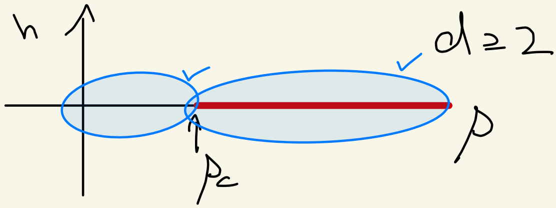



We prove i) in the phase-diagram theorem $h \neq 0$; $d \geq 2$. In fact we will study

$$h \in \mathbb{H}^+ = \{ z \in \mathbb{C} : \mathrm{Re}(z) > 0 \} \text{ and}$$
$$\mathbb{H}^- = \{ z \in \mathbb{C} : \mathrm{Re}(z) < 0 \}$$

**Thm** (Uniqueness in the asymmetric model)
Let $\beta > 0$: $h \longmapsto \psi(h)$ can be extended from $\{ h \in \mathbb{R} : h > 0 \}$ to an analytic function on $\mathbb{H}^+$, respectively, $\{ h \in \mathbb{R} : h < 0 \}$ — " — $\mathbb{H}^-$.

$\psi$ can be computed using the thermodynamic limit with free boundary condition.

Hence $\psi$ has complex derivative on $\mathbb{H}^+$, $\mathbb{H}^-$, and also a real derivative on $\{h > 0\}$ and $\{h < 0\}$ and hence we have uniqueness of Gibbs states $\forall h \neq 0$.

We work with $\psi^{\phi}_{\Lambda}(h) = \frac{1}{|\Lambda|} \log Z^{\phi}_{\Lambda;\beta,h}$ for convenience. In thermodynamic limit $\Lambda \Uparrow \mathbb{Z}^d$ $\psi(h)$ is independent of boundary on $\mathbb{R}$ and hence also $\psi(h)$ is — " — on $\mathbb{C}$.

1) $Z^{\phi}_{\Lambda;\beta,h}$ is real analytic as a finite sum of
   — " — functions $e^{\pm h}$

2) Since $Z^{\phi}_{\Lambda;\beta,h} > 0$ $\forall h \in \mathbb{R}$ also $\psi_\Lambda^{\phi}$ is real analytic in $h$.

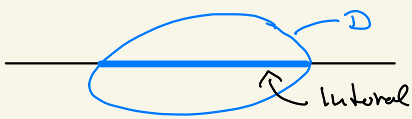
3) However in $\Lambda \Uparrow \mathbb{Z}^d$, $\psi(h)$ is not even differentiable for all $h$.

4) In $\mathbb{C}$ $Z^{\phi}_{\Lambda;\beta,h}$ might vanish $\Rightarrow$ $\psi_\Lambda(h)$?

**Thm (Lee - Yang)**
Let $\beta \geq 0$: Let $D \subset \mathbb{C}$ open, simply connected and such that $D \cap \mathbb{R}$ is an interval of $\mathbb{R}$. Assume that

$$\forall \Lambda \Subset \mathbb{Z}^d \qquad Z^{\phi}_{\Lambda;\beta,} \neq 0 \quad \forall h \in D.$$

Then, $h \mapsto \psi(h)$ admits an analytic continuation to $D$.




Proof: Let $\Lambda_n \Uparrow \mathbb{Z}^d$.

1) Since $Z^{\phi}_{\Lambda_n;\beta,h}$ has no zeros, there exist a „branch of the logarithm of $Z^{\phi}_{\Lambda_n;\beta,h}$ on $D$"

$h \longmapsto \log Z^{\phi}_{\Lambda_n;\beta,h}$ equal $D \cap \mathbb{R}$ to the previously studied object.

[ the branch of the logarithm is not unique : $g(z) + 2ik\pi \quad \forall k \in \mathbb{Z}$ ]

In particular

$$g_n(h) = e^{\frac{1}{|\Lambda_n|} \log Z^{\phi}_{\Lambda_n;\beta,h}}$$

1) $g_n(h) = e^{\psi^{\phi}_{\Lambda_n}(h)}$ on $D \cap \mathbb{R}$

$$\xrightarrow[n\uparrow\infty]{} g(h) = e^{\psi(h)}$$

pressure

2) $(g_n)_{n\in\mathbb{N}}$ is locally uniformly bounded

[ $\forall z \in D \ \exists M > 0 \ \exists U(z)$

$|f(x)| \leq M \quad \forall x \in U(z)$

$\forall f \in \mathcal{A}$ ]

$$\left| Z^{\phi}_{\Lambda_n;\beta,h} \right| \leq \sum_{\omega \in \Omega_{\Lambda_n}} e^{-H^{\phi}_{\Lambda_n,\beta;\mathrm{Re}(h)}(\omega)}$$

$$\leq e^{2d\beta|\Lambda_n| + |\mathrm{Re}(h)||\Lambda_n| + \log 2 |\Lambda_n|}$$

$$\Rightarrow \quad |g_n(h)| \leq \exp\left( 2d\beta + |\mathrm{Re}(h)| + \log 2 \right) \quad \forall h \in D$$

Vitali's Convergence Theorem (convergence from set with cluster points to convergence on whole domain)

$\Rightarrow \exists$ analytic $g$ such that $g_n \longrightarrow g$ locally uniformly.

By Hurwitz Theorem

$g$ has no zeros on $D$ since $g_n(h) \neq 0$ and $g \not\equiv 0$.

Finally, since $g \neq 0$ on $D$

$\exists$ analytic branch of the logarithm on $D$

$\log g$ which is real for real variables,

hence $\log g = \varphi$ on $D \cap \mathbb{R}$.

$\square$

**Thm** (Lee-Yang Circle Theorem)

For $\mathbb{D} = \mathbb{H}^+$ or $\mathbb{D} = \mathbb{H}^-$

$$Z^{\phi}_{\Lambda,\beta,h} \neq 0 \quad \forall h \in \mathbb{D} \ \forall \Lambda \subset\subset \mathbb{Z}^d .$$

Notation $\quad z = e^{-2h}$ ; $\quad h \in \mathbb{H}^+ \iff z \in \mathcal{U} = \{ z \in \mathbb{C} : |z| < 1 \}$

It suffices to show that zeros of

$$z \mapsto Z^{\phi}_{\Lambda_n;\beta,z} \quad \text{lie on unit circle.}$$

Proof: 1) for $\beta = 0$ $\quad Z^{\phi}_{\Lambda_n;0,z} = \left( z^{-\frac{1}{2}} + z^{\frac{1}{2}} \right)^{|\Lambda_n|} \neq 0 \ \forall z \in \mathbb{C}$

2) Let $\beta > 0$. We consider subgraphs of $\mathbb{Z}^d$, $(V,E)$ via edges $E \subset \mathcal{E}$ with $V = \{ i \in \mathbb{Z}^d : \exists j \in \mathbb{Z}^d$ such that $\{ij\} \in E \}$

(no isolated points)

Then
$$Z^{\phi}_{V;\beta,h} = \sum_{\omega \in \Omega_V} \prod_{\{ij\} \in E} e^{\beta \omega_i \omega_j} \prod_{i \in V} e^{h \omega_i}$$

$$= e^{\beta|E| + h|V|} \underbrace{\sum_{\omega \in \Omega_V} \prod_{\{ij\}} e^{\beta(\omega_i \omega_j - 1)} \prod_{i \in V} e^{h(\omega_i - 1)}}_{= \sum_{X \subset V} a_E(X) z^{|X|} = \mathcal{P}_E(z)} .$$

Any $\omega \in \Omega_V$ is one-to-one with $X(\omega) = \{ i \in V : \omega_i = -1 \}$

$$a_E(\emptyset) = a_E(V) = 1; \quad a_E(X) = \prod_{\substack{\{ij\}\in E: \\ i\in X;\ j\in V\setminus X}} e^{-2\beta}$$

It suffices to show $\mathcal{Z}_E(z) \neq 0 \quad \forall z \in \mathcal{U} \quad (h \in \mathbb{H}^+)$

Consider $(z_i)_{i\in V}$ instead of $z$ and

$$\hat{\mathcal{P}}_E(z_V) = \sum_{X\subset V} a_E(X) \prod_{i\in X} z_i$$

[ $\hat{\mathcal{P}}_E(z_V) = \mathcal{P}(z)$, $z_i \equiv z$ ]

We show if $|z_i| < 1 \ \forall i \in V$

(*) $\Rightarrow \hat{\mathcal{P}}_E(z_V) \neq 0$ via induction over the number of edges in $E$

$\boxed{\text{IB}}$ $|E| = 1$ : $\hat{\mathcal{P}}_E(z_{ij}) = z_i z_j + e^{-2\beta}(z_i + z_j) + 1$

$w_i = w_j = -1$

$$\Rightarrow \quad 0 = \hat{\mathcal{P}}_E(z_{ij}) \iff z_i = -\underbrace{\frac{e^{-2\beta} z_j + 1}{z_j + e^{-2\beta}}}_{\varphi(z_j)}$$

Möbius transform

[ Intermezzo: if $|z| < 1 \Rightarrow |\varphi(z)| > 1$ [ $|z| = 1 \Rightarrow |\varphi(z)| = 1$ ]

$$|\varphi(z)| > 1 \iff 0 < |az + 1|^2 - |z + a|^2$$

$0 < e^{-2\beta} < 1$

$$= \left((az_1 + 1)^2 + (az_2)^2\right) - \left((z_1 + a)^2 + z_2^2\right)$$

$$= (az_1)^2 + \cancel{2az_1} + 1 + (az_2)^2 - z_1^2 - \cancel{2z_1 a} - a^2 - z_2^2$$

$$= (a^2 - 1)(z_1^2 + z_2^2) + (1 - a^2)$$

$$= (1 - a^2)(1 - |z|^2)$$

$>0$ $\quad >0$ if $|z|<1$

In particular if $|z_i|<1$ ; $|z_j|<1 \Rightarrow \hat{\mathcal{P}}_E(z_{ij}) \neq 0$

$\boxed{\text{IS}}$ Assume that $(*)$ holds for $E$ ;
and let $b=\{ij\}\in \mathcal{E}\setminus E$. Three cases

$\boxed{\text{Case 1}}$ $V\cap\{ij\}=\emptyset$ :

$$\hat{\mathcal{P}}_{E\cup\{ij\}}(z_{V\cup\{ij\}}) = \hat{\mathcal{P}}_E(z_V)\,\hat{\mathcal{P}}_{\{ij\}}(z_{ij}) \Rightarrow (*)$$

$\boxed{\text{Case 2}}$ $V\cap\{ij\}=\{i\}$ : Via Asano-Contraction

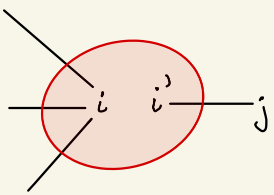
Consider $\{i',j\}$ such that $V\cap\{i',j\}=\emptyset$, then

$$\underbrace{\hat{\mathcal{P}}_{E\cup\{i',j\}}(z_{V\cup\{i',j\}})}_{\text{no roots in } \mathcal{U}} = \hat{\mathcal{P}}^{--} z_i z_{i'} + \hat{\mathcal{P}}^{+-} z_{i'} + \hat{\mathcal{P}}^{-+} z_i + \hat{\mathcal{P}}^{++}$$

Polynomials in $z_i$ and $z_k$, $k\in V\setminus\{i\}$

Asano contraction

$$\hat{\mathcal{P}}^{--} z_i + \hat{\mathcal{P}}^{++}$$




set $z_i = z_{i'} = z$

$$\Rightarrow \quad z_{0,1} = \frac{-\hat{\mathcal{P}}^{+-} \pm \sqrt{(\hat{\mathcal{P}}^{+-})^2 - 4\hat{\mathcal{P}}^{--}\hat{\mathcal{P}}^{++}}}{2\,\hat{\mathcal{P}}^{++}}$$

$$\Rightarrow |z_0 z_1| = \left|\frac{\hat{\mathcal{P}}^{++}}{\hat{\mathcal{P}}^{--}}\right| \geq 1 \quad \Rightarrow \quad |\hat{\mathcal{P}}^{++}| \geq |\hat{\mathcal{P}}^{--}|$$

$\Rightarrow \quad \hat{p}^{--} z + \hat{p}^{++}$ also has no roots in $\mathcal{U}$.

It remains to prove that

$$\hat{p}_{E \cup \{ij\}}(z_{V \cup \{j\}}) = \hat{p}^{--} z_i + \hat{p}^{++}$$

For this $\tilde{V} = (V \setminus \{i\}) \cup \{j\}$ and $\sigma_1 \sigma_2 \in \{-,+\}$

the $\hat{p}^{\sigma_1\sigma_2} = \sum_{X \subset \tilde{V}} a^{\sigma_1\sigma_2}_{E \cup \{i',j\}}(X) \prod_{k \in X} z_k$

$$a^{--}_{E \cup \{i',j\}}(X) = \left( \mathbb{1}\{X \ni j\} + \mathbb{1}\{X \not\ni j\} e^{-2\beta} \right) a_E(X \cup \{i\})$$

$$a^{+-}_{E \cup \{i',j\}}(X) = \left( \quad \text{—''—} \quad \right) a_E(X)$$

$$a^{-+}_{E \cup \{i',j\}}(X) = \left( \mathbb{1}\{X \not\ni j\} + \mathbb{1}\{X \ni j\} e^{-2\beta} \right) a_E(X \cup \{i\})$$

$$a^{++}_{E \cup \{i',j\}}(X) = \left( \quad \text{—''—} \quad \right) a_E(X)$$

Similar for $\hat{p}_{E \cup \{ij\}}(z_{V \cup \{j\}}) = \hat{p}^{-} z_i + \hat{p}^{+}$ with $\sigma \in \{+,-\}$

$$\hat{p}^{\sigma} = \sum_{X \in \tilde{V}} a^{\sigma}_{E \cup \{ij\}}(X) \prod_{k \in X} z_k$$

$$a^{-}_{E \cup \{ij\}}(X) = \left( \mathbb{1}\{X \ni j\} + \mathbb{1}\{X \not\ni j\} e^{-2\beta} \right) a_E(X \cup \{i\})$$

$$a^{+}_{E \cup \{ij\}}(X) = \left( \mathbb{1}\{X \not\ni j\} + \mathbb{1}\{X \ni j\} e^{-2\beta} \right) a_E(X)$$

$\boxed{\text{Case 3}}$ similar $\square$

# The Phase-Diagram of the Ising Model