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Burgers' Equation

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Declaration

I hereby declare that this thesis is my original work and it has been written by me in its entirety. I have acknowledged all the sources of information which have been used in the thesis. This thesis has not been submitted to any other University or Institution.

Place, Date

Leonie Metz

Contents

Chapter 1

Introduction

There is a strong interest in studying nonlinear partial differential equations and nowadays computer capacities allow to compute approximations for such problems. Burgers' equation is one famous example for a nonlinear partial differential equation which is suitable for the application in various important areas in physics as well as in applied mathematics such as fluid mechanics, gas dynamics or traffic flow, see [BAM18].
In 1915, the english mathematician Harry Bateman firstly introduced the equation in [Bat15]. Later in 1948, the Dutch physicist Johannes Martinus Burgers models mathematically the theory of turbulence under the use of this equation in [Bur48]. Afterwards it was named in the honor of him as “Burgers' equation”.

For better understanding, the equation is mathematically analyzed in Chapter 2. Research looks at two versions of the differential equation: the viscid Burgers' equation, which is studied in Chapter 2.1 and the inviscid Burgers' equation, which is studied in Chapter 2.2.
For the viscid equation Eberhard Hopf and Julian David Cole independently introduced a transformation to convert Burgers' equation into a linear heat equation and solved it exactly for an arbitrary initial condition. Hence, the transformation is famously known as the Hopf–Cole transformation and studied in Section 2.1.1. In Section 2.1.2 there will be presented two methods to solve the heat equation and thus the Hopf-Cole-converted initial value problem of the viscid Burgers' equation. For the inviscid equation one can partly solve it with the method of characteristics, which is studied in Section 2.2.1. In some cases there will arise shocks, which we will analyze in Section 2.2.2 considering the breaking time, weak solutions, the Rankine-Hugoniot jump condition, the entropy condition, the Riemann problem and the vanishing viscosity approach.

Another important aspect of Burger's equation is that it allows us to compare the quality of numerical methods applied to a nonlinear equation. In Chapter 3 we will therefore have a look at the basics of finite difference methods and try to approximately solve the two versions of Burgers' equation.

Goals of this thesis are thus to present a survey on the mathematical analysis and look at the existence, construction and (non-)uniqueness and features of solutions and to introduce some selected finite difference methods for computing an approximate solution of Burgers' equation.

Chapter 2

Mathematical Analysis

To better understand Burgers' equation we will start by analyzing the two versions mathematically.

2.1 Viscid Burgers' Equation

The viscid Burgers' equation is the following nonlinear parabolic equation of second order and in one space dimension

$$u_t + uu_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \epsilon\frac{d^2u}{dx^2} = \epsilon u_{xx} \tag{2.1.1}$$

with $x \in \mathbb{R}$, $t > 0$ and the diffusion coefficient $\epsilon \in \mathbb{R}$. The diffusion coefficient must be positive (i.e., $\epsilon > 0$) in order that the initial value problem with initial condition

$$u(x, 0) = u_0(x) \tag{2.1.2}$$

will be well-posed in forward time. On the left-hand side of (2.1.1) we find a nonlinear advection term uu_x and a time-dependent term u_t. The right-hand side of (2.1.1) models the effect of linear diffusion or viscosity. The nonlinear advection term has a shocking up effect, which causes waves to break, but the viscid diffusion term suppresses the wave-breaking and smoothes out shock discontinuities, see [Sal16, p.1, 4].
Therefore Burgers' equation is also known as a 1D-version of the incompressible Navier-Stokes equation, see [Olv14, p.315].

Like with any other differential equation the question how to solve it is present.

2.1.1 Hopf-Cole Transformation

The Hopf-Cole transformation converts the nonlinear viscous Burgers' equation to the linear heat equation, which can be explicitly solved then.
We follow Olver [Olv14, p.318-319] in his approach to construct the famous transformation.

Remark 2.1.1. *To convert a nonlinear differential equation into a linear one is quite challenging and in most cases impossible. The reverse is trivial and achieved* *by simply changing the dependent variables nonlinearly. Indeed, we can convert the nonlinear Burgers' equation into the linear heat equation and vice versa.*

To understand, how the Hopf-Cole transformation is constructed, we start by looking at the linear heat equation

$$v_t = \epsilon v_{xx} \tag{2.1.3}$$

with $\epsilon > 0$. The simplest possible nonlinear change of the dependent variable is an exponential one, i.e.,

$$v(x,t) = e^{\alpha\phi(x,t)},$$

so

$$\phi(x,t) = \frac{1}{\alpha}\ln v(x,t)$$

with α in $\mathbb{R}$, $\alpha \neq 0$ as a constant.
We now see that $\phi(x,t)$ is real, if and only if $v(x,t)$ is a positive solution to the heat equation.
Therefore the initial data $v(x,0)$ has to be positive, i.e., $v(x,0) > 0$.
Because if $v(x,0) > 0$ it follows via the Maximum Principle that $v(x,t) > 0$ for all $t > 0$, see [MUS06, p.17].
To find now the partial differential equation which is satisfied by ϕ we use the product and chain rule to differentiate. We get

$$v_t = \alpha\phi_t e^{\alpha\phi}, \qquad v_x = \alpha\phi_x e^{\alpha\phi}, \qquad v_{xx} = (\alpha\phi_{xx} + \alpha^2{\phi_x}^2)e^{\alpha\phi}.$$

We now substitute the first and last expression into the heat equation (2.1.3) and cancel the common exponential factor out

$$\alpha\phi_t e^{\alpha\phi} = \epsilon(\alpha\phi_{xx} + \alpha^2{\phi_x}^2)e^{\alpha\phi},$$

$$\phi_t = \epsilon\phi_{xx} + \epsilon\alpha{\phi_x}^2. \tag{2.1.4}$$

We conclude that $\phi(x,t)$ satisfies the potential Burgers' equation (2.1.4).
The second step is now to differentiate the potential Burgers' equation with respect to x

$$\phi_{tx} = \epsilon\phi_{xxx} + 2\epsilon\alpha\phi_{xx}\phi_x. \tag{2.1.5}$$

Now set

$$\phi_x = u. \tag{2.1.6}$$

It follows that

$$u_t = \epsilon u_{xx} + 2\epsilon\alpha u u_x,$$

which is Burgers' equation for $\alpha = -\frac{1}{2\epsilon}$.
With this process, we have found the Hopf-Cole transformation.

Theorem 2.1.2 (Hopf-Cole transformation)**.** *If $v(x,t) > 0$ is any possible solution to the linear heat equation $v_t = \epsilon v_{xx}$, then*

$$u(x,t) = \frac{\partial}{\partial x}[-2\epsilon\ln v(x,t)] = -2\epsilon\frac{v_x}{v}$$

solves Burgers' equation $u_t + uu_x = \epsilon u_{xx}$.

Now, one has to check if indeed all solutions to the Burgers' equation arise in this way. We have to run the argument in reverse, compare [Olv14, p.319]. We choose a potential function which satisfies (2.1.6)

$$\tilde{\phi}(x,t) = \int_0^x u(y,t)\ dy.$$

If $u(x,t)$ is a solution to Burgers' equation, $\tilde{\phi}(x,t)$ satisfies (2.1.5)

$$\tilde{\phi}_{tx} = \epsilon\tilde{\phi}_{xxx} + 2\epsilon\alpha\tilde{\phi}_{xx}\tilde{\phi}_x.$$

We integrate both sides with respect to x and substitute in the process $z = \tilde{\phi}_x$ and $dz = \tilde{\phi}_{xx}dx$ and get

$$\int \tilde{\phi}_{tx}dx = \int \epsilon\tilde{\phi}_{xxx} + 2\epsilon\alpha\tilde{\phi}_{xx}\tilde{\phi}_x dx \Longrightarrow$$
$$\tilde{\phi}_t = \epsilon\tilde{\phi}_{xx} + \epsilon\alpha\tilde{\phi}_x^2 + c(t),$$

with $c(t)$ as an integration constant.
Now we see a problem. Unless $c(t) \equiv 0$, our potential function $\tilde{\phi}$ does not satisfy the potential Burgers' equation (2.1.4). We have to modify the potential function

$$\phi(x,t) = \tilde{\phi}(x,t) - C(t),$$

where $C'(t) = c(t)$.
Then, it follows

$$\phi_t = \tilde{\phi}_t - c(t) = \epsilon\tilde{\phi}_{xx} + \epsilon\alpha\tilde{\phi}_x^2 = \epsilon\phi_{xx} + \epsilon\alpha{\phi_x}^2$$

and $\phi(x,t)$ is a solution to the potential Burgers' equation (2.1.4). It follows that

$$v(x,t) = e^{\frac{-\phi(x,t)}{2\epsilon}}$$

is a positive solution to the heat equation and therefore the solution $u(x,t)$ of Burgers' equation can be obtained through Theorem 2.1.2. In [Olv14, p. 319], it is summed up: "We conclude that every solution to Burgers' equation comes from a positive solution to the heat equation via the Hopf–Cole transformation."

Example 2.1.3. *[Olv14, p.320] As an example let us look at the following solution of the heat equation*

$$v(x,t) = a + be^{-\epsilon\omega^2 t}\cos\omega x$$

and let $a > |b|$ in order that $v(x,t) > 0$ for $t \geq 0$. We find the solution $u(x,t)$ of Burgers' equation via the Hopf-Cole transformation 2.1.2 as follows.

We compute

$$v_x(x,t) = -b\omega e^{-\epsilon\omega^2 t}\sin\omega x.$$

Then

$$\begin{aligned} u(x,t) &= \frac{\partial}{\partial x}[-2\epsilon \ln v(x,t)] \\ &= -2\epsilon \frac{v_x}{v} \\ &= \frac{2\epsilon b\omega \sin \omega x}{a e^{\epsilon \omega^2 t} + b \cos \omega x} \end{aligned}$$

is the solution.

To solve the initial value problem (2.1.1), (2.1.2), we also have to transform the initial condition (2.1.2) via Hopf-Cole transformation

$$v(x,0) = e^{-\frac{\phi(x,0)}{2\epsilon}} = e^{-\frac{1}{2\epsilon}\int\limits_0^x u(y,0)\ dy} = e^{-\frac{1}{2\epsilon}\int\limits_0^x u_0(y)\ dy} \equiv h(x). \tag{2.1.7}$$

To sum up, we have now reduced the initial value problem of the Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \epsilon \frac{d^2 u}{dx^2}, \qquad\qquad u(x,0) = u_0(x),$$

to the following initial value problem of the heat equation

$$v_t = \epsilon v_{xx}, \qquad\qquad v(x,0) = e^{-\frac{1}{2\epsilon}\int\limits_0^x u_0(y)\ dy}. \tag{2.1.8}$$

In the next step we will show how to solve the heat equation fundamentally and then solve the Hopf-Cole-converted initial value problem (2.1.8).

Remark 2.1.4. *We had a look at the Hopf-Cole transformation for the infinite space. We can also solve viscid Burgers' equation in finite space with constant or time-dependent Dirichlet boundary conditions via the method, but, for example, in finite space with constant total flux at one boundary and homogeneous Dirichlet at the other we cannot solve viscid Burgers' equation via the Hopf-Cole transformation, see [Bes10, p.3457].*

2.1.2 Heat Equation

Solving with Fourier Transform

We are going to solve the initial value problem for the heat equation

$$v_t - \epsilon v_{xx} = 0, \qquad\qquad v(x,0) = v_0(x)$$

with $x \in \mathbb{R}$ and $t > 0$ by using Fourier transforms.

Definition 2.1.5. *[Olv14, p.264] We define*

$$\hat{f}(k) := \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x) e^{-ikx} dx \tag{2.1.9}$$

as the Fourier transform of the function $f(x)$ and

$$\hat{f}(k) = \mathcal{F}[f(x)] \tag{2.1.10}$$

as the Fourier transform operator.

In our case we need the definition of the Fourier transform with respect to x:

$$\hat{v}(k,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} v(x,t) e^{-ikx} dx. \tag{2.1.11}$$

We now want to take the Fourier transform of each term of the heat equation. Since we are working with derivatives, we need the "derivative theorem" for Fourier transforms:

Theorem 2.1.6 (Derivative Theorem)**.** *[Olv14, p.275] The Fourier transform of the derivative $f'(x)$ of a function is obtained by multiplication of its Fourier transform by ik:*

$$\mathcal{F}[f'(x)] = ik\hat{f}(k). \tag{2.1.12}$$

In our case this translates to

$$\mathcal{F}[v_x(x,t)] = ik\mathcal{F}[v(x,t)] = ik\hat{v}(k,t). \tag{2.1.13}$$

Then by (2.1.13), we have that

$$\mathcal{F}[\frac{\partial^2 v}{\partial x^2}] = \mathcal{F}[\frac{\partial}{\partial x}(\frac{\partial v}{\partial x})] = ik\mathcal{F}[\frac{\partial v}{\partial x}] = (ik)^2\hat{v}(k,t) = -k^2\hat{v}(k,t).$$

We also have that

$$\mathcal{F}[\frac{\partial v}{\partial t}] = \frac{\partial \hat{v}}{\partial t}$$

because we took Fourier transforms with respect to x and not to t.

To sum up, taking Fourier transforms of $v_t - \epsilon v_{xx} = 0$ gives

$$\frac{\partial \hat{v}}{\partial t} + \epsilon k^2 \hat{v}(k,t) = 0. \tag{2.1.14}$$

This behaves like a linear first order ordinary differential equation. We consider an integrating factor method for solving. We choose

$$e^{\int \epsilon k^2 dt} = e^{\epsilon k^2 t}$$

as the integrating factor and multiply with (2.1.14)

$$e^{\epsilon k^2 t}\frac{\partial \hat{v}}{\partial t} + \epsilon k^2 e^{\epsilon k^2 t}\hat{v}(k,t) = 0.$$

The two terms on the left side are a perfect derivative. We can rewrite as

$$\frac{\partial}{\partial t}[e^{\epsilon k^2 t}\hat{v}] = 0.$$

If we integrate both sides with respect to t, we get

$$e^{\epsilon k^2 t}\hat{v} = c_1(k).$$

By rearranging, we get

$$\hat{v}(k,t) = c_1(k)e^{-\epsilon k^2 t}.$$

Now we have to apply Fourier transformation to the initial condition $v(x,0) = v_0(x)$, giving

$$\hat{v}(k,0) = \hat{v}_0(k).$$

So we get

$$c_1(k) = \hat{v}_0(k)$$

and hence

$$\hat{v}(k,t) = \hat{v}_0(k)e^{-\epsilon k^2 t}.$$

We can now use the definition of the inverse Fourier transform to find the function $g(x,t)$ whose Fourier transform is $\hat{g}(k,t) = e^{-\epsilon k^2 t}$.

Definition 2.1.7. *[Olv14, p.265] To reconstruct the function $f(x)$, we define*

$$f(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dk \tag{2.1.15}$$

as the inverse Fourier transform and

$$f(x) = \mathcal{F}^{-1}[\hat{f}(k)] \tag{2.1.16}$$

as the inverse of the Fourier transform operator.

In our case this translates to

$$g(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(k,t)e^{ikx}dk. \tag{2.1.17}$$

Then we get

$$\begin{aligned}
g(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\epsilon k^2 t}e^{ikx}dk && (2.1.18)\\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\epsilon t[k^2 - \frac{ix}{\epsilon t}k]}dk && (2.1.19)\\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\epsilon t[(k-\frac{ix}{2\epsilon t})^2 + \frac{x^2}{4\epsilon^2 t^2}]}dk && (2.1.20)\\
&= \frac{1}{2\pi}e^{-\frac{x^2}{4\epsilon t}} \int_{-\infty}^{\infty} e^{-\epsilon t(k-\frac{ix}{2\epsilon t})^2}dk && (2.1.21)\\
&= \frac{1}{2\pi}e^{-\frac{x^2}{4\epsilon t}} \int_{-\infty}^{\infty} e^{-\epsilon t y^2}dy && (2.1.22)\\
&= \frac{1}{2\pi}e^{-\frac{x^2}{4\epsilon t}} \frac{1}{\sqrt{\epsilon t}} \int_{-\infty}^{\infty} e^{-z^2}dz && (2.1.23)\\
&= \frac{1}{2\sqrt{\epsilon t \pi}}e^{-\frac{x^2}{4\epsilon t}}. && (2.1.24)
\end{aligned}$$

In the first two steps we are just using the definition (2.1.17) and simplifying the expression. In the third step we are completing the square in the exponent. In the fourth step we take the factor without k in it out in front. In the fifth step we are substituting $y = k - \frac{ix}{2\epsilon t}$, hence $dy = dk$. In the sixth step we are going to substitute $z^2 = \epsilon t y^2$, so $z = \sqrt{\epsilon t}y$, hence $dz = \sqrt{\epsilon t}dy$. In the seventh step, we can solve the integral which is equal to $\sqrt{\pi}$ and simplify again.
To sum up, we find that

$$\hat{v}(k,t) = \hat{v}_0(k)e^{-\epsilon k^2 t} = \hat{v}_0(k)\hat{g}(k,t),$$

where $\hat{v}_0(k)$ is the Fourier transform of $v_0(x)$ and $\hat{g}(k,t)$ is the Fourier transform of $g(x,t) = \frac{1}{2\sqrt{\epsilon t\pi}}e^{-\frac{x^2}{4\epsilon t}}$.
Now we can use the convolution theorem for Fourier transforms.

Definition 2.1.8. *[Olv14, p.281] The convolution of scalar functions $f(x)$ and $g(x)$ is the scalar function $h = f * g$ defined by the formula*

$$h(x) = f * g(x) = \int_{-\infty}^{\infty} f(r)g(x-r)dr. \tag{2.1.25}$$

Theorem 2.1.9. *[Jam11, p.26] The Fourier transform of the convolution $h(x) = f(x) * g(x)$ of two functions is the product of their Fourier transforms:*

$$\hat{h}(k) = \hat{f}(k)\hat{g}(k). \tag{2.1.26}$$

In our case, this means that $\hat{v} = \mathcal{F}[v_0 * g]$, and so after applying inverse Fourier transforms on both sides, we get

$$v(x,t) = (v_0 * g)(x,t) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\epsilon t\pi}}e^{-\frac{(x-r)^2}{4\epsilon t}}v_0(r)dr, \tag{2.1.27}$$

which is the fundamental solution.
If we now use (2.1.7) as initial condition instead of $v_0(x)$, we get

$$v(x,t) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\epsilon t\pi}}e^{-\frac{(x-r)^2}{4\epsilon t}}e^{-\frac{1}{2\epsilon}\int_0^r u_0(y)dy}dr = \frac{1}{2\sqrt{\epsilon t\pi}}\int_{-\infty}^{\infty} e^{-[\frac{(x-r)^2}{4\epsilon t}+\frac{1}{2\epsilon}\int_0^r u_0(y)dy]}dr. \tag{2.1.28}$$

And so the solution for the initial value problem for viscid Burgers' equation via the Hopf-Cole transformation is given by Theorem 2.1.2

$$u(x,t) = -2\epsilon\frac{v_x}{v}.$$

Digression: Solving with Fourier Series

By using Fourier series, we present another way of computing the exact solution of the heat equation

$$\frac{\partial v}{\partial t} = \epsilon\frac{d^2v}{dx^2}$$

with

$$v : [0, L] \times (0, T), \qquad v(0, t) = 0, \qquad v(L, t) = 0, \qquad v(x, 0) = v_0(x).$$

We will follow Dawkins [Daw18] and use separation of variables. That means that we assume

$$v(x, t) = w(x)G(t).$$

We plug that into our heat equation and get

$$\begin{aligned} \frac{\partial}{\partial t}(w(x)G(t)) = \epsilon \frac{\partial^2}{\partial x^2}(w(x)G(t)) \Longrightarrow \\ w(x)\frac{dG}{dt} = \epsilon G(t)\frac{d^2w}{dx^2}. \end{aligned}$$

Notice that we have now two ordinary derivatives. Now we separate the variables by dividing both sides by $w(x)G(t)$, divide by ϵ and get

$$\frac{1}{\epsilon G}\frac{dG}{dt} = \frac{1}{w}\frac{d^2w}{dx^2} = -\lambda,$$

where $-\lambda$ is the separation constant. This means both functions of x and t are equal only if they are the same constant. Now we split into the two ordinary differential equations and get

$$\frac{dG}{dt} = -\epsilon\lambda G, \qquad \frac{d^2w}{dx^2} = -\lambda w.$$

We check that our product solution satisfies the boundary conditions

$$\begin{aligned} v(0, t) = w(0)G(t) = 0, \\ v(L, t) = w(L)G(t) = 0. \end{aligned}$$

If $G(t) = 0$ for all t, then $v(x, t) = 0$ is the trivial solution. So we have to assume $w(0) = 0$ and $w(L) = 0$ to avoid the trivial solution.
To sum up, we have gotten a first order differential equation and a second order boundary value problem

$$\frac{dG}{dt} = -\epsilon\lambda G, \qquad \frac{d^2w}{dx^2} + \lambda w = 0, \; w(0) = 0, \; w(L) = 0.$$

Now we want to solve the spatial boundary value problem. This is an eigenvalue problem. Since we do not know λ, we have to consider three different cases, where $\lambda > 0, \lambda = 0, \lambda < 0$.

Let us start assuming $\lambda > 0$. The characteristic polynomial of the differential equation is

$$s^2 + \lambda = 0.$$

It follows that

$$s_{1,2} = \pm\sqrt{-\lambda}$$

and because $\lambda > 0$, it follows that these roots are complex and we can write

$$s_{1,2} = \pm\sqrt{\lambda}i.$$

The general solution of the differential equation is then

$$w(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

Applying the first boundary condition we get

$$0 = w(0) = c_1.$$

Applying the second boundary condition as well gives us

$$0 = w(L) = c_2 \sin(L\sqrt{\lambda}).$$

For a nontrivial solution we assume $c_2 \neq 0$. It follows that

$$\sin(L\sqrt{\lambda}) = 0$$

and therefore

$$L\sqrt{\lambda} = n\pi$$

for $n = 1, 2, 3, \ldots$. So we get as eigenvalues and functions

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad w_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

for $n = 1, 2, 3, \ldots$.

For the second case, we assume $\lambda = 0$. The boundary value problem becomes

$$w'' = 0,\ w(0) = 0,\ w(L) = 0.$$

Integrating gives as general solution

$$w(x) = c_1 + c_2 x.$$

Applying the first boundary condition we get

$$0 = w(0) = c_1.$$

Applying the second boundary condition we get

$$0 = w(L) = c_2 L.$$

And it follows

$$c_2 = 0.$$

Together we have the trivial solution, so $\lambda = 0$ cannot be an eigenvalue.

For the third case, we assume $\lambda < 0$. The characteristic polynomial and roots are the same as in the first case where $\lambda > 0$. Because of $\lambda < 0$ it follows that

$$s_{1,2} = \pm\sqrt{-\lambda}$$

are real, distinct roots. The general solution is

$$w(x) = c_1 \cosh(\sqrt{-\lambda}x) + c_2 \sinh(\sqrt{-\lambda}x).$$

Applying the first boundary condition gives us

$$0 = w(0) = c_1.$$

Applying the second boundary condition, we get

$$0 = w(L) = c_2 \sinh(L\sqrt{-\lambda}).$$

Because we are assuming $\lambda < 0$, we know that

$$L\sqrt{-\lambda} \neq 0$$

and therefore

$$\sinh(L\sqrt{-\lambda}) \neq 0.$$

It follows that

$$c_2 = 0.$$

Together we have the trivial solution, so there are no negative eigenvalues for this boundary value problem.
To sum up, we have only

$$\lambda_n = (\frac{n\pi}{L})^2, \qquad w_n(x) = \sin(\frac{n\pi x}{L})$$

for $n = 1, 2, 3, \ldots$ as eigenvalues and eigenfunctions.

Now we want to solve the time differential equation

$$\frac{dG_n}{dt} = -\epsilon\lambda_n G_n.$$

It is a linear, first order differential equation and we know that the general solution is

$$G_n(t) = ce^{-\epsilon\lambda_n t} = ce^{-\epsilon(\frac{n\pi}{L})^2 t}.$$

Now we can write down the overall solution

$$v_n(x,t) = A_n \sin(\frac{n\pi x}{L})e^{-\epsilon(\frac{n\pi}{L})^2 t}$$

for $n = 1, 2, 3, \ldots$ and A_n is an arbitrary constant.
Because of the Principle of Superposition

$$v(x,t) = \sum_{n=1}^{M} A_n \sin(\frac{n\pi x}{L})e^{-\epsilon(\frac{n\pi}{L})^2 t}$$

is also a solution to the partial differential equation, which satisfies the boundary conditions and the initial condition

$$v(x,0) = \sum_{n=1}^{M} A_n \sin(\frac{n\pi x}{L}).$$

We can take the limit as M goes to infinity

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{L}) e^{-\epsilon(\frac{n\pi}{L})^2 t}.$$

The solution satisfies any initial condition which can be written in the form

$$v(x,0) = v_0(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{L}).$$

We see it is the Fourier sine series, which we can write down for any piecewise smooth function on $0 \leq x \leq L$.
It follows that

$$A_n = \frac{2}{L} \int_0^L v_0(x) \sin(\frac{n\pi x}{L}) dx$$

for $n = 1, 2, 3, \ldots$.

After analyzing and solving the viscid Burgers' equation, we want to do the same with the second version of Burgers' equation.

2.2 Inviscid Burgers' Equation

In Chapter 2.1 we introduced the viscid Burgers' equation (2.1.1). If we now assume that $\epsilon = 0$ we get the first order quasilinear hyperbolic equation

$$u_t + uu_x = 0, \tag{2.2.1}$$

which is called "inviscid Burgers' equation". The equation can also be written in the form of "conservation law". Conservation laws include equations that model the conservation laws of physics, i.e., mass, momentum, energy, et cetera, see [Sar02].

Definition 2.2.1. *([Olv14, p.38]) A conservation law, in one space dimension, is an equation of the form*

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0. \tag{2.2.2}$$

The function T is known as the conserved density, while X is the associated flux.

If we write inviscid Burgers' equation as conservation law, we get

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \tag{2.2.3}$$

with $T = u$ and $X = \frac{1}{2}u^2$.

We can solve it with the method of characteristics only for some specific initial values, otherwise there will occur shocks.

2.2.1 Method of Characteristics

General Idea

We follow [Sar02, p.1-2]. The method of characteristics is a method to solve partial differential equations, which are of first order type or hyperbolic. Let us consider the first order linear partial differential equation

$$a(x,t)u_x + b(x,t)u_t + c(x,t)u = 0 \tag{2.2.4}$$

with initial condition

$$u(x,0) = u_0(x). \tag{2.2.5}$$

The goal is to reduce the partial differential equation with a coordinate transformation from (x,t) with $x \in \mathbb{R}$ and $t > 0$ to (x_0, s) to a system of ordinary differential equations on hypersurfaces $[x(s), t(s)] : 0 < s < \infty$, which are called "characteristics". Now we choose

$$\frac{dx}{ds} = a(x,t), \qquad\qquad \frac{dt}{ds} = b(x,t). \tag{2.2.6}$$

Then we have

$$\frac{du}{ds} = \frac{dx}{ds}u_x + \frac{dt}{ds}u_t = a(x,t)u_x + b(x,t)u_t. \tag{2.2.7}$$

The partial differential equation becomes the ordinary differential equation

$$\frac{du}{ds} + c(x,t)u = 0.$$

To solve (2.2.4), we firstly have to solve (2.2.6) and find the constants of integration by setting $t(0) = 0$ and $x(0) = x_0$. We have a transformation from (x,t) to (x_0, s), $x = x(x_0, s)$ and $t = t(x_0, s)$. Then we have to solve the ordinary differential equation (2.2.7) with initial condition $u(0) = x_0$. We obtain the solution $u(x_0, s)$ and can substitute to get the solution $u(x,t)$ to the partial differential equation.

Example: Linear Transport Equation

As an easier to begin with example, we consider a conservation law $u_t + [f(u)]_x = 0$ with initial condition $u(x,0) = u_0(x)$ and with $u = u(x,t)$, where $x \in \mathbb{R}$, $t > 0$ and $u : \mathbb{R} \times (0,\infty) \to \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be $f(u) = au$ with $a > 0$ as wave speed or velocity of propagation.
Then after differentiating $f(u)$ with respect to x we get for the conservation law

$$u_t + au_x = 0. \tag{2.2.8}$$

Equation (2.2.8) is called the "transport equation".

Now we want to solve this partial differential equation with the method of characteristics. Assume $x = x(t)$. Now consider the following by use of chain rule

$$\frac{d}{dt}(u(x(t),t)) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt}.$$

If

$$\frac{dx}{dt} = a, \tag{2.2.9}$$

then

$$\frac{d}{dt}(u(x(t),t)) = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0. \tag{2.2.10}$$

These are now two ordinary differential equations. If we integrate (2.2.9), it follows that

$$x(t) = at + x_0,$$

with x_0 as integration constant. If we solve for t we get

$$t = \frac{1}{a}(x - x_0)$$

which is a line with the slope $\frac{1}{a}$. So the characteristics are straight parallel lines for any x_0, see Figure 2.1.

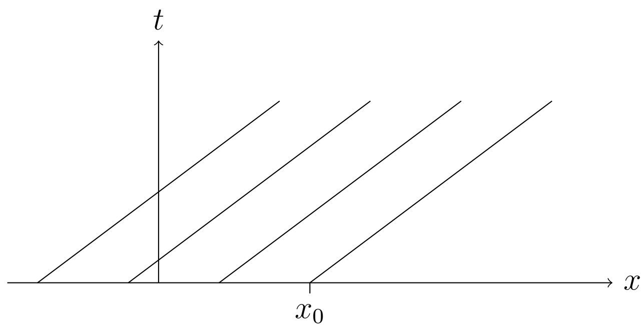

Figure 2.1: Characteristics are straight parallel lines

This means that on each of these characteristic lines the derivative of $u(x(t),t)$ with respect to t is zero, see (2.2.10). So the solution $u(x(t),t)$ is constant along this line. We can also see this by solving the other ordinary differential equation (2.2.10) by integrating. We get

$$u(x(t),t) = C.$$

Now we want to determine C by plugging in $t = 0$. We get

$$u(x(0),0) = u_0(x(0)) = u_0(x_0) = C.$$

Our solution is therefore

$$\begin{aligned} u(x(t),t) &= u_0(x_0), \\ u(x,t) &= u_0(x - at). \end{aligned}$$

Since $a > 0$, this is a translation of $u_0(x)$ to the right, see Figure 2.2.

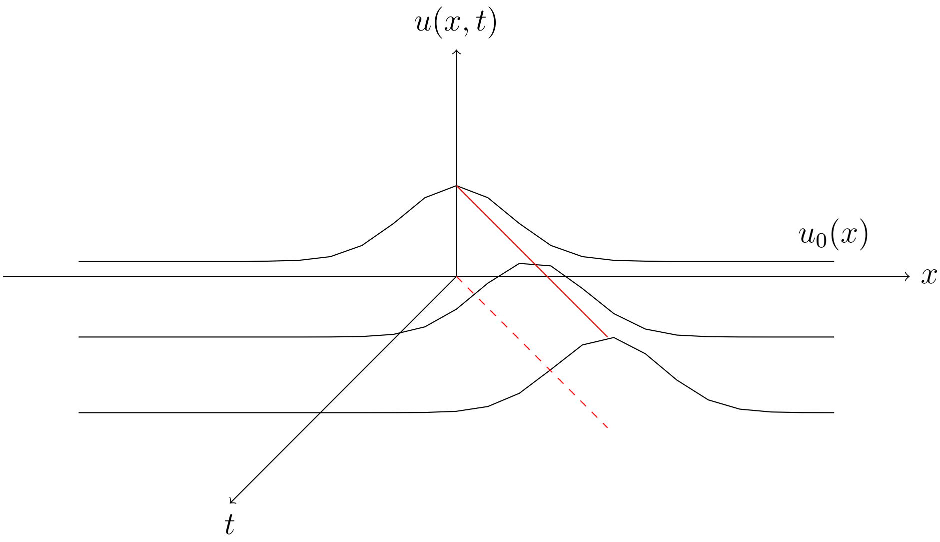

Figure 2.2: Visualization of the solution $u(x,t)$ of the transport equation (2.2.8)

Because of the great work of the method of characteristics for the linear transport equation, we will try the method now for the nonlinear transport equation which is also known as inviscid Burgers' equation.

Use for Inviscid Burgers' Equation

We will consider the initial value problem

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \qquad u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \qquad t > 0.$$

Let $x = x(t)$ and consider the following by use of chain rule

$$\frac{d}{dt}(u(x(t),t)) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt}.$$

By comparing we see that if

$$\frac{dx}{dt} = u(x(t),t), \tag{2.2.11}$$

then

$$\frac{d}{dt}(u(x(t),t)) = 0. \tag{2.2.12}$$

This is a system of differential equations, which we can solve.
If we integrate (2.2.12), it follows that $u(x(t),t)$ is constant along the characteristic, i.e.,

$$u(x(t),t) = C,$$

where C is a constant. For $t = 0$ we get

$$u(x(0),0) = u(x_0,0) = u_0(x_0).$$

Therefore

$$u(x(t),t) = u_0(x_0).$$

From (2.2.11) we now know that

$$\frac{dx}{dt} = u_0(x_0).$$

If we integrate that, we get

$$x(t) = u_0(x_0)t + D,$$

where D is a constant. For $t = 0$ we get

$$x(0) = D = x_0.$$

Therefore

$$x(t) = u_0(x_0)t + x_0. \tag{2.2.13}$$

This shows that the characteristics are straight lines. The solution $u(x,t)$ is constant along this line. The slope $\frac{1}{u_0(x_0)}$ of our line depends on the initial data x_0. We can see this by writing

$$t = \frac{1}{u_0(x_0)}(x - x_0).$$

A point on this line is $(x(t), t)$, see Figure 2.3.

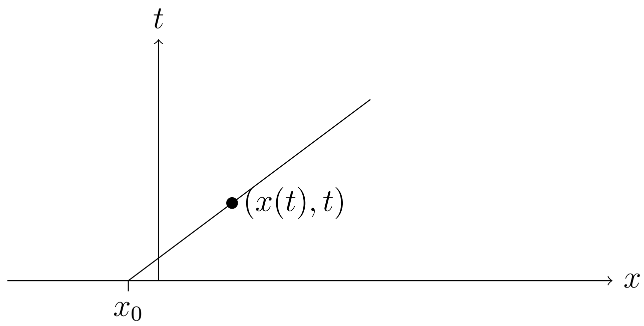

Figure 2.3: Characteristics are straight lines

Remark 2.2.2. *It is important to note that the characteristics might intersect because of the variable slope. At that intersection point the solution is not unique. There is a discontinuity and the appearance of a shock. One can have a look at weak solutions to solve that problem.*

Now we would substitute and the solution is then implicitly given by

$$u(x,t) = u_0(x_0) = u_0(x - ut), \qquad x_0 = x - u_0(x_0)t,$$

if the characteristics do not intersect.

Example 2.2.3. *We will consider the initial value problem*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \qquad u(x,0) = u_0(x) = \begin{cases} 1, & x < -1, \\ -x, & -1 \le x \le 0, \\ 0, & x > 0. \end{cases}$$

The initial condition $u_0(x)$ is pictured in Figure 2.4.

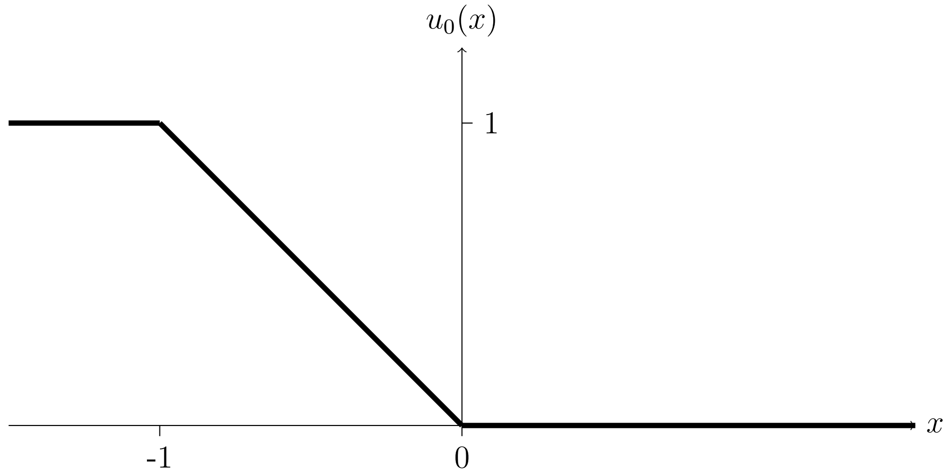

Figure 2.4: Initial condition $u_0(x)$

Let $x = x(t)$. We follow the characteristic method and find (2.2.13). *Then we consider three cases.*

Let $x_0 < -1$.
Then $u_0(x_0) = 1$, $x(t) = t + x_0$ and $u(x(t), t) = 1$.

Let $-1 \leq x_0 \leq 0$.
Then $u_0(x_0) = -x_0$ and $x(t) = -x_0 t + x_0$ which means $x_0 = \frac{x(t)}{1-t}$ and $u(x(t), t) = -x_0 = -\frac{x(t)}{1-t}$.

Let $x_0 > 0$.
Then $u_0(x_0) = 0$, $x(t) = x_0$ and $u(x(t), t) = 0$.

Our solution is

$$u(x(t),t) = \begin{cases} 1, & x_0 < -1, \\ -\frac{x(t)}{1-t}, & -1 \leq x_0 \leq 0, \\ 0, & x_0 > 0. \end{cases} = \begin{cases} 1, & x - t < -1, \\ -\frac{x(t)}{1-t}, & -1 \leq \frac{x}{1-t} \leq 0, \\ 0, & x > 0. \end{cases}$$

And rewriting a last time under the assumption that $t < 1$ gives

$$u(x,t) = \begin{cases} 1, & x < t - 1, \\ -\frac{x}{1-t}, & t - 1 \leq x \leq 0, \\ 0, & x > 0. \end{cases}$$

The solution $u(x,t)$ is pictured in Figure 2.6. We can see that $u(x,t)$ keeps moving to the right over time and has a discontinuity at $t = 1$.
The characteristic lines, which are firstly crossing at $t = 1$ are pictured in Figure 2.5.
For the first case we have characteristics with slope one, in the third case we have characteristics with slope zero and in the second case the slope of the characteristic

lines is changing and blowing up to infinity as t approaches one.

We need to figure out what happens for $t > 1$. We will have a look at the Riemann problem later.

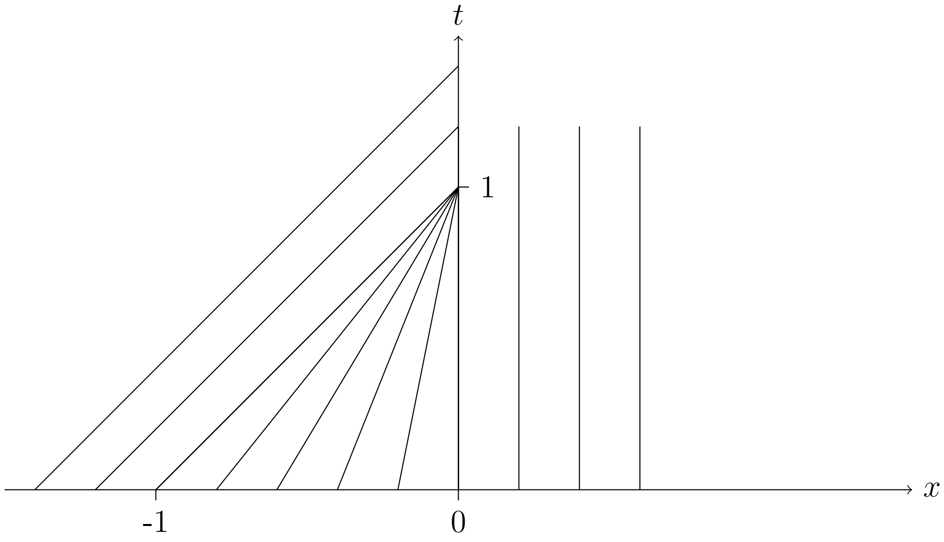

Figure 2.5: Characteristics in the (x,t) plane, which are firstly crossing at $t = 1$

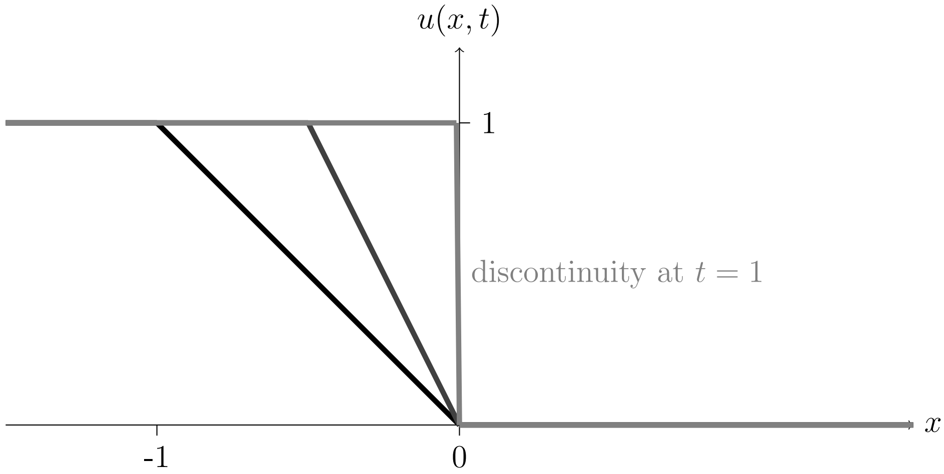

Figure 2.6: Solution $u(x,t)$

If the initial data is smooth, the method of characteristics can be used to determine the solution for small enough t such that the characteristics do not intersect. For larger t, after the characteristics have intersected, the partial differential equation will not have a “classical solution”, because we will obtain a multi-valued solution, or no solution at all, see [Sar02]. Because of the lack of a classical solution we will introduce the concept of “weak solutions”, which may contain discontinuities, may not be differentiable, and will require less smoothness to be considered a solution than a classical solution, see [Sar02], in the following Section 2.2.2.

2.2.2 Formation of Shocks

Breaking Time

At time T_b when the characteristics first cross, the function $u(x,t)$ has an infinite slope. The wave breaks and a shock forms. The time T_b is therefore called "breaking time". We can compute the breaking time as follows, see [Lan11, p.5].
Let us take two characteristic lines that arise from initial data x_1 and $x_2 = x_1 + \Delta x$. Because of (2.2.13) the two characteristics will cross when

$$x(t) = u_0(x_1)t + x_1 = u_0(x_2)t + x_2.$$

We assume $u_0 \in C^1(\mathbb{R})$ and solve for t. We get

$$t = -\frac{x_1 - x_2}{u_0(x_1) - u_0(x_2)} = \frac{\Delta x}{u_0(x_1) - u_0(x_1 + \Delta x)}.$$

When now $\Delta x \to 0$ the time converges because of the differential quotient to

$$t = -\frac{1}{u_0'(x_1)}.$$

To find the breaking time T_b we now search for the infimum value of t

$$T_b = \inf_{x \in \mathbb{R}, u_0'(x) < 0} \left[-\frac{1}{u_0'(x)} \right] = \frac{-1}{\inf_{x \in \mathbb{R}, u_0'(x) < 0} u_0'(x)}.$$

If x_0 produces the infimum T_b, then the slope of the solution will first become infinite at the location X_b, with

$$X_b = u_0(x_0)T_b + x_0,$$

where the characteristic starting at x_0 is at time T_b, see [Olv14, p.37].

Example 2.2.4. *[Log08, p.128] Let us consider* (2.2.1) *with initial condition* $u(x,0) = u_0(x) = e^{-x^2}$, *with* $x \in \mathbb{R}$, *which is a bell-shaped curve. We compute*

$$u_0'(x_1) = -2x_1 e^{-x_1^2}$$
$$u_0''(x_1) = -(4x_1^2 - 2)e^{-x_1^2}$$

It follows

$$u_0''(x_1) = 0 \iff x_1 = \sqrt{0,5}$$

This is where u_0' *has a minimum. Therefore, the breaking time is*

$$T_b = -\frac{1}{u_0'(\sqrt{0,5})} \approx 1,16.$$

The solution obtained via the characteristic method holds for $t < T_b$, but if we look at $u(x,t)$ in a physical context, a multi-valued solution is not acceptable, so we have to find restrictions for the solution for $t > T_b$. This is also known as the "shock fitting problem", see [Lan11, p.6]. To get a solution after the breaking time we have to allow discontinuities of $u(x,t)$. One can have a look at weak solutions to solve that problem.

Weak Solutions

Before we have that look, we define "classical solutions" or sometimes referred to as "strong solutions" of partial differential equations.

Definition 2.2.5. *([Eva10, pp.1, 7]) An expression of the form*

$$F(x, u(x), Du(x), \dots, D^{k-1}u(x), D^k u(x) = 0 \ (x \in U)$$

is called a k-th-order partial differential equation, where

$$F : U \times \mathbb{R} \times \mathbb{R}^n \times \dots \times \mathbb{R}^{n^{k-1}} \times \mathbb{R}^{n^k} \to \mathbb{R}$$

is given and

$$u : U \to \mathbb{R}$$

is the unknown.
We solve the partial differential equation if we find all u verifying the above.
We say u is a classical solution if it is at least k times continuously differentiable.

Remark 2.2.6. *For an initial value problem u has also to satisfy the following initial conditions*

$$D^i u(x_0) = u_i$$

with $i = 0, \dots, k-1$.

To validate discontinuous solutions to partial differential equations, we present the concept of weak solutions.
"Weak" therefore means that the solution holds for all appropriately chosen test functions $\varphi(x,t)$ [Joh21b, p.4], which we assume to be infinitely often differentiable and with compact support, i.e., they are different from zero only within some compact subset of space $(x,t) = \mathbb{R} \times [0,\infty)$, see [Cam, p.6].

Definition 2.2.7. *([Joh21a, p.4]) The space of infinitely often differentiable real functions with compact (closed and bounded) support in* $\Omega = \mathbb{R} \times [0,\infty)$ *is denoted by*

$$C_0^\infty = \{v : v \in C^\infty(\Omega), supp(v) \subset \Omega\},$$

where

$$supp(v) = \overline{\{x \in \Omega : v(x) \neq 0\}}.$$

In particular, functions from C_0^∞ *vanish in this case for* $x \to \pm\infty$ *and* $t \to \infty$.

We obtain the weak solution by multiplying the strong form of the equation with the test function and integrating the equation on $\Omega = \mathbb{R} \times [0,\infty)$ by parts as follows

$$\begin{aligned} 0 = \int_0^\infty \int_{-\infty}^\infty (u_t + [f(u)]_x)\varphi dx dt &= \int_{-\infty}^\infty u\varphi|_0^\infty \, dx - \int_0^\infty \int_{-\infty}^\infty u\varphi_t dx dt \\ &\quad + \int_0^\infty \varphi f(u)|_{-\infty}^\infty \, dt - \int_0^\infty \int_{-\infty}^\infty f(u)\varphi_x dx dt \\ &= -\int_0^\infty \int_{-\infty}^\infty u\varphi_t dx dt - \int_0^\infty \int_{-\infty}^\infty f(u)\varphi_x dx dt - \int_{-\infty}^\infty u\varphi|_{t=0} \, dx. \end{aligned}$$

After inserting the initial condition

$$u(x,0) = u_0(x)$$

we get

$$\int_0^\infty \int_{-\infty}^\infty \varphi_t u + \varphi_x f(u) dx dt + \int_{-\infty}^\infty \varphi(x,0)u(x,0)dx = 0.$$

Nearly all boundary terms arisen through integration by parts drop out due to φ having compact support, see [LeV92, p.28]. They vanish at infinity. The remaining term holds the initial condition of the partial differential equation.

Definition 2.2.8. *([Cam, p.6]) $u(x,t)$ is a weak solution of the conservation law $u_t + [f(u)]_x = 0$ if for any infinitely differentiable function $\varphi(x,t)$ with compact support*

$$\int_0^\infty \int_{-\infty}^\infty \varphi_t u + \varphi_x f(u) dx dt = -\int_{-\infty}^\infty \varphi(x,0)u(x,0)dx \tag{2.2.14}$$

holds. Such a function $\varphi(x,t)$ is called "test function".

The main aspect is, that in the weak form, the derivatives are acting only on $\varphi(x,t)$, which is smooth by assumption and not on the solution $u(x,t)$ which now doesn't need to be continuous for the integral to be well-defined, see [Olv14, p.432].

Remark 2.2.9. *If a classical solution to a problem exists, it also satisfies the definition of a weak solution.*

But if we include discontinuous solutions, we cannot guarantee the uniqueness of the solution. The uniqueness can be restored by using physical criteria. We need conditions for the shock.

Rankine-Hugoniot Jump Condition

Therefore we present the "Rankine-Hugoniot jump condition", which states that the shock speed equals the average of the solution values on either side. It is named after the nineteenth-century Scottish physicist William Rankine and French engineer Pierre Hugoniot, although these conditions first appeared in a paper by George Stokes in 1849, see [Olv14, p.40].

Definition 2.2.10. *([Cam, p.5]) For hyperbolic conservation laws $u_t + [f(u)]_x = 0$ the shock speed $\frac{ds}{dt}$ is determined by*

$$\frac{ds}{dt} = \frac{f(u_L) - f(u_R)}{u_L - u_R}. \tag{2.2.15}$$

This equation is called the Rankine-Hugoniot Jump Condition.

This condition determines the position of a shock at a given time [Lan11, p.6]. The argument above is valid for a general hyperbolic conservation law of the form $u_t + [f(u)]_x = 0$. For (2.2.3) we get the following proposition.

Proposition 2.2.11. *([Olv14, p.41]) Let $u(x,t)$ be a solution to the nonlinear transport equation (inviscid Burgers' equation) that has a discontinuity at position $x = s(t)$, with finite, unequal left- and right-hand limits*

$$u_L = u(s(t)^-, t) = \lim_{x \to s(t)^-} u(x,t), \qquad u_R = u(s(t)^+, t) = \lim_{x \to s(t)^+} u(x,t)$$

on either side of the shock discontinuity. Then, to maintain conservation of mass, the speed of the shock must equal the average of the solution values on either side:

$$\frac{ds}{dt} = \frac{u_L + u_R}{2}. \tag{2.2.16}$$

Proof. We can either use Definition 2.2.10 and insert $f(u) = \frac{1}{2}u^2$

$$\frac{ds}{dt} = \frac{f(u_L) - f(u_R)}{u_L - u_R} = \frac{\frac{1}{2}u_L^2 - \frac{1}{2}u_R^2}{u_L - u_R} = \frac{1}{2}\frac{u_L^2 - u_R^2}{u_L - u_R} = \frac{u_L + u_R}{2}$$

or we follow Olver in his proof in [Olv14, p.41f.]. □

This is the first condition that the shock of our discontinuous solution will have to satisfy.

The concept of weak solutions and the Rankine-Hugoniot jump condition are still not enough to guarantee uniqueness. We have to single out the unique, physically meaningful weak solution. Therefore we need an additional condition.

Entropy Condition

There exist different definitions of the "entropy condition". We will only work with the following.

Definition 2.2.12. *([LeV92, p.36]) A discontinuity propagating with speed $\frac{ds}{dt}$ given by $\frac{ds}{dt} = \frac{f(u_L)-f(u_R)}{u_L - u_R}$ satisfies the entropy condition if*

$$f'(u_L) > \frac{ds}{dt} > f'(u_R). \tag{2.2.17}$$

The definition is the mathematical translation of: In every physical process the entropy of the system is nondecreasing, see [Lan11, p.7].
For the inviscid Burgers' equation this means if the discontinuity propagates with the above speed, then $u_L > u_R$.

After introducing weak solutions, the Rankine-Hugoniot jump condition and entropy condition, we now want to use our new knowledge for an example.

Example: Riemann Problem

The conservation law with piecewise constant initial values and one discontinuity is called the "Riemann problem". In this thesis we consider the initial value problem of the Burgers' equation (2.2.1) with the following initial data:

$$u(x,0) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0. \end{cases} \tag{2.2.18}$$

The solution depends on the relation between u_L and u_R. We will consider two cases, see [Cam, p.8-9].

Case 1: In the first case we assume that $u_L > u_R$.
The characteristics cover the whole (x,t) space and cross. At the crosspoints the solution is multi-valued. The characteristics flow into the shock, see Figure 2.8. In this case there exists a unique solution

$$u(x,t) = \begin{cases} u_L, & x < \frac{ds}{dt}t, \\ u_R, & x > \frac{ds}{dt}t, \end{cases} \tag{2.2.19}$$

where $\frac{ds}{dt} = \frac{u_L+u_R}{2}$ is the shock speed, see Figure 2.9. It satisfies the entropy condition 2.2.12. This is the vanishing viscosity solution.

Proof. We will follow Cameron in her proof in [Cam, p.8-9].
Let $\varphi(x,t)$ be a test function, i.e., with compact support. First suppose that the support U of φ lies entirely in one of the sets $\{x < \frac{ds}{dt}t\}$ or $\{x > \frac{ds}{dt}t\}$. Since $u(x,t)$ is constant in each of these sets, it satisfies the inviscid Burgers' equation on the support U of φ. It is a weak solution, because (2.2.14) holds. Now suppose that the support U of φ is divided into two sets U_L and U_R by the line $x = \frac{ds}{dt}t$, see Figure 2.7.

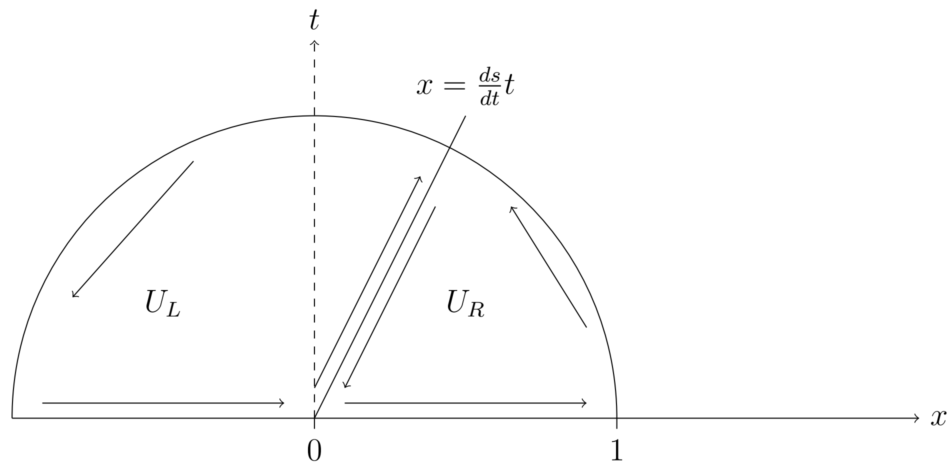

Figure 2.7: Visualization for the proof ([Cam, p.8-9])

Then we have

$$\int_0^\infty \int_{-\infty}^\infty \varphi_t u + \varphi_x f(u) dx dt = \int\int_{U_L} \varphi_t u + \varphi_x f(u) dx dt + \int\int_{U_R} \varphi_t u + \varphi_x f(u) dx dt.$$

Now we can apply the Green identity

$$\int\int_D (P_x - Q_t) dx dt = \int_{\delta D} P dt + Q dx \tag{2.2.20}$$

by noting that u is constant within U_L and U_R, therefore $(\varphi u)_t = \varphi_t u$ and $(\varphi f(u))_x = \varphi_x f(u)$ and continue with

$$
\begin{aligned}
&= \int\limits_{\delta U_L} \varphi(\frac{u_L^2}{2}dt - u_L dx) + \int\limits_{\delta U_R} \varphi(\frac{u_R^2}{2}dt - u_R dx) \\
&= \int\limits_{x=\frac{ds}{dt}t} \varphi(\frac{u_L^2}{2\frac{ds}{dt}} - u_L)dx - \int\limits_{x=\frac{ds}{dt}t} \varphi(\frac{u_R^2}{2\frac{ds}{dt}} - u_R)dx - \int\limits_{-\infty}^{0} \varphi u_L dx - \int\limits_{0}^{\infty} \varphi u_R dx \\
&= \int\limits_{x=\frac{ds}{dt}t} \varphi(\frac{u_L^2 - u_R^2}{2\frac{ds}{dt}} - (u_L - u_R))dx - \int\limits_{-\infty}^{\infty} \varphi(x,0)u(x,0)dx.
\end{aligned}
$$

In the last equality, the first integral is zero for any test function φ if $\frac{ds}{dt} = \frac{u_L+u_R}{2}$. The solution (2.2.19) is the unique weak solution for the Riemann problem in the case $u_L > u_R$. $\square$

Remark 2.2.13. *To graphically construct weak solutions to problems with shocks, one can use the "equal area rule", see [Sar02]. It starts with the multi-valued solution constructed by the method of characteristics, replaces the multi-valued parts by a vertical shock line and obtains a single-valued solution, which has the same area under its graph, see Figure 2.10 . It is a result of the conservation law, see [Olv14, p.40].*

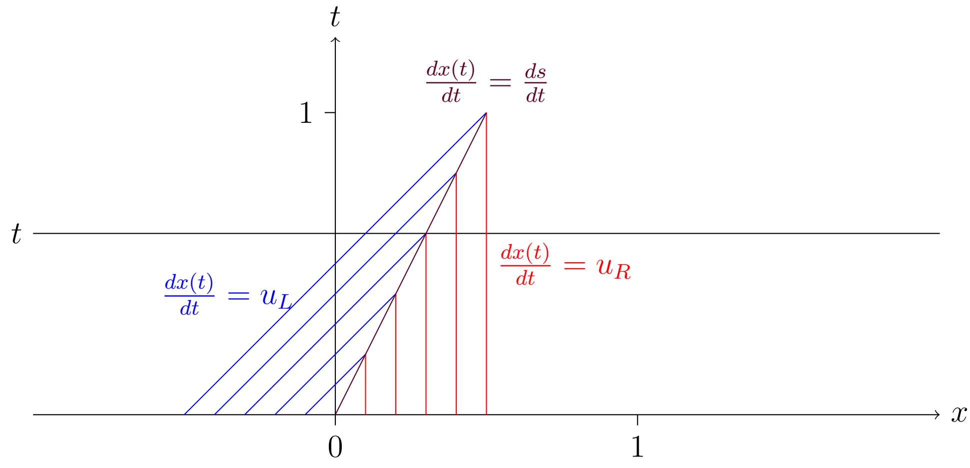

Figure 2.8: Case 1: Characteristics in the (x, t) plane for $u_L = 1 > 0 = u_R$

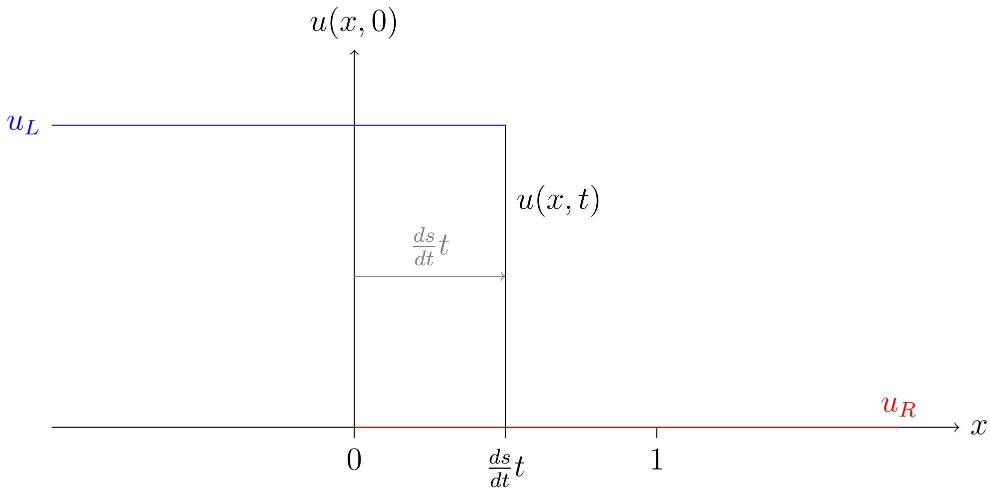

Figure 2.9: Case 1: Solution $u(x, t)$ for $u_L = 1 > 0 = u_R$

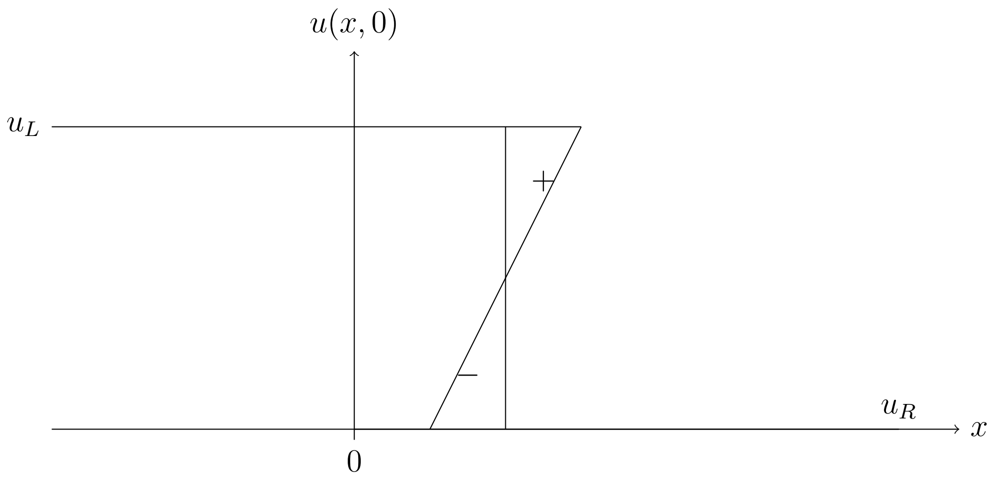

Figure 2.10: Equal area rule and solution $u(x, t)$ with shock

Case 2: In the second case we assume that $u_L < u_R$.
The characteristics do not cross and do not cover the whole (x,t) space. In this case we can construct infinitely many weak solutions. There are at least two which satisfy the Rankine-Hugoniot jump condition.
Firstly,

$$u_1(x,t) = \begin{cases} u_L, & x < \frac{ds}{dt}t, \\ u_R, & x > \frac{ds}{dt}t, \end{cases} \tag{2.2.21}$$

is also a solution which satisfies the Rankine-Hugoniot jump condition, see Figure 2.12. But it is unstable and does not satisfy the entropy condition. Note that the characteristics flow out of the shock, see Figure 2.11.

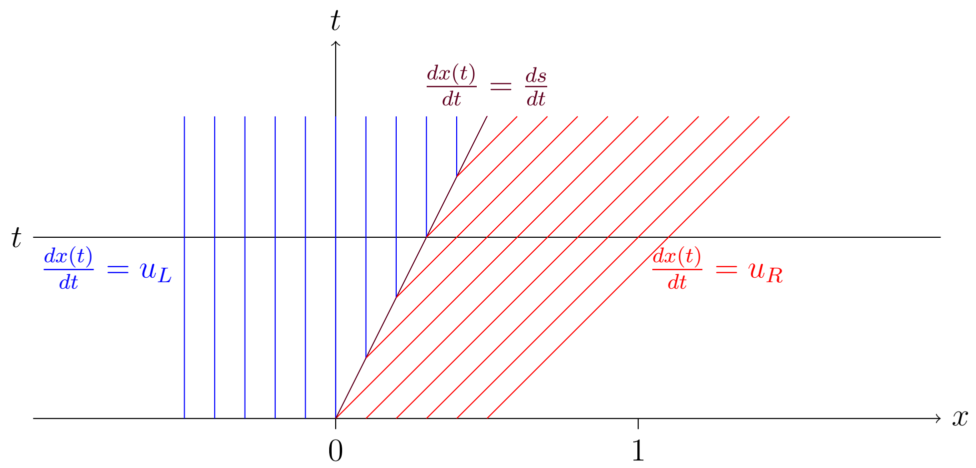

Figure 2.11: Case 2a: Characteristics in the (x,t) plane for $u_L = 0 < 1 = u_R$

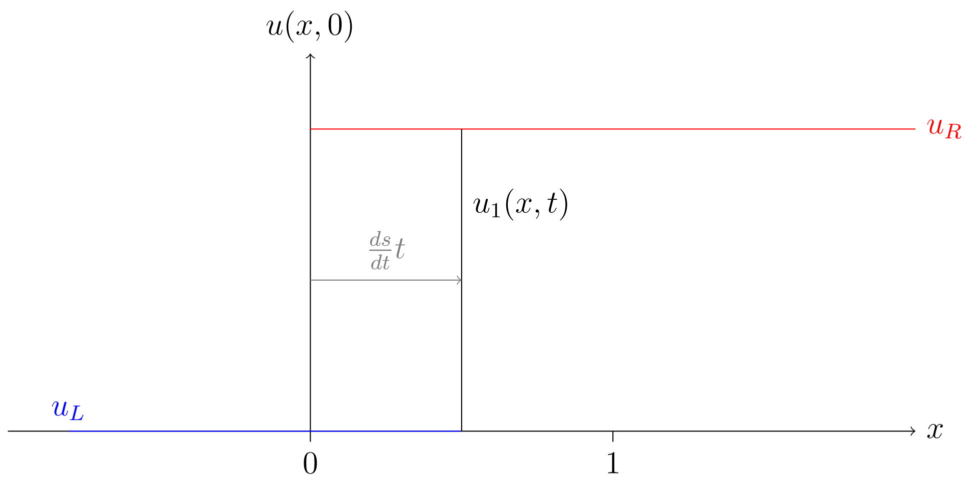

Figure 2.12: Case 2a: Solution $u_1(x,t)$ for $u_L = 0 < 1 = u_R$

Secondly, we will look at the "rarefaction wave" as a solution. It is given by

$$u_2(x,t) = \begin{cases} u_L, & x < u_L t, \\ \frac{x}{t}, & u_L t \le x \le u_R t, \\ u_R, & x > u_R t, \end{cases} \tag{2.2.22}$$

and is continuous but not smooth, see Figure 2.14. It does satisfy the entropy condition and is therefore the vanishing viscosity solution.

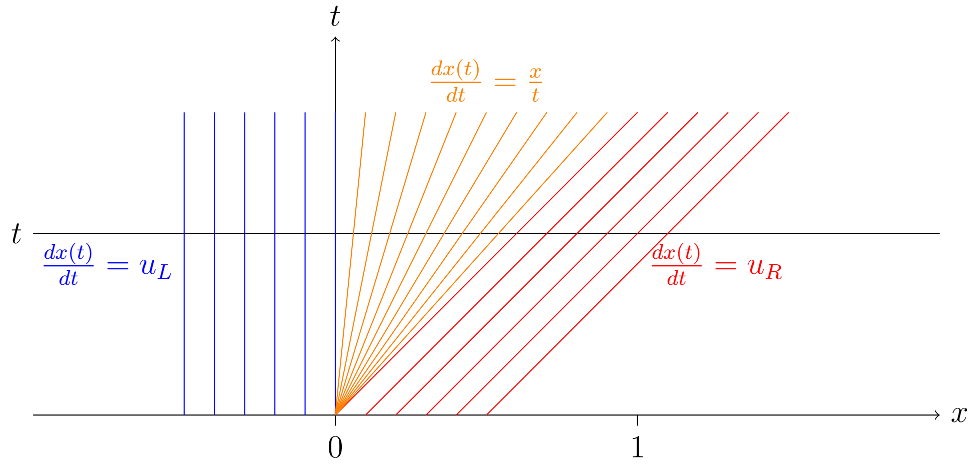

Figure 2.13: Case 2b: Characteristics in the (x, t) plane for $u_L = 0 < 1 = u_R$

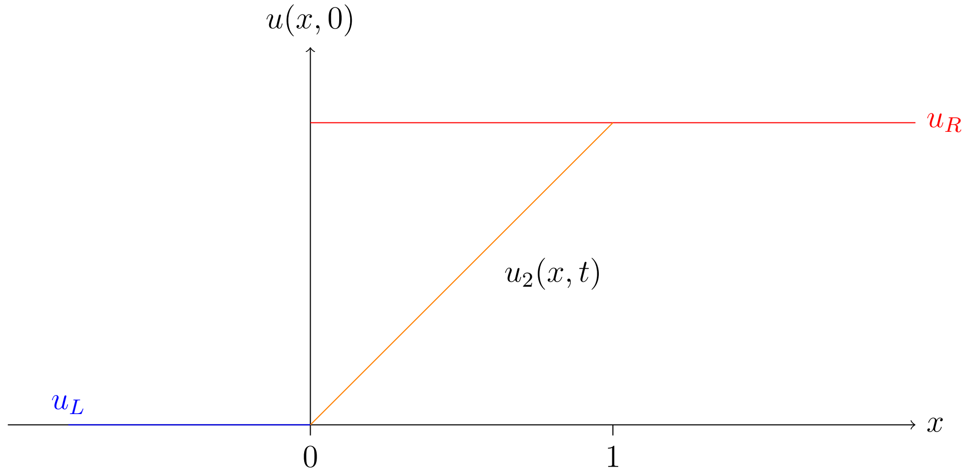

Figure 2.14: Case 2b: Solution $u_2(x, t)$ for $u_L = 0 < 1 = u_R$

Vanishing Viscosity Approach

There is a more intuitive approach in constructing the discontinuous entropy solution than fitting a shock following the Rankine-Hugoniot jump condition and entropy condition. It is called the "vanishing viscosity approach", see [Lan11, p.9].
The inviscid Burgers' equation (2.2.1) is a model of the viscid Burgers' equation (2.1.1) valid for small ϵ and smooth $u(x, t)$, see [Duy18, p.6]. When the model breaks down, we have to return to (2.1.1) by adding the viscosity dispersion term ϵu_{xx}. Adding this term suppresses the wavebreaking, because dispersion acts against the steepening effect of the nonlinearity, see [Lan11, p.9]. If the initial data is smooth and ϵ very small, then before the wave breaks, ϵu_{xx} is negligible compared to the other terms. Hence the solutions to both partial differential equations look almost the same. As the wave begins to break, the u_{xx} term grows much faster than the

u_x one. And it prevents the breakdown of solutions that occurs for the hyperbolic problem, see [Duy18, p.6]. We expect to get a smooth solution of (2.1.1) even with discontinuous data.
So the vanishing viscosity approach shows that the smooth solution of (2.1.1) approaches a shock wave and therefore becomes a discontinuous solution as $\epsilon \to 0$. Because the inviscid Burgers' equation (2.2.1) is some kind of idealization of a model where there is always some degree of viscosity, the only relevant solutions are those that we get by using this approach, see [Lan11, p.9].

For this thesis, we will stop the mathematical analysis of Burgers' equation here, because we also want to have a brief look at numerical methods to solve Burgers' equation approximately.

Chapter 3

Finite Difference Methods

With numerical methods mathematicians try to solve equations approximately or visualize the problems at hand.
Burgers' equation is an interesting nonlinear partial differential equation, with which one can test and compare the quality of different numerical methods.
In this thesis we want to introduce some finite difference methods to be a starting point for more research and practical work, i.e., coding.

3.1 Basics of Finite Difference Methods

"The finite difference approximation is the oldest of the methods applied to obtain numerical solutions of differential equations, and the first application is attributed to Leonhard Euler (1707–1783) in 1768" [Hir07, p.147].

The idea of finite difference methods is to estimate a derivative at finitely many steps by the ratio of two differences according to the definition of the derivative, see [Hir07, p.147].
The derivative at point x for a function $u(x)$ is defined, see [Hir07, p.147], by

$$u_x = \frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x}.$$

To get a finite difference, we remove the limit. To improve the approximation, we try to reduce Δx.

But for any finite value of Δx, the "truncation error" is introduced and it goes to zero for Δx going to zero, see [Hir07, p.147]. The "order of accuracy" of the finite difference approximation, can be obtained from Taylor expansion and is the power of Δx with which this error tends to zero, see [Hir07, p.147]. We will call it "order of consistency".

The analysis of finite differences is based on Taylor expansion, see [Hir07, p.147]. When we expand $u(x + \Delta x)$ around $u(x)$ we get

$$u(x + \Delta x) = u(x) + \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{3!} \frac{\partial^3 u}{\partial x^3} + \dots .$$

We can rewrite this as

$$\frac{u(x+\Delta x)-u(x)}{\Delta x} = u_x(x) + \underbrace{\frac{\Delta x}{2}u_{xx}(x) + \frac{\Delta x^2}{6}u_{xxx}(x) + \ldots}_{\substack{truncation\\ error}}. \tag{3.1.1}$$

Now we want to have a look at some difference formulas for first derivatives. First we have to discretize the domain. For the sake of simplicity, let us take a bounded interval of the x-axis, i.e., $[0,1]$ and discretize the space into $N+1$ discrete mesh points x_i, for $i = 0 \ldots N$. The value of the function $u(x)$ at the points x_i will be labeled by u_i, i.e., $u_i = u(x_i)$. Let the spacing between the discrete points be constant and equal to Δx. Without loss of generality, we can consider that $x_i = i\Delta x$, see [Hir07, p.149].

For the first derivative $(u_x)_i = (\frac{\partial u}{\partial x})_i$ we get the following finite difference approximations, see [Hir07, p.150]:

$$\begin{aligned}(u_x)_i = (\frac{\partial u}{\partial x})_i &= \frac{u_{i+1}-u_i}{\Delta x} - \frac{\Delta x}{2}(u_{xx})_i - \frac{\Delta x^2}{6}(u_{xxx})_i + \ldots \\ &= \frac{u_{i+1}-u_i}{\Delta x} + \mathcal{O}(\Delta x).\end{aligned} \tag{3.1.2}$$

This is a "forward difference" approximation, since it involves the point $(i+1)$. The approximation is of first order consistency. By replacing Δx with $-\Delta x$, we can also get

$$\begin{aligned}(u_x)_i = (\frac{\partial u}{\partial x})_i &= \frac{u_i-u_{i-1}}{\Delta x} + \frac{\Delta x}{2}(u_{xx})_i - \frac{\Delta x^2}{6}(u_{xxx})_i + \ldots \\ &= \frac{u_i-u_{i-1}}{\Delta x} + \mathcal{O}(\Delta x).\end{aligned} \tag{3.1.3}$$

This is a "backward difference" approximation, since it involves the point $(i-1)$. The approximation is of first order consistency. Both are "one-sided difference" formulas. If we add them up, we can also get

$$\begin{aligned}(u_x)_i = (\frac{\partial u}{\partial x})_i &= \frac{u_{i+1}-u_{i-1}}{2\Delta x} - \frac{\Delta x^2}{6}(u_{xxx})_i + \ldots \\ &= \frac{u_{i+1}-u_{i-1}}{2\Delta x} + \mathcal{O}(\Delta x^2).\end{aligned} \tag{3.1.4}$$

This is a "central difference" approximation, since it involves the points to the left and right of point i. The approximation is of second order consistency. These three approximations are represented geometrically in Figure 3.1.

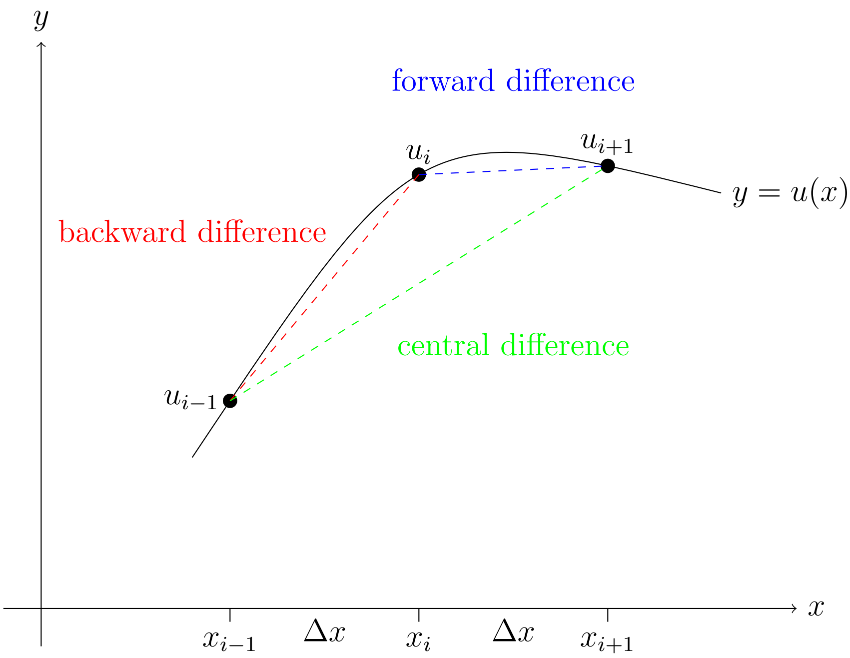

Figure 3.1: Illustration of the finite difference approximations of first order derivatives

Remark 3.1.1. *A first order finite difference approximation is exact for a linear function, because the truncation error of the approximation is proportional to the second derivative. A second order finite difference approximation is exact for a parabolic function, because the truncation error of the approximation is proportional to the third derivative, see [Hir07, p.151].*

Now we want to have a look at some difference formulas for second derivatives

$$\begin{aligned} (u_{xx})_i = (\frac{\partial^2 u}{\partial x^2})_i &= \frac{(u_x)_{i+1} - (u_x)_i}{\Delta x} \\ &= \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + \mathcal{O}(\Delta x^2). \end{aligned} \tag{3.1.5}$$

This is a central difference approximation, obtained by using backward difference approximations for $(u_x)_{i+1}$ and $(u_x)_i$. It is of second order consistency, as can be seen from Taylor series expansion. We get

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} = (u_{xx})_i + \frac{\Delta x^2}{12}(\frac{\partial^4 u}{\partial x^4})_i + \dots .$$

Finite difference approximations of higher order derivatives can be obtained by repeatedly using approximations for first order derivatives, see [Hir07, p.151].

For any partial differential equation of a mathematical model, we will have a large number of possible numerical schemes, see [Hir07, p.145]. For instance, for time-dependent problems there exist "explicit" and "implicit" methods. A method is explicit if the discretized equation contains only one unknown at level $(i + 1)$ and implicit if it contains more than one unknowns at level $(i + 1)$ see [Hir07, p.156]. These methods differ regarding accuracy, stability and error properties, see [Hir07, p.157].

For example, for a partial differential equation like Burgers' equation, we discretize the solution domain, fulfill the equation at discrete time points, replace derivatives by finite differences, and formulate an algorithm.

3.2 FDMs for the Inviscid Burgers' Equation

Finding a numerical solution to the inviscid Burgers' equation is challenging based on the allowed discontinuities for solutions of the equation, see [Lan11, p.12]. Even consistent and stable schemes might propagate discontinuities with wrong speeds and the methods might converge to the wrong weak solution, see [Cam, p.9]. We need some conditions to avoid that to happen.

Let us consider (2.2.1) with $x \in \mathbb{R}$, $t > 0$ and $u(x,0) = u_0(x)$. We discretize the solution domain $\{(x,t) : x \in [0,1], t \in [0,\infty)\}$ into cells described by the node set (x_i, t_j) in which $x_i = ih$, $t^j = jk$ and $i = 0:1:N; j = 0:1:M, Nh = 1, Mk = t_f$, see Figure 3.2. The spatial mesh size is $h \equiv \Delta x$, the time step is $k \equiv \Delta t$ and t_f is the final time, see [KBÖ98, p.253].

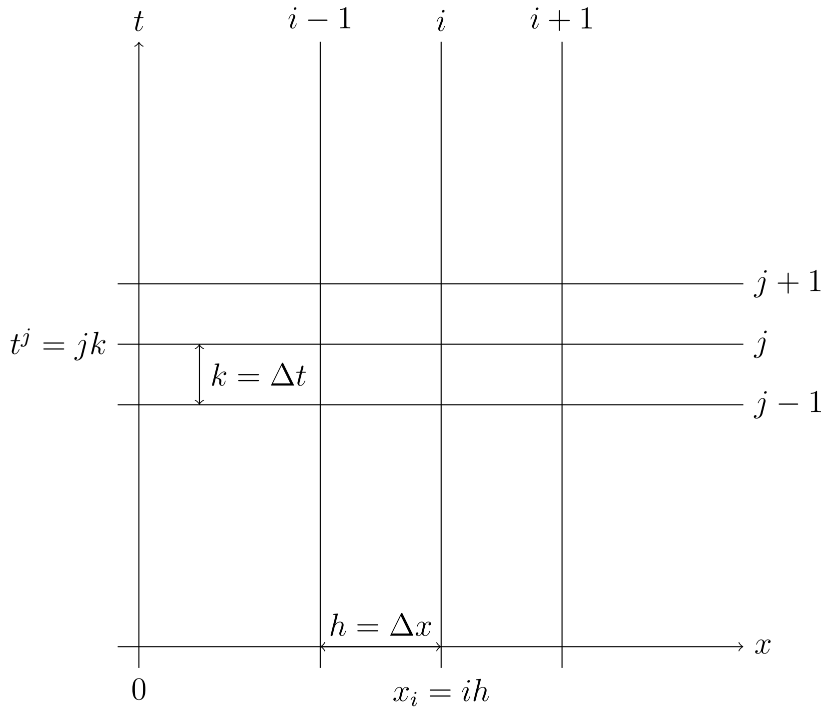

Figure 3.2: Discretization of the (x,t) space, where x is space and t is time

Burgers' equation is nonlinear. Since it is time-dependent, there are three possibilities in solving. We can handle the nonlinear term implicitly. Then we need a fix point iteration, i.e., Newton's method, to solve the nonlinear equation in the new time step, which is quite expensive, see [Lar17].
We can handle the nonlinear term semi-implicitly, which means we take u for one

factor from the old time step. Then we only have to solve a linear problem in the new time step.
We can handle the nonlinear term explicitly. Then we also only have to solve a linear problem in the new time step.
For the sake of simplicity, we will only focus on the last possibility.

A natural finite difference method obtained by a forward in time and backward in space discretization of the derivatives is

$$U_i^{j+1} = U_i^j - \frac{k}{h} U_i^j (U_i^j - U_{i-1}^j). \tag{3.2.1}$$

It is an "up-wind nonconservative" scheme, see [Lan11, p.12]
This method will in general not converge to a discontinuous weak solution of (2.2.1) with a more fine mesh, although it can be shown that it is consistent with (2.2.1) and sufficient for smooth solutions, see [Lan11, p.12].

Example 3.2.1. *([Cam, p.10], [LeV92, p.123]) Consider Burgers' equation* (2.2.1) *with the initial data*

$$u(x,0) = u_0(x) = \begin{cases} 1, & x < 0, \\ 0, & x \geq 0, \end{cases}$$

which gives $U_i^0 = 1$ for $i < 0$ and $U_i^0 = 0$ for $i \geq 0$ for (3.2.1)*. Then*

$$U_i^1 = \begin{cases} 1 - \frac{k}{h} 1(1-1) = 1, & i < 0, \\ 0 - \frac{k}{h} 0(0 - U_{i-1}^0) = 0, & i \geq 0, \end{cases}$$

which means $U_i^1 = U_i^0$. This happens in every step, therefore, $U_i^j = U_i^0$ for all i, regardless of the step sizes h and k. The method propagates the discontinuity with a wrong speed $\frac{ds}{dt} = 0$. The numerical solution converges to the function $u(x,t) = u_0(x)$. This is not a weak solution.

We conclude, a non-conservative method can give a solution, where the shock propagates at a wrong speed.

To resolve this issue, we have a short look at what it means to be a hyperbolic conservation law $u_t + [f(u)]_x = 0$. The conserved quantity

$$\int_L^R u(x,t)dx$$

can only change through the boundaries, see [Cam, p.11]. This is because of the flux.

$$\int_L^R u(x,t+b)dx - \int_L^R u(x,t)dx = \int_0^b f(u(L,t+\tau))d\tau - \int_0^b f(u(R,t+\tau))d\tau.$$

We impose a condition on our numerical method, to prevent the method from converging to non-solutions. The method will have to be in "conservation form".

Definition 3.2.2. *([Cam, p.11], [LeV92, p.124]) A numerical method is in a conservation form if it can be rewritten in the form*

$$U_i^{j+1} = U_i^j - \frac{k}{h}[F(U_{i-p}^j, \dots, U_{i+q}^j) - F(U_{i-p-1}^j, \dots, U_{i+q-1}^j)] \tag{3.2.2}$$

for some function F which is called the "numerical flux". A method that can be written in a conservation form is called "conservative".

With a conservative method the problem with non-conservative methods cannot occur, because a wrong shock speed would lead to an incorrect flux and thus conservation would not be maintained.
The easiest way to write a numerical method, i.e., finite difference method, in conservation form is to use the conservative form of the partial differential equation rather than the quasilinear one, see [LeV92, p.125].

A method in conservation form (3.2.2) is "consistent" with the conservation law if the following consistency conditions hold, see [Cam, p.12]:

$$F(\bar{u}, \dots, \bar{u}) = f(\bar{u}), \tag{3.2.3}$$
$$|F(v_1, \dots, v_r) - F(\bar{u}, \dots, \bar{u})| \leq K \max\{|v_1 - \bar{u}|, \dots, |v_r - \bar{u}|\} \tag{3.2.4}$$

where $K \geq 0$ is the Lipschitz constant.
The numerical flux function F reduces to the flux function f for the case of constant flow, see [LeV92, p.126]. Also F should approach $f(\bar{u})$ smoothly, if the arguments of F approach some constant value $\bar{u}$, see [LeV92, p.126]. Therefore we need the Lipschitz continuity of F.

If we consider the conservation law $u_t + [f(u)]_x = 0$ and a finite difference discretization, we get the "up-wind conservative" method

$$U_i^{j+1} = U_i^j - \frac{k}{h}[f(U_i^j) - f(U_{i-1}^j)]. \tag{3.2.5}$$

This is the form of (3.2.2) for $p = 0, q = 0, F(U, V) = f(U)$, see [Lan11, p.13].
The upwind flux is consistent, see [LeV92, p.126]. For (2.2.3) we have

$$U_i^{j+1} = U_i^j - \frac{k}{h}[\frac{1}{2}(U_i^j)^2 - \frac{1}{2}(U_{i-1}^j)^2]. \tag{3.2.6}$$

The "Lax-Friedrichs method" is a forward in time, centered in space finite difference scheme, which is named after the two mathematicians Peter David Lax and Kurt Otto Friedrichs:

$$U_i^{j+1} = \frac{1}{2}(U_{i-1}^j + U_{i+1}^j) - \frac{k}{2h}[f(U_{i+1}^j) - f(U_{i-1}^j)]. \tag{3.2.7}$$

This is the form of (3.2.2) for $p = 0, q = 1, F(U, V) = \frac{h}{2k}(U - V) + \frac{1}{2}(f(U) + f(V))$, see [Lan11, p.13].
One can show that the method is consistent, see [Cam, p.12].
For (2.2.3) we have

$$U_i^{j+1} = \frac{1}{2}(U_{i-1}^j + U_{i+1}^j) - \frac{k}{2h}[\frac{1}{2}(U_{i+1}^j)^2 - \frac{1}{2}(U_{i-1}^j)^2]. \tag{3.2.8}$$

All methods from above are of first order. Now we consider the "Lax-Wendroff method" which is of second order. It takes the form

$$\begin{aligned} U_i^{j+1} =& U_i^j - \frac{k}{2h}(f(U_{i+1}^j) - f(U_{i-1}^j)) + \\ & \frac{k^2}{2h^2}[A_{i+\frac{1}{2}}(f(U_{i+1}^j) - f(U_i^j)) - A_{i-\frac{1}{2}}(f(U_i^j) - f(U_{i-1}^j))] \end{aligned} \tag{3.2.9}$$

where $A_{i\pm\frac{1}{2}}$ denotes the Jacobian matrix $A(u) = f'(u)$ evaluated at $\frac{1}{2}(U_i^j + U_{i\pm1})$. Since we have to evaluate the Jacobian matrix it is more expansive, see [LeV92, p.127].
For (2.2.3) we have $f'(u) = u$, so

$$\begin{aligned} U_i^{j+1} = U_i^j - & \frac{k}{2h}(\frac{1}{2}(U_{i+1}^j)^2 - \frac{1}{2}(U_{i-1}^j)^2) + \\ & \frac{k^2}{2h^2}[(\frac{1}{2}(U_i^j + U_{i+1}^j))(\frac{1}{2}(U_{i+1}^j)^2 - \frac{1}{2}(U_i^j)^2) - \\ & (\frac{1}{2}(U_i^j + U_{i-1}^j))(\frac{1}{2}(U_i^j)^2 - \frac{1}{2}(U_{i-1}^j)^2)]. \end{aligned} \tag{3.2.10}$$

We can also rewrite the Lax-Wendroff method in conservative form, see [LeV92, p.127].

Remark 3.2.3. *Lax and Wendroff have proven that if a conservative and consistent method converges to some function $u(x,t)$, then this function is a weak solution of the conservation law, see [LeV92, p.129f.], [Cam, p.13]. For convergence, we do not only need consistency, but also some form of stability. But even if the method converges to a weak solution, it is non-unique, so we would need a discrete analog of the entropy condition, see [Cam, p.13-14], which we will not further discuss in this thesis.*

There are many more finite difference and other numerical methods for inviscid Burgers' equation. We want to have a look at a few methods for viscid Burgers' equation now.

3.3 FDMs for the Viscid Burgers' Equation

There exist a variety of numerical techniques based on finite-difference, finite-element and boundary element methods in attempting to solve viscid Burgers' equation particularly for small values of the viscosity ϵ, see [KBÖ98, p.252]. In this thesis we will only introduce some finite difference methods.

3.3.1 An Explicit Finite Difference Method

Let us consider (2.1.1) with $a < x < b$, $t > 0$, with initial condition $u(x, 0) = u_0(x)$ and boundary conditions $u(a,t) = D_1(t)$, $u(b,t) = D_2(t)$.
For example, let us take $u(x, 0) = \sin(\pi x)$ as initial condition with $0 < x < 1$ and

as homogeneous boundary conditions let us take $u(0,t) = u(1,t) = 0,\ t > 0$.
Via the Hopf-Cole transformation (2.1.2)

$$u(x,t) = -2\epsilon \frac{v_x}{v}$$

the viscid Burgers' equation transforms into the linear heat equation

$$v_t = \epsilon v_{xx},$$

with initial condition

$$v(x,0) = e^{-\frac{1}{2\epsilon}\int\limits_0^x u_0(y)\ dy} = e^{-\frac{1}{2\epsilon}\frac{1-\cos(\pi x)}{\pi}}$$

for $0 < x < 1$ and the boundary conditions

$$v_x(0,t) = v_x(1,t) = 0.$$

We discretize the solution domain $\{(x,t) : x \in (0,1), t \in [0,\infty)\}$ into cells described by the node set (x_i, t_j) in which $x_i = ih$, $t^j = jk$ and $i = 0 : 1 : N; j = 0 : 1 : M, Nh = 1, Mk = t_f$. The spatial mesh size is $h \equiv \Delta x$, the time step is $k \equiv \Delta t$ and t_f is the final time, see [KBÖ98, p.253].

An explicit finite difference approximation is given by, see [KBÖ98, p.253]

$$v_i^{j+1} = (1 - 2\epsilon\frac{k}{h^2})v_i^j + 2\epsilon\frac{k}{h^2}v_{i+1}^j,\ i = 0, \tag{3.3.1}$$

$$v_i^{j+1} = \epsilon\frac{k}{h^2}v_{i-1}^j + (1 - 2\epsilon\frac{k}{h^2})v_i^j + \epsilon\frac{k}{h^2}v_{i+1}^j,\ i = 1 : 1 : N-1, \tag{3.3.2}$$

$$v_i^{j+1} = 2\epsilon\frac{k}{h^2}v_{i-1}^j + (1 - 2\epsilon\frac{k}{h^2})v_i^j,\ i = N, \tag{3.3.3}$$

for $j = 0 : 1 : M$. For stability one can use Von Neumann's approach, see [Smi87, p.80 ff.] with $k \leq \frac{h^2}{2\epsilon}$. With the Hopf-Cole transformation we get the following explicit finite difference solution for the problem above

$$u(x_i, t^j) = -\frac{\epsilon}{h}\left(\frac{v_{i+1}^j - v_{i-1}^j}{v_i^j}\right), i = 1 : 1 : N-1, j = 0 : 1 : M.$$

3.3.2 Douglas Finite Difference Method

Let us consider (2.1.1) with $(x,t) \in (0,1) \times (0,T]$, with initial condition $u(x,0) = u_0(x)$ and boundary conditions $u(0,t) = D_1(t)$, $u(1,t) = D_2(t)$, where u_0, D_1 and D_2 are sufficiently smooth functions.

Via the Hopf-Cole transformation (2.1.2)

$$u(x,t) = -2\epsilon \frac{v_x}{v}$$

the viscid Burgers' equation transforms into the linear heat equation

$$v_t = \epsilon v_{xx},$$

with initial condition

$$v(x,0) = e^{-\frac{1}{2\epsilon}\int_0^x u_0(y)\ dy}$$

for $x \in (0,1)$ and the boundary conditions

$$v_x(0,t) = v_x(1,t) = 0,$$

for $t \in (0,T]$.

We discretize the solution domain into a uniform mesh dividing [0,1] in N sub-intervals and [0,T] into M sub-intervals, described by the node set (x_i, t^j) in which $x_i = ih$, for $i = 1 : 1 : N$ and $t^j = jk$, for $j = 0 : 1 : M$, where the spatial mesh size is $h = 1/N$ and the time step is $k = T/M$, see [PVV09, p.2207].

The Douglas finite difference approximation is given by, see [PVV09, p.2207-2208]

$$\begin{aligned}&(1-6r)v_{i-1}^{j+1} + (10+12r)v_i^{j+1} + (1-6r)v_{i+1}^{j+1} = (1+6r)v_{i-1}^j + \\ &(10-12r)v_i^j + (1+6r)v_{i+1}^j,\ i = 0 : 1 : N, \qquad (3.3.4)\\ &v_{i-1}^j = v_{i+1}^j,\ i = 0, N \qquad (3.3.5)\end{aligned}$$

where $r = \epsilon\frac{k}{h^2}$ and v_i^j is the discrete approximation to $v(x_i, t^j)$. The approximate solution of Burgers' equation is then given by

$$u_i^j(x,t) = -\epsilon\frac{v_{i+1}^j - v_{i-1}^j}{hv_i^j}.$$

It is shown in [PVV09, p.2208] that the method is unconditionally stable and has consistency of $\mathcal{O}(h^4) + \mathcal{O}(k^2)$. There is no restriction on the time step.

3.3.3 An Implicit Exponential Finite Difference Method

Let us consider (2.1.1) with $a < x < b$, $t > 0$, with initial condition $u(x,0) = u_0(x)$ and boundary conditions $u(a,t) = D_1(t)$, $u(b,t) = D_2(t)$.
We discretize the solution domain $\{(x,t) : x \in (a,b), t \in [0,\infty)\}$ into cells described by the node set (x_i, t_j) in which $x_i = ih$, $t^j = jk$ and $i = 0 : 1 : N; j = 0 : 1 : M$. The spatial mesh size is $h \equiv \Delta x$ and the time step is $k \equiv \Delta t$, see [IB13, p.548].

We rearrange (2.1.1) to obtain

$$\frac{\partial u}{\partial t} = \epsilon\frac{d^2u}{dx^2} - u\frac{\partial u}{\partial x}.$$

Dividing by u, see [AI21, p.85], gives

$$\frac{\partial \ln u}{\partial t} = \frac{1}{u}(\epsilon\frac{d^2u}{dx^2} - u\frac{\partial u}{\partial x}).$$

An implicit exponential finite difference approximation is given by, see [IB13, p.548]

$$U_i^{j+1} = U_i^j exp\left(\frac{\epsilon\Delta t}{(\Delta x)^2}\left[-\frac{\Delta x U_i^j}{2\epsilon}\frac{(U_{i+1}^{j+1} - U_{i-1}^{j+1})}{U_i^j} + \frac{(U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1})}{U_i^j}\right]\right), \qquad (3.3.6)$$

which is valid for $1 \leq i \leq N-1$. U_i^j is the exponential finite difference approximation to the exact solution $u(x,t)$ of Burgers' equation. Equation (3.3.6) is a system of nonlinear equations. We can solve it by Newton's method.

Chapter 4

Conclusion and Outlook

In the thesis at hand, we have been introducing, analyzing and approximating Burgers' equation and its solutions, which helped us in understanding the topics conservation laws, hyperbolic and parabolic partial differential equations, viscosity, diffusion, advection, nonlinearity, shock formation and numerical approximation by finite difference methods.

By studying Burger's equation we have found out that there exists the Hopf-Cole transformation for converting the viscid version of Burgers' equation in infinite space and some finite spaces into the linear heat equation, which can be explicitly solved then, for example by using Fourier transforms.
The inviscid version of Burgers' equation can be solved via the method of characteristics only for smooth initial values and small enough t, such that the characteristics do not intersect. We can compute the breaking time, where the characteristics firstly intersect. For larger t, after the characteristics have intersected and a shock has formed, the partial differential equation has no classical solution and we have introduced the concept of weak solutions, to allow such discontinuities. But if we include discontinuous solutions, we cannot guarantee the uniqueness of the solution. The uniqueness can be restored by using physical criteria, such as the Rankine-Hugoniot jump condition and the entropy condition. We have looked at the Riemann problem as an example, to test what we have found out by our analysis and we have also learned about the vanishing viscosity approach as an alternative way in constructing the discontinuous entropy solution.
Then the question presented how to approximate solutions of Burgers' equation and the focus of this thesis shifted to numerical methods, particularly finite difference methods. Since we only introduced some finite difference methods in this thesis, it would be interesting to compare them between each other in terms of stability, accuracy and effort and actually writing some code for different examples in further work.
One could also have a look at other numerical methods, for example, finite element methods, finite volume methods et cetera to approximate solutions of Burgers' equation. It would also be interesting to extend the research on an alternative for the Hopf-Cole transformation, to fully cover every possible case of boundary conditions and moreover, study the Navier-Stokes equations as an advanced extension.

In summary, one can say that Burgers' equation is indeed a very interesting example of a partial differential equation, which serves as an easier model for more complex problems.

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