\cdot\overline{u_{n+1}},q)=0,&\\
(u_{n+1},v)=(\overline{u_{n+1}}-\alpha^2\Delta\overline{u_{n+1}},v)+(\lambda,\nabla\cdot v).&
\end{aligned}\right\}\tag{5.2}$$

We will solve (5.2) for every timestep by linearizing the equation to obtain an Oseen system, that we can solve by a fixed point iteration. We hereby replace $(\overline{u_{n+1}}^{(k+1)}\cdot\nabla)u_{n+1}^{(k+1)}$ by $(\overline{u_{n+1}}^{(k)}\cdot\nabla)u_{n+1}^{(k+1)}$, where k is the iterate in this timestep. As a result we get the final numerical scheme:

$$\begin{aligned}
(u_{n+1}^{(k+1)},v)+\frac{1}{2}\Delta t_n\Big(2\nu(\mathbb{D}(u_{n+1}^{(k+1)}),\mathbb{D}(v))+((\overline{u_{n+1}}^{(k)}\cdot\nabla)u_{n+1}^{(k+1)},v)+(P_{n+1}^{(k+1)},\nabla\cdot v)\Big)&\\
=(u_n,v)+\frac{1}{2}\Delta t_n\Big(2\nu(\mathbb{D}(u),\mathbb{D}(v))+((\overline{u_n}\cdot\nabla)u_n,v),v)+(P_n,\nabla\cdot v)\Big)&\\
+\frac{1}{2}\Delta t_n(f_{n+1}+f_n,v),&\\
(\nabla\cdot u_{n+1}^{(k+1)},q)=(\nabla\cdot\overline{u_{n+1}}^{(k+1)},q)=0,&\\
(u_{n+1}^{(k+1)},v)=(\overline{u_{n+1}}^{(k+1)}-\alpha^2\Delta\overline{u_{n+1}}^{(k+1)},v)+(\lambda,\nabla\cdot v).&
\end{aligned}$$

The Crank-Nicolson scheme was applied with an equidistant time step of $\Delta t_n = 0.002$. In the x- and z-direction we used a uniform grid, but it is an isotropic because of the y-direction, where the spacing between the nodes becomes finer and finer the closer we approach the wall. The nodes y_i are given by:

$$y_i = 1 - cos(\frac{i\pi}{N_y}), \qquad i = 1,\cdots N_y,$$

where N_y is the number of nodes in the y-direction.

5.3 Statistics of interests

We want to compare the results of our computational experiments for NS-α with the statistical results of direct numerical simulations for NSE. For that purpose we will introduce some statistics and mean values we are interested in.

Let $u^h(t,x,y,z) = (U(t,x,y,z), V(t,x,y,z), W(t,x,y,z))$ be the solution computed with the scheme above and let N_x, N_y, N_z denote the number of gridpoints in each x-, y-, z-direction respectively. In the context of the NSE we always consider mean values, let $\langle\cdot\rangle_s$ be the average in space, then we can define the spatial mean velocity at time t_n in the plane $y = const$ as:

$$\left\langle u^h(t_n,x,y,z)\right\rangle_s \;=\; \frac{1}{N_xN_z}\sum_{i=1}^{N_x}\sum_{j=1}^{N_z} u^h(t_n,x_i,y,z_j).$$

This will be done for all planes $y = const$. If we average these spatial means in time, recall $\Delta t_n = 0.002$ for all times, we get the mean velocity profile.

Definition (Mean velocity profile)**.**

$$u^h_{mean}(y) = \frac{1}{N_t + 1} \sum_{n=0}^{N_t} \left\langle u^h(t_n, x, y, z) \right\rangle_s .$$

We will present results for the first component of u^h, namely U.

The simulated friction velocity u^h_τ is defined as the average of the computed friction velocities at both walls, where the friction velocity at each wall is approximated by a one-sided difference:

$$u^h_\tau = \frac{1}{2} \left(\frac{u^h_{mean}(y^+_{min})}{y^+_{min}} - \frac{u^h_{mean}(2 - y^+_{min})}{2 - y^+_{min}} \right),$$

where y^+_{min} is the minimum height of a cell. This will be used later to normalize the secondary statistics.
Second order statistics in turbulent channel flows are the off-diagonal Reynolds stresses and the root mean square turbulence intensities. Since the definition of the Reynolds stresses aren't unique in the literature we will define them again, to specify what we are referring to. The Reynolds stresses can be defined by:

$$\mathbb{R}_{i,j} = \left\langle \left\langle u_i u_j \right\rangle_s \right\rangle_t - \left\langle \left\langle u_i \right\rangle_s \right\rangle_t \left\langle \left\langle u_j \right\rangle_s \right\rangle_t . \tag{5.3}$$

The discrete Reynolds stresses are defined analogously. We can interpret the Reynolds Stress Tensor as the force that determines how the average $\langle u_i \rangle$ develops. The diagonal entries of the stress tensor are called normal stresses, the off-diagonal stresses are called shear stresses. Obviously the normal and shear stresses depend on the choice of the coordinate system, therefore distinguishing between isotropic and anisotropic stresses instead is more useful. Since the trace of a tensor is invariant under coordinate transformation, we define isotropic stress as the scaled trace, namely $\frac{1}{3}\sum_{i=1}^{3} \mathbb{R}_{ii}$. Therefore the deviation from isotropy can be expressed by $\mathbb{R}_{ij} - \frac{1}{3}\sum_{i=1}^{3} \mathbb{R}_{ii}$. Finally we can define the second order statistics we are interested in.

Since we want to compare our results to reference data coming from DNS, we will have to compare the statistics of u^{DNS} to the statistics of u. Let us define the discrete normalized off-diagonal Reynolds stresses by:

$$\mathbb{R}^{h,*}_{i,j} = \frac{\mathbb{R}^{NS\alpha}_{i,j}}{(u^h_\tau)^2}.$$

This leads to the following definition:

Definition (Root mean square turbulence intensities)**.** The root mean square turbulence intensities are computed by:

$$u^h_{rms} = \left(\mathbb{R}^{h,*}_{1,1}\right)^{\frac{1}{2}} = \frac{1}{u^h_\tau}\left(\mathbb{R}^{NS\alpha}_{1,1} - \frac{1}{3}\sum_{i=1}^{3}\mathbb{R}^{NS\alpha}_{i,i}\right)^{\frac{1}{2}}.$$

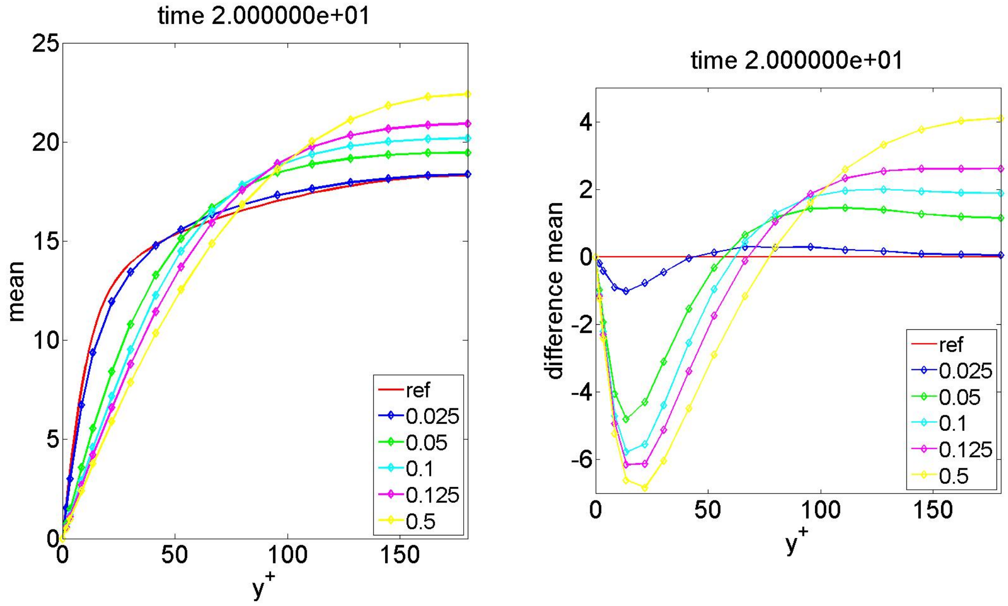
5.4 Results

Recall that the filter width α is proportional to the cell width h_K, where K is a cell. The cell size varies in wall normal direction so does the filter width. We will perform computational tests with different values for the filter width constant, namely $0.2, 0.3, ..., 0.9, 1.0, 1.2, 1.5, 2.0$. Additionally we will test three different rules to compute h_K, denote the length of the edges of a cell K by h_x, h_y and h_z. We used the following three rules:

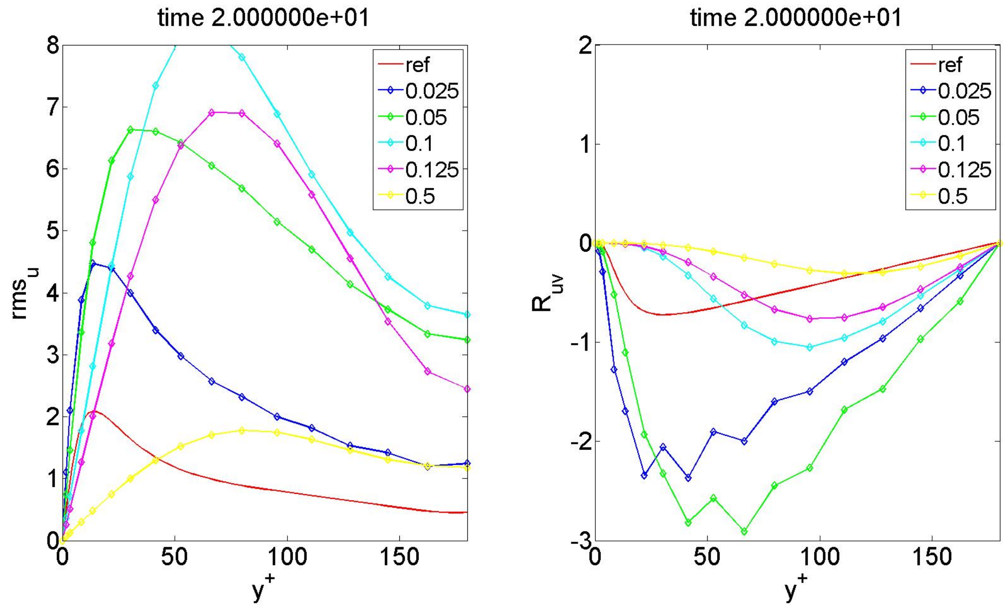
- Geometric mean: $geom(K) = \sqrt[3]{h_x h_y h_z}$.
- Diameter : $diam(K) = \sqrt{h_x^2 + h_y^2 + h_z^2}$.
- Shortest edge: $edge(K) = min\{h_x, h_y, h_z\}$

In the first stage of our experiments we will present results for values of C in the range of $C = 0.025, 0.05, 0.075, 0.1, 0.125, 0.15, 2, 3, ..., 1.0, 1.025, 1.05, 1.2, 1.5, 2.0$. They were computed for each type of measuring the gridsize. The Results are presented in figures 1,2 and 3. We classify the results by first comparing the mean profile and select acceptable values of C. Out of these values we choose the one that has the best results in terms of the secondary statistics.

For the geometric measure (Figure 1), the value $C = 0.025$ was by far the best value. It was the closest to the reference mean velocity profile from DNS calculation. The computed root mean square intensities (rms_u) have the correct form, but highly overpredict the reference values. The overprediction is smallest for the value $C = 0.025$. The results for R_{uv} aren't good in particular, since they show the same overprediction as for the rms_u. The form of the curve is similar to the reference values $C = 0.025$ and $C = 0.05$, but the oscillations near the wall are quite pronounced. These are both effects that will be lessened considerably on the finer grid. In conclusion $C = 0.025$ seems to be the best value here.
For the diameter measure (Figure 2) the acceptable values for the filter width constant were larger then in the first case. Values $C = 0.05$,$C = 0.1$ and $C = 0.125$ produce acceptable results when considering the mean. The root mean square intensities are still being overpredicted for all values, but $C = 0.1$ and $C = 0.125$ seem to work best for rms_u and R_{uv}. We observe that for values smaller then $C = 0.1$ our results tend to move away from the reference values.
For the shortest edge measure we can see that every value for C is somehow decent. Considering the rms_u and R_{uv} results we observe that the form of our tests coincide with the reference, but for $C = 0.05$ we can see a tendency to move further away from the reference.

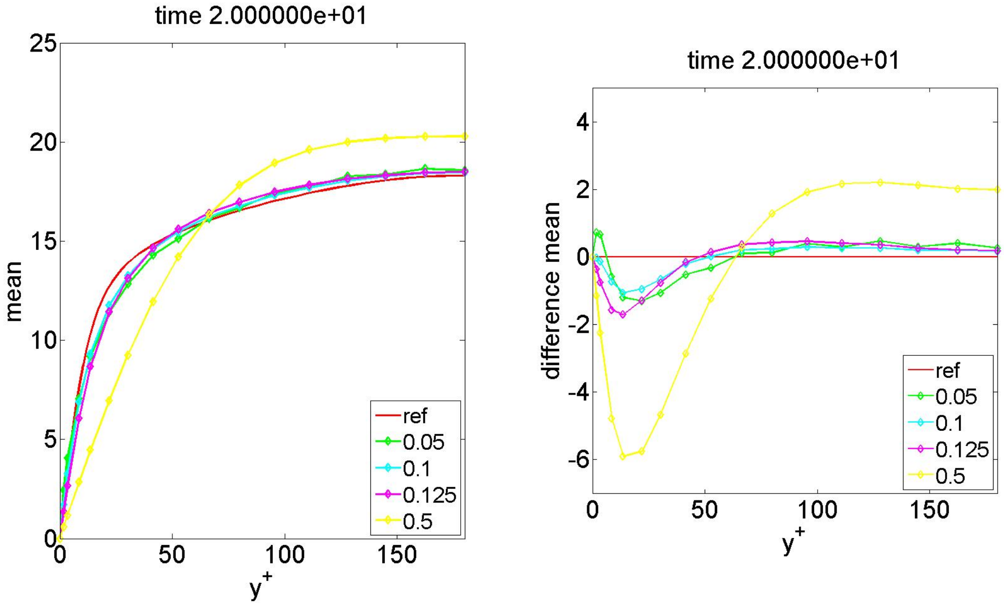


Mean and difference mean

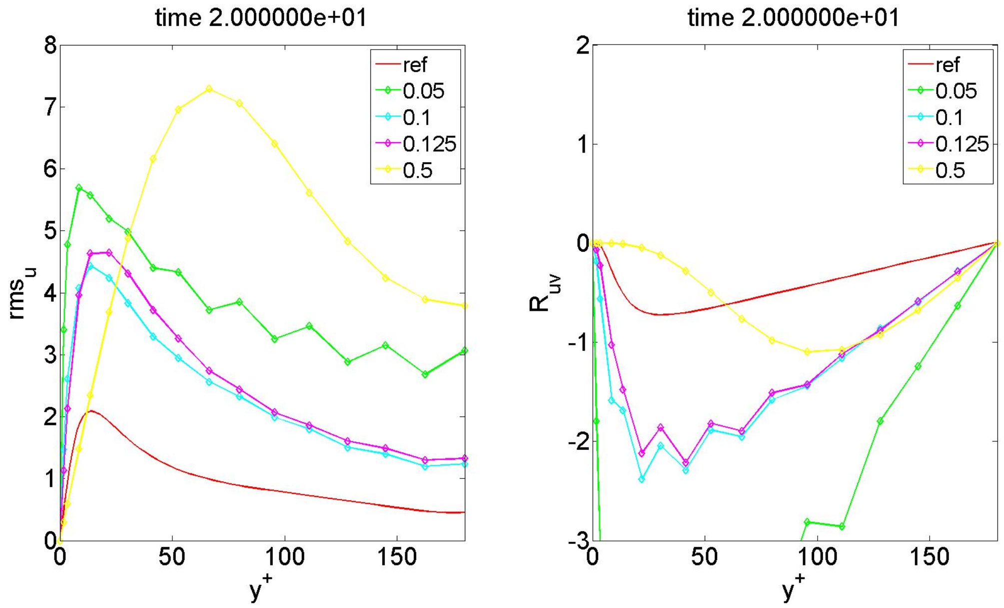


rms and R_{uv}

Figure 1: h = geom(K)

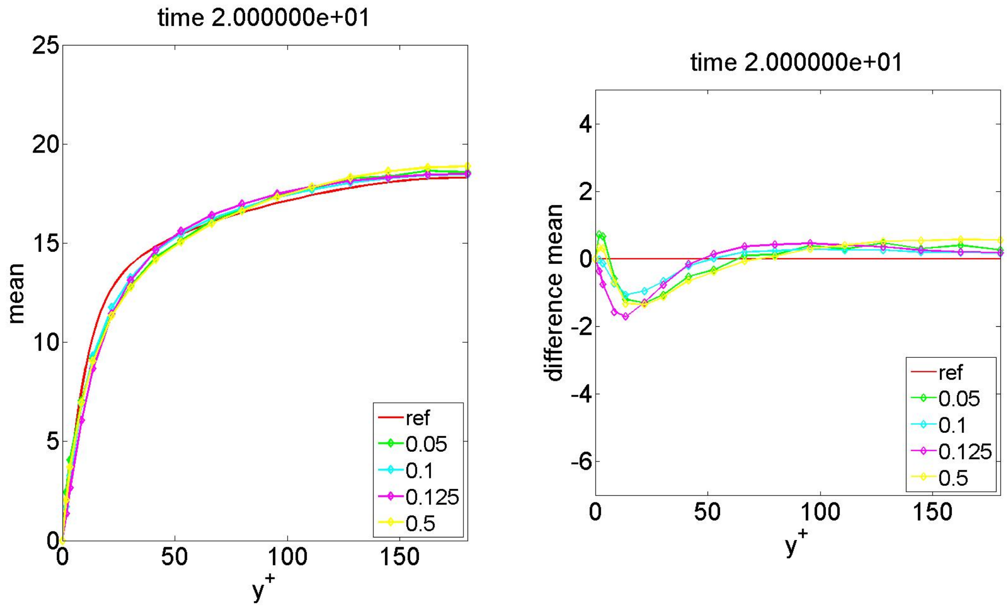


Mean and difference mean

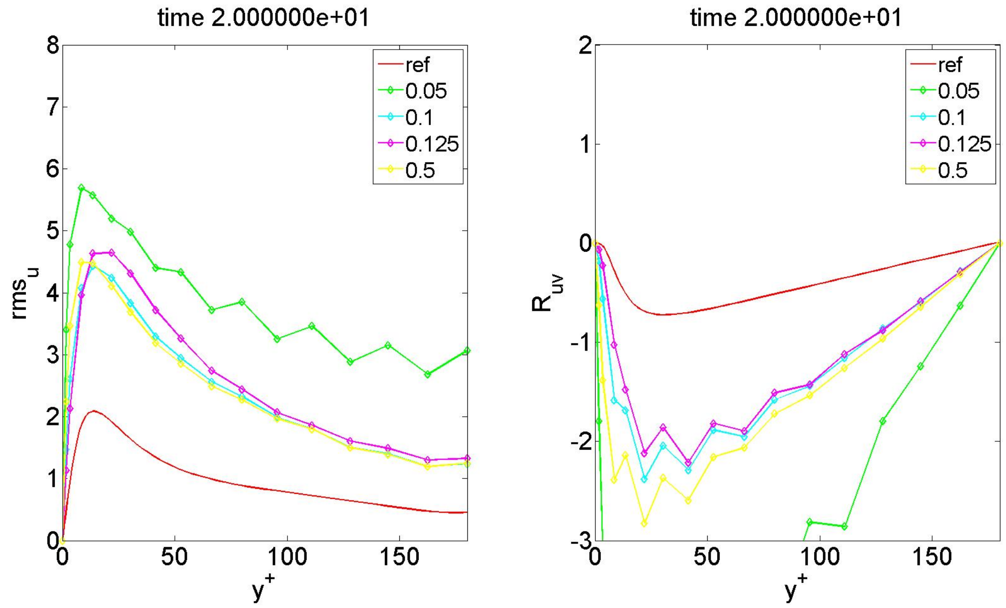


rms and R_{uv}

Figure 2: h = diam(K)



Mean and difference mean



rms and R_{uv}

Figure 3: h = edge(K)

In hindsight, our calculations show, that there is no best value for the filter width constant and the measurement of h. No value for C satisfies all statics well enough to be considered the best fitting value. In terms of the first order statistics, namely the mean value comparison, $C = 0.025$ for the geometric measure and $C = 0.1$ and $C = 0.125$ for the diameter and shortest edge measure gives good results. Considering second order statistics these values don't come close to the reference. All in all, the dependence on α can be seen very clearly.

6 Summary and Outlook

In [FHT02] it is shown that the dimension of the global attractor is decreasing, this theoretical statements suggests that the NS-α model of turbulence is easier to compute. Our own numerical experiments did not provide us with a good value for the filter width constant. More testing might have been necessary for that. It might be interesting to consider the question of what would be a good result.
A comparison of NS-α and NS-ω is done in [LMNR10]. In Chapter 5 an interesting idea emerges: NS-α and NS-ω perform well in different flow regions. This would lead to a combination of both models which is a very interesting perspective for further research.
It is also interesting to improve the numerical approximation schemes to improve numerical results. In [MNOR11] it was possible to drastically improve the results by introducing a combination of a stabilization of grad-div type and an adapted approximate deconvolution of the filtering operation.
It is needless to say that the potential of NS-α has not been exhausted yet.

Nevertheless, more numerical experiments using the NS-α model and studies that compare the results of experiments using the NS-α model to the results obtained by other turbulence models are needed.

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