(11) Phase diagram

1) $\langle \cdot \rangle^{+}_{\beta,h}$; $\langle \cdot \rangle^{-}_{\beta,h}$; Uniqueness of Gibbs states ?

2) Non-uniqueness $\longrightarrow$ multiple (thermodynamic) equilibria of the system.

Def (Phase transition)
If there exist two distinct Gibbs states for (β, h), we say that there is a first-order phase transition at (β, h).

Recall the alternative (equivalent) definition of a first-order phase transition at (β, h) as a point of non-differentiability of the pressure ψ.

Thm (Phase diagram)

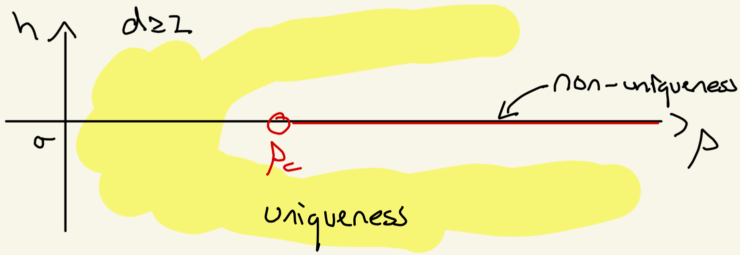
i) $\forall d \geq 1$, $h \neq 0$; $\beta \geq 0$ $\quad \exists!$ Gibbs state

ii) $d = 1$; $\forall h \in \mathbb{R}$; $\beta \geq 0$ $\quad \exists!$ Gibbs state

iii) $\forall d \geq 2$; $h = 0$; $\exists\, \beta_c = \beta_c(h) \in (0, \infty)$ such that

a) $\forall \beta < \beta_c$ $\quad \exists!$ Gibbs state

b) $\forall \beta > \beta_c$; $\langle \cdot \rangle^{+}_{\beta,0} \neq \langle \cdot \rangle^{-}_{\beta,0}$ and thus a first-order phase transition occurs.



Remarks: 1) In fact uniqueness holds at $(\beta_c, 0)$

2) In fact in the non-uniqueness region, there are infinitely many Gibbs states.

For the prove of ii) we relate the uniqueness of Gibbs state to the differentiability of the pressure.

Thm (Characterizations of uniqueness)

$\forall \beta \geq 0$ and $h \in \mathbb{R}$, then the following statements are equivalent:

a) $\exists!$ Gibbs state at (β, h)

b) $\langle \cdot \rangle^+_{\beta,h} = \langle \cdot \rangle^-_{\beta,h}$

c) $\langle \sigma_0 \rangle^+_{\beta,h} = \langle \sigma_0 \rangle^-_{\beta,h}$

Proof: 1) a) $\Rightarrow$ b) $\Rightarrow$ c) ✓

2) c) $\Rightarrow$ b) Note that $\sum_{i \in A} \eta_i - \eta_A$ is non-decreasing $\forall A \subset\subset \mathbb{Z}^d$ (← occupation variables)

$$\Rightarrow \mu^-_{\Lambda;\beta,h}\Big(\sum_{i\in A} \eta_i - \eta_A\Big) \overset{\text{FKG}}{\leq} \mu^+_{\Lambda;\beta,h}\Big(\sum_{i\in A} \eta_i - \eta_A\Big)$$

$$\overset{\Lambda \uparrow \mathbb{Z}^d}{\Rightarrow} \sum_{i\in A}\Big(\langle \eta_i \rangle^+_{\beta,h} - \langle \eta_i \rangle^-_{\beta,h}\Big) \geq \langle \eta_A \rangle^+_{\beta,h} - \langle \eta_A \rangle^-_{\beta,h} \overset{\text{FKG}}{\geq} 0$$

$$\overset{\text{c), transl. inv.}}{\Rightarrow} \langle \eta_A \rangle^+_{\beta,h} = \langle \eta_A \rangle^-_{\beta,h} \quad \forall A \subset\subset \mathbb{Z}^d \Rightarrow \text{b)}$$

3) b) $\Rightarrow$ a) For any Gibbs state $\langle \cdot \rangle_{\beta,h}$ by FKG

$$\langle n_A \rangle^-_{\beta,h} \le \langle n_A \rangle_{\beta,h} \le \langle n_A \rangle^+_{\beta,h} \quad \forall A \subset\subset \mathbb{Z}^d$$

$$\overset{b)}{\Longrightarrow} \quad \langle n_A \rangle^+_{\beta,h} = \langle n_A \rangle_{\beta,h} = \langle n_A \rangle^+_{\beta,h}$$

$$\Longrightarrow \quad \langle \cdot \rangle^+_{\beta,h} = \langle \cdot \rangle_{\beta,h} = \langle \cdot \rangle^+_{\beta,h}$$ □

Recall $m_\Lambda^\eta(\beta,h) = \mu^\eta_{\Lambda;\beta,h}(m_\Lambda)$ is expected magnetization density.

Prop (Magnetization $\Longleftrightarrow$ expected spin at origin)

1) $\forall \Lambda \Uparrow \mathbb{Z}^d$ $\quad m^+(\beta,h) = \lim\limits_{\Lambda \Uparrow \mathbb{Z}^d} m^+_\Lambda(\beta,h)$ and

$\quad m^-(\beta,h) = \lim\limits_{\Lambda \Uparrow \mathbb{Z}^d} m^-_\Lambda(\beta,h)$ exist

2) $m^+(\beta,h) = \langle \sigma_0 \rangle^+_{\beta,h}$; $m^-(\beta,h) = \langle \sigma_0 \rangle^-_{\beta,h}$

3) $h \mapsto m^+(\beta,h)$ is right-continuous and non-decreasing

$\quad h \mapsto m^-(\beta,h)$ is left-continuous and non-decreasing

4) $h \ge 0$: $\quad \beta \mapsto m^+(\beta,h)$ is non-decreasing

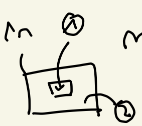
$\quad h \le 0$: $\quad \beta \mapsto m^-(\beta,h)$ is non-increasing

Proof: (Only for +)

1) & 2) : $\Lambda_n \Uparrow \mathbb{Z}^d$

$$\langle \sigma_0 \rangle^+_{\beta,h} = \langle m_{\Lambda_n} \rangle^+_{\beta,h} \overset{\text{FKG}}{\le} \liminf_{n \uparrow \infty} m^+_{\Lambda_n}(\beta,h)$$

(where $m_{\Lambda_n} = \frac{1}{|\Lambda_n|} \sum_{i \in \Lambda_n} \sigma_i$)



$$m^+_{\Lambda_n}(\beta,h) = \frac{1}{|\Lambda_n|} \sum_{\substack{i \in \Lambda_n: \\ i + B_k \subset \Lambda_n}} \mu^+_{\Lambda_n;\beta,h}(\sigma_i) + \frac{1}{|\Lambda_n|} \sum_{\substack{i \in \Lambda_n \\ i + B_k \not\subset \Lambda_n}} \mu^+_{\Lambda_n;\beta,h}(\sigma_i)$$

$$\overset{\text{FKG}}{\leq} \mu^{+}_{B_K;\beta,h}(\sigma_0) + 2\,\frac{|B_K|\,|\partial\Lambda_n|}{|\Lambda_n|}$$

$$\Rightarrow \limsup_{n\uparrow\infty} m^{+}_{\Lambda_n}(\beta,h) \leq \langle\sigma_0\rangle^{+}_{\beta,h}$$

$$\frac{\sum_\omega \sigma_0(\omega) e^{\beta\Sigma + h\Sigma}}{\sum_\omega e^{\beta\Sigma + h\Sigma}}$$

3) $$\frac{\delta}{\delta h}\mu^{+}_{\Lambda;\beta,h}(\sigma_0) = \sum_{i\in\Lambda}\Big[\underbrace{\mu^{+}_{\Lambda;\beta,h}(\sigma_0\sigma_i) - \mu^{+}_{\Lambda;\beta,h}(\sigma_0)\mu^{+}_{\Lambda;\beta,h}(\sigma_i)}_{\text{Co-Variance / GKS-inequality}}\Big] \geq 0$$

$$\Rightarrow h \mapsto \mu^{+}_{\Lambda;\beta,h}(\sigma_0) \text{ non-decreasing}$$

$$\overset{\text{monotonicity}}{\Rightarrow} h \mapsto \langle\sigma_0\rangle^{+}_{\beta,h} \text{ non-decreasing}$$

$$\mu^{+}_{\Lambda_n;\beta,h_m}(\sigma_0) = a_{n,m} \qquad \Lambda_n \Uparrow \mathbb{Z}^d \text{ and } h_m \downarrow h$$

$\uparrow$ bounded and non-increasing

$$\Rightarrow \lim_{m\uparrow\infty}\langle\sigma_0\rangle^{+}_{\beta,h_m} = \lim_{m\uparrow\infty}\lim_{n\uparrow\infty} a_{n,m} = \lim_{n\uparrow\infty}\lim_{m\uparrow\infty} a_{n,m}$$
$$= \lim_{n\uparrow\infty}\mu^{+}_{\Lambda_n;\beta,h}(\sigma_0) = \langle\sigma_0\rangle^{+}_{\beta,h}$$

4) $$\frac{\delta}{\delta\beta}\mu^{+}_{\Lambda;\beta,h}(\sigma_0) = \sum_{\{i,j\}\in\mathcal{E}^b_\Lambda}\Big(\underbrace{\mu^{+}_{\Lambda;\beta,h}(\sigma_0\sigma_i\sigma_j) - \mu^{+}_{\Lambda;\beta,h}(\sigma_0)\mu^{+}_{\Lambda;\beta,h}(\sigma_i\sigma_j)}_{\text{GKS inequality}}\Big) \geq 0$$

$$\Rightarrow \beta \mapsto \langle\sigma_0\rangle^{+}_{\beta,h} \text{ is non-decreasing}$$ □

Thm (Pressure vs. expected spin at origin)

$\forall \beta\geq 0;\ h\in\mathbb{R}$

$$\frac{\delta}{\delta h^{+}}\psi(\beta,h) = m^{+}(\beta,h) = \langle\sigma_0\rangle^{+}_{\beta,h}$$

$$\frac{\delta}{\delta h^{-}}\psi(\beta,h) = m^{-}(\beta,h) = \langle\sigma_0\rangle^{-}_{\beta,h}$$

Proof: $(h_k) \subset B_\beta^c \quad h_k \downarrow h$

$$\frac{\delta}{\delta h^+} \psi(\beta, h) = \lim_{h_k \downarrow h} m(\beta, h_k) = \lim_{h_k \downarrow h} m^+(\beta, h_k)$$
$$= m^+(\beta, h)$$

Similar for $+ \to -$. $\square$

Remarks: 1) This implies the equivalence

$$\frac{\delta}{\delta h} \psi(\beta, h) \text{ exists} \iff \exists! \text{ Gibbs state at } (\beta, h)$$

2) This proves ii) in phase diagram since in $d=1$

$$\frac{\delta}{\delta} \psi \text{ always exists.}$$

Remark: 1) $m^*(\beta) = \lim_{h_k \downarrow 0} m(\beta, h_k) = \lim_{h_k \downarrow 0} m^+(\beta, h_k) = m^*(\beta, 0)$

$= \langle \sigma_0 \rangle^+_{\beta,0}$

↑ Spontaneous magnetization; $(h_k) \subset B^c_\beta$

2) $\langle \sigma_0 \rangle^-_{\beta,0} = - \langle \sigma_0 \rangle^+_{\beta,0}$

↑ symmetrie $\Rightarrow$ at $h=0$

Uniqueness $\Leftrightarrow$ $m^*(\beta) = 0$

Def (Critical temperature)

$$\beta_c(d) = \inf\{\beta \geq 0 : m^*(\beta) > 0\}$$
$$= \sup\{\beta \geq 0 : m^*(\beta) = 0\}$$

Remark: 1) $\beta_c = \beta_c(d)$ is well defined since $\beta \mapsto \langle \sigma_0 \rangle^+_{\beta,0}$ is monotone

2) $\forall \beta < \beta_c$: $m^*(\beta) = 0$

$\forall \beta > \beta_c$: $m^*(\beta) > 0$

3) By FKG: $\langle \sigma_0 \sigma_i \rangle^+_{\beta,0} \geq \left(\langle \sigma_0 \rangle^+_{\beta,0}\right)^2 = m^*(\beta)^2 \; \forall i$

$\Rightarrow$ for $\beta > \beta_c(d)$ $\inf_{i \in \mathbb{Z}^d} \langle \sigma_0 \sigma_i \rangle^+_{\beta,0} > 0$

long-range order

however $\lim_{|i| \uparrow \infty} \left(\langle \sigma_0 \sigma_i \rangle^+_{\beta,0} - \langle \sigma_0 \rangle^+_{\beta,0} \langle \sigma_i \rangle^+_{\beta,0} \right) = 0$

asymptotic decorrelation

We prove iii) a); i.e., we prove $\beta_c(d) > 0$ $d \geq 2$ via high-temperature representation.

$$e^{\beta \sigma_i \sigma_j} = \cosh(\beta) + \sigma_i \sigma_j \sinh(\beta)$$
$$= \cosh(\beta)(1 + \tanh(\beta)\sigma_i\sigma_j)$$

$\cosh = \frac{1}{2}(e^x + e^{-x})$, $\sinh = \frac{1}{2}(e^x - e^{-x})$, $\sigma_i\sigma_j \in \{\pm 1\}$

$$e^{-H_{\Lambda;\beta,0}(\omega)} = \prod_{\{ij\}\in\mathcal{E}^b_\Lambda} e^{\beta\omega_i\omega_j}$$
$$= \cosh(\beta)^{|\mathcal{E}^b_\Lambda|} \prod_{\{ij\}\in\mathcal{E}^b_\Lambda} (1 + \tanh(\beta)\omega_i\omega_j)$$

$$Z^+_{\Lambda;\beta,0} = \sum_{\omega\in\Omega^+_\Lambda} e^{\beta\sum_{\{ij\}\in\mathcal{E}^b_\Lambda}\omega_i\omega_j}$$
$$= \sum_{\omega\in\Omega^+_\Lambda} \cosh(\beta)^{|\mathcal{E}^b_\Lambda|} \prod_{\{ij\}\in\mathcal{E}^b_\Lambda} (1+\tanh(\beta)\omega_i\omega_j)$$

$\underbrace{(1+a)(1+b)}_{1+a+b+ab}\cdots$

$$= \cosh(\beta)^{|\mathcal{E}^b_\Lambda|} \sum_{\omega\in\Omega^+_\Lambda} \sum_{E\subset\mathcal{E}^b_\Lambda} \prod_{\{ij\}\in E} \omega_i\omega_j \tanh(\beta)$$
$$= \cosh(\beta)^{|\mathcal{E}^b_\Lambda|} \sum_{E\subset\mathcal{E}^b_\Lambda} \tanh(\beta)^{|E|} \sum_{\omega\in\Omega^+_\Lambda} \underbrace{\prod_{\{ij\}\in E} \omega_i\omega_j}_{\prod_{i\in\Lambda}\omega_i^{I(i,E)}}$$

$$I(i,E) = \#\{ j\in\mathbb{Z}^d : \{i,j\}\in E\}$$

incidence number

here $\sum_{\omega_i = \pm 1} \omega_i^{I(i,E)} = \begin{cases} 2 & \text{if } I(i,E) \text{ is even} \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow \quad Z^+_{\Lambda;\beta,0} = 2^{|\Lambda|} \cosh(\beta)^{|\mathcal{E}^b_\Lambda|} \sum_{E \in \mathcal{E}^{+;\text{even}}_\Lambda} \tanh(\beta)^{|E|}$$

$\mathcal{E}^{+;\text{even}}_\Lambda = \{ E \subset \mathcal{E}^b_\Lambda : I(i,E) \text{ even } \forall i \in \Lambda \}$

$$\Rightarrow \quad \mu^+_{\Lambda;\beta,0}(\sigma_0) = \left(Z^+_{\Lambda;\beta,0}\right)^{-1} 2^{|\Lambda|} \cosh(\beta)^{|\mathcal{E}^b_\Lambda|} \sum_{E \in \mathcal{E}^{+;0}_\Lambda} \tanh(\beta)^{|E|}$$

$$= \frac{\sum_{E \in \mathcal{E}^{+;0}_\Lambda} \tanh(\beta)^{|E|}}{\sum_{E \in \mathcal{E}^{+;\text{even}}_\Lambda} \tanh(\beta)^{|E|}}$$

$\mathcal{E}^{+;0}_\Lambda = \{ E \subset \mathcal{E}^b_\Lambda : I(i,E) \text{ is even } \forall i \in \Lambda \setminus \{0\} \text{ but } I(0,E) \text{ odd} \}$

For $E \subset \mathcal{E}^b_\Lambda$ let $\Delta(E)$ denote the set of edges sharing no endpoint with an edge in E

$\forall E \in \mathcal{E}^{+;0}_\Lambda$ we have $E = E_0 \cup E'$ where

$E_0 \neq \emptyset$ the connected component of E containing 0

$E' \in \mathcal{E}^{+,\text{even}}_\Lambda$ satisfying $E' \in \Delta(E_0)$

$$\Rightarrow \quad \mu^{+}_{\Lambda,\beta,0}(\sigma_0) = \sum_{\substack{E_0 \in \mathcal{E}^{+,0}_{\Lambda}: \\ \text{connected}; E_0 \ni \sigma}} \tanh(\beta)^{|E_0|} \underbrace{\frac{\sum_{E \in \mathcal{E}^{+,\text{even}}_{\Lambda};\, E' \subset \Delta(E_0)} \tanh(\beta)^{|E'|}}{\sum_{E \in \mathcal{E}^{+,\text{even}}_{\Lambda}} \tanh(\beta)^{|E|}}}_{\leq 1}$$

Assume $\Lambda = B_n$ and note

that $E_0 \cap B_n^c \neq \emptyset$ since

$\sum_{i \in \mathbb{Z}^d} I(i, E_0) = 2|E_0|$ and $I(0, E_0)$ odd

hence $\exists\, i$ with $I(i, E_0)$ odd, but $i \in B_n^c$

$\Rightarrow |E_0| \geq n$

$$\Rightarrow \mu^{+}_{B_n,\beta,0}(\sigma_0) \leq \sum_{\ell \geq n} \underbrace{\tanh(\beta)}_{\leq \beta}{}^{\ell} \; \underbrace{\#\{E_0 \in \mathcal{E}^{+,0}_{B_n} : E_0 \text{ connected}, E_0 \ni \sigma, |E_0| = \ell\}}_{\substack{\leq \#\{\text{path starting on the origin of length } 2\ell\} \\ \leq (2d)^{2\ell}}}$$

$$\leq \sum_{\ell \geq n} (4d^2\beta)^{\ell} \leq e^{-cn}$$

$\beta < \frac{1}{4d^2}$ $\qquad c \doteq c(\beta, d) > 0$

$$\Rightarrow \langle \sigma_0 \rangle^{+}_{\beta,0} = 0$$

$\square$