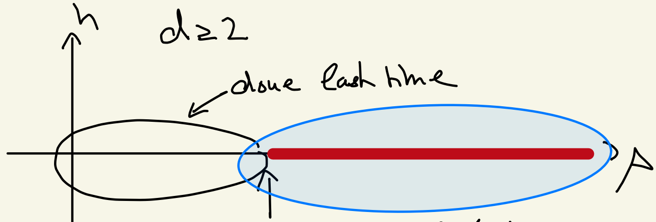
Recall



$$\beta_c(d) = \inf\{\beta > 0 : m^*(\beta) > 0\}$$

spontaneous magnetization $= \langle \sigma_0 \rangle^+_{\beta,0}$

We prove iii) b) in the phase-diagram theorem:

"spontaneous symmetry breaking" $h = 0$, $d \geq 2$

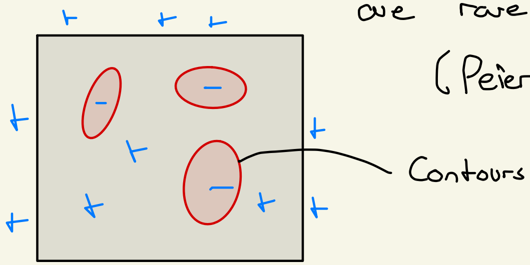
$$\iff \beta_c(d) < \infty$$

It suffices to find numbers $\delta(\beta)$ with $\delta(\beta) \downarrow 0$ as $\beta \uparrow \infty$ such that

$$\sup_\Lambda \mu^+_{\Lambda;\beta,0}(\sigma_0 = -1) \leq \delta(\beta)$$

$$\begin{aligned}
\mu^+_{\Lambda;\beta,0}(\sigma_0) &= \mu^+_{\Lambda;\beta,0}(\sigma_0 = +1) - \mu^+_{\Lambda;\beta,0}(\sigma_0 = -1) \\
&= 1 - 2\mu^+_{\Lambda;\beta,0}(\sigma_0 = -1) \\
&\geq 1 - 2\delta(\beta) \overset{!}{>} 0 \quad (\Rightarrow \text{non-uniqueness})
\end{aligned}$$

Key idea: Contours = Borderlines between + and − spins are rare when β is large (Peierls' argument)



$d=2$:

$$H_{\Lambda;\beta,0}(\omega) = -\beta \sum_{\{i,j\}\in\mathcal{E}^b_\Lambda} \omega_i\omega_j = -\beta|\mathcal{E}^b_\Lambda| + \sum_{\{i,j\}\in\mathcal{E}^b_\Lambda} \beta(1-\omega_i\omega_j)$$

$$\omega \in \Omega^+_\Lambda$$

cancel with normalization

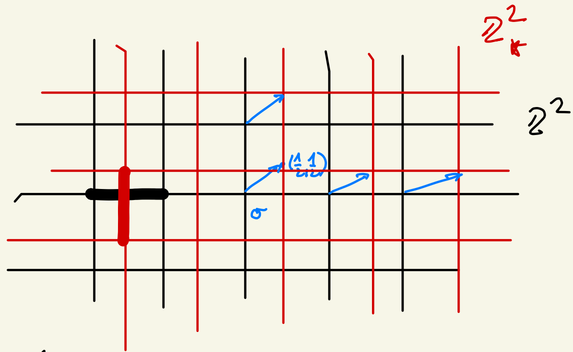
$$= 2\beta\,\#\{\{i,j\}\in\mathcal{E}^b_\Lambda : \omega_i\neq\omega_j\}$$

edges between unequal spins.

Dual Lattice

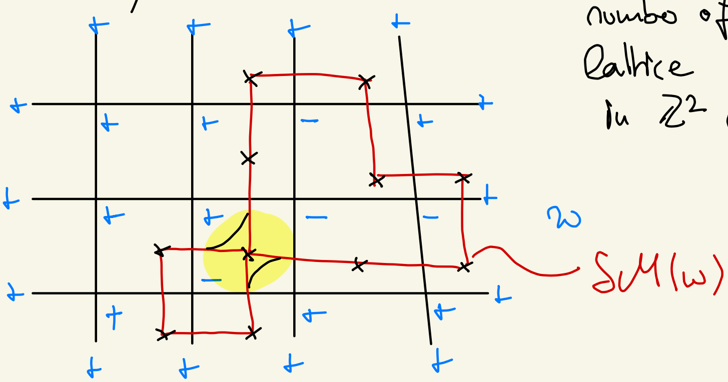
$$\mathbb{Z}^2_* = \mathbb{Z}^2 + \left(\tfrac{1}{2},\tfrac{1}{2}\right)$$

$$\begin{array}{ccc} e & \longleftrightarrow & e_* \\ \cap & & \cap \\ \mathcal{E} & & \mathcal{E}_* \end{array}$$

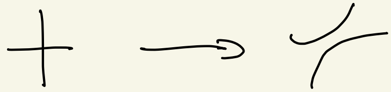


$$\omega\in\Omega^+_\Lambda \longmapsto \mathcal{M}(\omega) = \{e_*\in\mathcal{E}_* : \text{for } e(e_*)=\{i,j\}\in\mathcal{E},\ \omega_i\neq\omega_j\}$$

$$\Rightarrow H_{\Lambda;\beta,0}(\omega) = -\beta|\mathcal{E}^b_\Lambda| + 2\beta\underbrace{|\delta\mathcal{M}(\omega)|}_{\text{number of edges in dual lattice separating two sites in } \mathbb{Z}^2 \text{ with unequal spins,}}$$



$x\in\mathbb{Z}^2_*$ has $0,2,4$ adjacent edges in $\delta\mathcal{M}$



creates closed simple paths in $\delta\mathcal{M}(\omega)$

$$\delta\mathcal{M}(\omega) = \gamma_1 \cup \dots \cup \gamma_n$$

$\uparrow$ contours

$\Gamma(\omega) = \{\gamma_1 - \gamma_n\}$ set of contours in ω.

$|\gamma|$ = number of edges in contour

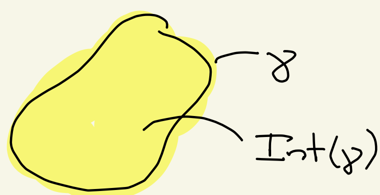
$$\Rightarrow \mathcal{H}_{\Lambda;\beta,0}(\omega) = -\beta|\mathcal{E}_\Lambda^b| + 2\beta \sum_{\gamma\in\Gamma(\omega)} |\gamma|$$

$$Z^+_{\Lambda;\beta,0} = e^{\beta|\mathcal{E}_\Lambda^b|} \sum_{\omega\in\Omega_\Lambda^+} \prod_{\gamma\in\Gamma(\omega)} e^{-2\beta|\gamma|}$$

$$\Rightarrow \mu^+_{\Lambda;\beta,0}(\omega)\underset{\Omega_\Lambda^+}{\cap} = \frac{\prod_{\gamma\in\Gamma(\omega)} e^{-2\beta|\gamma|}}{\sum_{\omega\in\Omega_\Lambda^+} \prod_{\gamma\in\Gamma(\omega)} e^{-2\beta|\gamma|}}$$

Note that any $\omega \in \Omega^+_{B_n}$ with $\omega_0 = -1$ must have an odd number of contours surrounding 0.

Any γ has a finite interior $\mathrm{Int}(\gamma)$ and an exterior



$$\Rightarrow \mu^{+}_{B_n;\beta;0}(\sigma_0 = -1) \leq \mu^{+}_{B_n;\beta;0}(\exists\, \gamma \in \Gamma : \underline{\mathrm{Int}}(\gamma) \ni \sigma)$$

$$\leq \sum_{\gamma :\, \mathrm{Int}(\gamma) \ni \sigma} \mu^{+}_{B_n;\beta;0}(\Gamma \ni \gamma)$$

Lemma (Bounds on Ising contours)

$\forall \beta > 0 \quad \forall$ contours γ:

$$\mu^{+}_{B_n;\beta,0}(\Gamma \ni \gamma) \leq e^{-2\beta|\gamma|}$$

<u>Proof:</u> $$\mu^{+}_{B_n;\beta,0}(\Gamma \ni \gamma) = \sum_{\omega : \Gamma(\omega) \ni \gamma} \mu^{+}_{B_n;\beta,0}(\omega)$$

$$= e^{-2\beta|\gamma|} \frac{\sum_{\omega : \Gamma(\omega) \ni \gamma} \prod_{\gamma' \in \Gamma(\omega) \setminus \gamma} e^{-2\beta|\gamma'|}}{\underbrace{\sum_{\omega \in \Omega^{+}_{\Lambda}} \prod_{\gamma' \in \Gamma(\omega)} e^{-2\beta|\gamma'|}}_{\circledast}}$$

<u>Trick:</u> for $\omega : \Gamma(\omega) \ni \gamma$ define

ω_γ such that $(\omega_\gamma)_i = \begin{cases} -\omega_i & \text{for } i \in \mathrm{Int}(\gamma) \\ \omega_i & \text{else} \end{cases}$

we flip spins in the interior of γ and hence removing the contour γ in ω_γ while keeping all other contours of ω in ω_γ.

Hence $$\sum_{\omega \in \Gamma(\omega) \ni \gamma} \prod_{\gamma' \in \Gamma(\omega) \setminus \gamma} e^{-2\beta|\gamma'|}$$

$$= \sum_{\omega_\gamma} \prod_{\gamma' \in \Gamma(\omega)} e^{-2\beta|\gamma'|} \implies (\circledast) \leq 1 \qquad \square$$

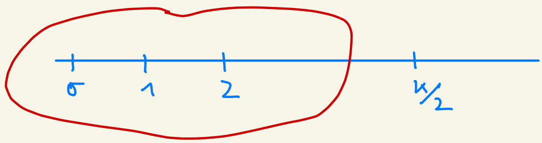
Hence

$$\mu^{+}_{B_n;\beta,0}(\sigma_0=-1) \overset{\text{Lemma}}{\le} \sum_{\gamma:\ \mathrm{Int}(\gamma)\ni\sigma} e^{-2\beta|\gamma|}$$

$$= \sum_{k\ge 4}\ \sum_{\gamma:\ \mathrm{Int}(\gamma)\ni\sigma:\ |\gamma|=k} e^{-2\beta|\gamma|}$$

$$= \sum_{k\ge 4} e^{-2\beta k}\,\#\{\gamma:\ \mathrm{Int}(\gamma)\ni\sigma;\ |\gamma|=k\}$$

1)



2) # contours of length k starting at σ

$$\le 4\cdot 3^{k-1}$$

↑ first step ← subsequent steps

$$\le \sum_{k\ge 4} e^{-2\beta k}\,\frac{k}{2}\,4\cdot 3^{k-1}$$

$$\le \frac{2}{3}\sum_{k\ge 4} k\left(3e^{-2\beta}\right)^k =: \delta(\beta)\downarrow 0 \text{ as } \beta\uparrow\infty$$

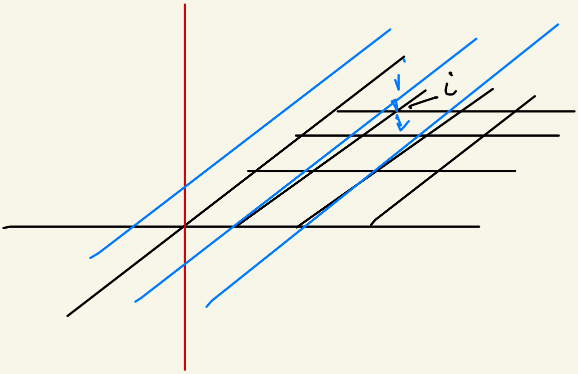
$\Rightarrow$ $\beta_c(2) < \infty$ via Peierls' argument

$d\ge 3$ (We do $d=3$ via GKS inequality)

$$B_n^3=\{-n,\dots,n\}^3;\quad B_n^2=\{-n,\dots,n\}^2$$

Claim

$$\mu^+_{B_n^3;\beta,0}(\sigma_0) \ge \mu^+_{B_n^2;\beta,0}(\sigma_0)$$



$i \in \{ j \in \mathbb{Z}^3 : j_0 = 0 \} = \mathbb{Z}^2$

j = neighbor of i in the layer above or below i

$|ij| = 1$

$$\frac{\delta}{\delta J_{ij}} \mu^+_{B_n^3;\beta,0}(\sigma_0) = \mu^+_{B_n^3;\beta,0}(\sigma_0\sigma_i\sigma_j) - \mu^+_{B_n^3;\beta,0}(\sigma_0)\,\mu^+_{B_n^3;\beta,0}(\sigma_i\sigma_j) \ge 0$$

↑ generalization edge-dependent Hamiltonian $\beta = J_{ij}$ ↑ GKS

$\Rightarrow$ Increasing expectation in J_{ij}

as $J_{ij} \downarrow 0$ for all edge between $\mathbb{Z}^2$ and $\mathbb{Z}^2 + \{0,0,1\}$, $\mathbb{Z}^2 - \{0,0,1\}$

(decoupling of $\mathbb{Z}^2$ in $\mathbb{Z}^3$)

$$\mu^+_{B_n^3;\beta,0}(\sigma_0) \ge \mu^+_{B_n^2;\beta,0}(\sigma_0)$$

$$\Rightarrow \quad \beta_c(3) \le \beta_c(2) < \infty$$