9 Infinite-Volume Gibbs states

1) Pressure provides important macroscopic / thermodynamic properties of the Ising model in infinite volume, e.g. magnetization; or higher derivatives of ψ

2) We like to also evaluate the infinite-volume Ising model w.r.t. local observable; e.g. expected spin at origin.

Def (local observable)

A function $f: \Omega \to \mathbb{R}$ is local if $\exists \Lambda \subset\subset \mathbb{Z}^d$, such that $f(\omega) = f(\omega')$ $\forall \omega, \omega' \in \Omega$ s.t. $\omega_i = \omega'_i$ $\forall i \in \Lambda$.
The smallest such set Λ is called the support of f, $\mathrm{supp}(f)$.

Ex: $\omega \mapsto m_\Lambda(\omega)$ is local for $\Lambda \subset\subset \mathbb{Z}^d$

$\omega \mapsto \omega_i$ is local for $i \in \mathbb{Z}^d$

Def (Infinite-volume state)

A mapping $f \mapsto \langle f \rangle$ is a state if

1) $\langle 1 \rangle = 1$ normalized

2) $\langle f \rangle \geq 0$ $\forall f \geq 0$ positive

3) $\langle f + \lambda g \rangle = \langle f \rangle + \lambda \langle g \rangle$ $\forall g, f, \lambda \geq 0$ linear

$\langle f \rangle$ is called the average of f under the state $\langle \cdot \rangle$

Def (Gibbs state)

Let $\Lambda_n \Uparrow \mathbb{Z}^d$ and $(\eta_n)_{n\geq 1}$ a sequence of boundary conditions. The Gibbs measure $\left(\mu^{\eta_n}_{\Lambda_n,\beta,h}\right)_{n\geq 1}$ converges to the $\langle \cdot \rangle$ $:\Longleftrightarrow$

$$\lim_{n \uparrow \infty} \mu^{\eta_n}_{\Lambda_n,\beta,h}(f) = \langle f \rangle \quad \forall \text{ local } f.$$

$\langle \cdot \rangle$ is called a Gibbs state at (β, h).

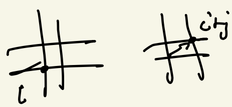
<u>Remark:</u> 1) The topology of local convergence label measures close if they are close tested against local observables

2) In fact $\forall \langle \cdot \rangle$, f $\exists \mu$ s.t. $\langle f \rangle = \mu(f)$
(Riesz theorem)

3) So convergence tested on local functions can be seen as weak-convergence of probability measures, where the continuity of the test-function is wrt the product topology

Important class of infinite-volume states are the translation-invariant states:

Let $\Theta_j : \mathbb{Z}^d \mapsto \mathbb{Z}^d$ $\quad i \mapsto \Theta_j i = i+j$



$\Theta_j : \Omega \mapsto \Omega$ $\quad \omega \mapsto \Theta_j \omega$ with $(\Theta_j \omega)_i = \omega_{i-j}$

Def (Translation-invariant states)

A state $\langle \cdot \rangle$ is called translation invariant if

$$\langle f \circ \Theta_j \rangle = \langle f \rangle \quad \forall \text{ local } f,\ j \in \mathbb{Z}^d$$

Do Gibbs states exist for the Ising model?

Thm (Extremal Ising states)

Let $\beta \geq 0$, $h \in \mathbb{R}$, then there exist translation-invariant states $\langle \cdot \rangle^{+}_{\beta,h}$ and $\langle \cdot \rangle^{-}_{\beta,h}$ such that

$$\lim_{n \uparrow \infty} \mu^{2^{\pm}}_{\Lambda_n;\beta,h} = \langle \cdot \rangle^{\pm}_{\beta,h}$$

independently from the converging sequence $\Lambda_n \Uparrow \mathbb{Z}^d$.

(10) Existence of extremal Ising models

Strategy: 1) Identify limits as monotone limits.
2) Use well-chosen class of local test functions.

$\forall A \subset\subset \mathbb{Z}^d$

$$\sigma_A := \prod_{j\in A} \underset{\in \{\pm 1\}}{\sigma_j} \quad ; \quad \eta_A := \prod_{j\in A} \underset{= \frac{1}{2}(1+\sigma_j) \in \{0,1\}}{\eta_j}$$

occupation variables

Lemma (σ_A and η_A basis for local functions)

$\forall f$ local $\exists (\hat{f}_A)_{A\subset \mathrm{supp}(f)}, (\tilde{f}_A)_{A\subset \mathrm{supp}(f)} \in \mathbb{R}$ such that

$$f = \sum_{A\subset \mathrm{supp}(f)} \hat{f}_A \sigma_A = \sum_{A\subset \mathrm{supp}(f)} \tilde{f}_A \eta_A .$$

Proof: 1) Orthogonality: $\forall B \subset\subset \mathbb{Z}^d$, $\omega, \tilde{\omega} \in \Omega$

$$2^{-|B|} \sum_{A\subset B} \sigma_A(\tilde{\omega})\sigma_A(\omega) = \mathbb{1}\{\omega_i = \tilde{\omega}_i, \ \forall i\in B\}$$

Indeed: 1) if $\omega_i = \tilde{\omega}_i$ $\forall i\in B$, then $\forall A\subset B$ $\sigma_A(\tilde{\omega})\sigma_A(\omega) = \prod_{i\in A} \underbrace{\omega_i\tilde{\omega}_i}_{=1} = 1$

2) $\exists i\in B$ $\omega_i \neq \tilde{\omega}_i$, then

$$\sum_{A\subset B} \sigma_A(\tilde{\omega})\sigma_A(\omega) = \sum_{A\subset B\setminus\{i\}} \left(\sigma_A(\tilde{\omega})\sigma_A(\tilde{\omega}) + \sigma_{A\cup\{i\}}(\tilde{\omega})\sigma_{A\cup\{i\}}(\omega) \right)$$

$$- \sigma_A(\bar{\omega})\,\sigma_A(\omega))$$

$$= 0$$

2) Let $B = \mathrm{supp}(f)$

$$f(\omega) = \sum_{\omega' \in \Omega_B} f(\omega')\, \underbrace{1\!\!1\{\omega'_i = \omega_i \ \forall i \in B\}}_{e^{-|B|} \sum_{A \subset B} \sigma_A(\omega)\sigma_A(\omega')}$$

$$= \sum_{A \subset B} \sigma_A(\omega) \Big(\underbrace{e^{-|B|} \sum_{\omega' \in \Omega_B} f(\omega')\sigma_A(\omega')}_{\hat{f}_A} \Big)$$

Similar for η_A $\square$

Using the Lemma, we can check convergence w.r.t. σ_A and η_A.

Key tool to derive monotone limits is via correlation inequalities

1) GKS inequality (Griffiths, Kelly, Sherman)
2) FKG inequality (Fortuin, Kasteleyn, Ginibre)

1) GKS inequalities: Idea: $\mu^{\eta^+}_{\Lambda;\beta,h}(\sigma_i) \geq 0$ $h \geq 0$ $i \in \Lambda$

+-boundary & non-negative external field should create bias for the expected spin at any site $i \in \Lambda$, $\forall \Lambda \subset\subset \mathbb{Z}^d$.

Even $\mu^{2^+}_{\Lambda;\beta,h}(\sigma_i=1;\sigma_j=1) \geq \mu^{2^+}_{\Lambda;\beta,h}(\sigma_i=1)\mu^{2^+}_{\Lambda;\beta,h}(\sigma_j=1)$

support each other — positive correllation!

Holds more generally:

$$\bar{J}=(J_{ij})_{\{i,j\}\in\mathcal{E}}\ ;\quad \bar{h}=(h_i)_{i\in\mathbb{Z}^d}$$

Let $H_{\Lambda;\bar{J},\bar{h}}(\omega) = -\sum_{\{i,j\}\in\mathcal{E}^b_\Lambda} J_{ij}\,\omega_i\omega_j - \sum_{i\in\Lambda} h_i\omega_i$

and let $\mu^{2}_{\Lambda,\bar{J},\bar{h}}$ the associated Ising model.

Thm (GKS)

Consider $\bar{J},\bar{h}$ and $\Lambda \subset\subset \mathbb{Z}^d$ with $h_i\geq 0\ \forall i$.

Then $\forall\, A,B\subset\Lambda$

$$\mu^{2^+}_{\Lambda;\bar{J},\bar{h}}(\sigma_A)\geq 0$$

$$\mu^{2^+}_{\Lambda;\bar{J},\bar{h}}(\sigma_A\sigma_B)\geq \mu^{2^+}_{\Lambda;\bar{J},\bar{h}}(\sigma_A)\,\mu^{2^+}_{\Lambda;\bar{J},\bar{h}}(\sigma_B)$$

These inequalities also hold for ϕ, ρ replacing 2^+.

No proof.

2) FKG inequality:

$\omega\leq\omega'\ :\Leftrightarrow\ \omega_i\leq\omega'_i\ \ \forall i\in\mathbb{Z}^d$

$E\subset\Omega$ increasing $:\Leftrightarrow$ if $\omega\in E \Rightarrow \omega'\in E\ \ \forall\omega\leq\omega'$

f non-decreasing $:\Leftrightarrow f(\omega)\leq f(\omega')\ \ \forall\omega\leq\omega'$

Idea: ① $\mu^{\eta^+}_{\Lambda;\beta,h}(E \cap F) \geq \mu^{\eta^+}_{\Lambda;\beta,h}(E)\,\mu^{\eta^+}_{\Lambda;\beta,h}(F)$

② $\mu(fg) - \mu(f)\mu(g) = \frac{1}{2}\int \underbrace{(f(x)-f(y))}_{\substack{\geq 0 \\ \leq 0}} \underbrace{(g(x)-g(y))}_{\substack{\geq 0 \\ \leq 0}} \mu(dx)\mu(dy)$

↑ both increasing or both decreasing

≥ 0

Note that:
$$\left.\begin{aligned} &\omega \longmapsto \omega_i = \sigma_i(\omega) \\ &\omega \longmapsto \eta_i(\omega) \\ &\omega \longmapsto \eta_A(\omega) \quad \forall A \subset\subset \mathbb{Z}^d \end{aligned}\right\} \text{ are non-decreasing}$$

Thm (FKG)
Consider $\bar{J}$ with $J_{ij} \geq 0$, $\bar{h}$; $\Lambda \subset\subset \mathbb{Z}^d$; $\eta \in \Omega$.
Then for two non-decreasing functions f and g
$$\mu^{\eta}_{\Lambda;\bar{J},\bar{h}}(f \cdot g) \geq \mu^{\eta}_{\Lambda;\bar{J},\bar{h}}(f)\,\mu^{\eta}_{\Lambda;\bar{J},\bar{h}}(g)$$

No proof:

Lemma (Monotonici for non-decreasing functions)
Let f be non-decreasing and $\Lambda_1 \subset \Lambda_2 \subset\subset \mathbb{Z}^d$,
Then for $\beta \geq 0$, $h \in \mathbb{R}$
$$\mu^{\eta^+}_{\Lambda_1;\beta,h}(f) \geq \mu^{\eta^+}_{\Lambda_2;\beta,h}(f).$$
This also holds for $\eta^+ \to \eta^-$ and non-decreasing $\to$ non-increasing.

Proof: Spatial Markov property

$$\mu^{\eta^+}_{\Lambda_1;\beta,h}(f) = \mu^{\eta^+}_{\Lambda_2;\beta,h}\big(f \mid \underbrace{\omega_i = 1 \ \forall i\in\Lambda_2\setminus\Lambda_1}_{\text{is non-decreasing}}\big)$$

$$\geq \mu^{\eta^+}_{\Lambda_2;\beta,h}(f)$$

$\square$

Lemma (Extremality)

Let f be non-decreasing; $\beta\geq 0$ and $h\in\mathbb{R}$

$$\mu^{\eta^-}_{\Lambda;\beta,h}(f) \leq \mu^{\eta}_{\Lambda;\beta,h}(f) \leq \mu^{\eta^+}_{\Lambda;\beta,h}(f)$$

$\forall\ \Lambda\subset\subset\mathbb{Z}^d$ and $\eta\in\Omega$ and even $\eta\mapsto\phi$, p.o. if $\sup(f)<1$.

Proof: Let $I(\omega) = e^{\beta\sum_{i\in\Lambda;\, j\notin\Lambda}\omega_i(1-\eta_j)}$ for $\omega\in\Omega^\eta_\Lambda$

non-decreasing.

$$\sum_{\omega\in\Omega^+_\Lambda} e^{-H_{\Lambda;\beta,h}(\omega)} = \sum_{\omega\in\Omega^\eta_\Lambda} e^{-H_{\Lambda;\beta,h}(\omega)}\, I(\omega)$$

further:
$$\sum_{\omega\in\Omega^+_\Lambda} e^{-H_{\Lambda;\beta,h}(\omega)} f(\omega) \geq \sum_{\omega\in\Omega^\eta_\Lambda} e^{-H_{\Lambda;\beta,h}(\omega)} f(\omega)\, I(\omega)$$

f might depend on the boundary.

$$\Rightarrow \mu^{\eta^+}_{\Lambda;\beta,h}(f) = \frac{\sum_{\omega\in\Omega^+_\Lambda} e^{-H_{\Lambda;\beta,h}(\omega)} f(\omega)}{\sum_{\omega\in\Omega^+_\Lambda} e^{-H_{\Lambda;\beta,h}(\omega)}}$$

$$\geq \frac{\sum e^{-H_{\Lambda;\beta,h}(\omega)} f(\omega) I(\omega)}{\sum e^{-H_{\Lambda;\beta,h}(\omega)} I(\omega)}$$

$$= \frac{\mu^2_{\Lambda;\beta,h}(\mathbb{I} f)}{\mu^2_{\Lambda;\beta,h}(\mathbb{I})} \overset{\text{FKG}}{\geq} \mu^2_{\Lambda;\beta,h}(f)$$

Similar for 2^-; ϕ; per -> as exercise. □

<u>Proof for existence of extremal Gibbs states</u>

Let f be local, then

$$\mu^{2^+}_{\Lambda_n;\beta,h}(f) = \sum_{A \subset \mathrm{supp}(f)} \tilde{f} \, \underbrace{\mu^{2^+}_{\Lambda_n;\beta,h}(2_A)}_{\leq \mu^{2^+}_{\Lambda_{n+1};\beta,h}(2_A) \geq 0} \quad \text{non-decreasing } \forall A$$

$\forall n \geq 1$

$\Rightarrow$ convergence:

$$\lim_{n\uparrow\infty} \mu^{2^+}_{\Lambda_n;\beta,h}(f) =: \langle f \rangle^+_{\beta,h} \text{ a Gibbs state.}$$

Independence of sequence via alternating and increasing sequence

$$\Delta_{2k-1} \subset \{\Lambda^{(1)}_n : n \geq 1\}; \quad \Delta_{2k} \subset \{\Lambda^{(2)}_n : n \geq 1\}; \quad \Delta_k \subsetneq \Delta_{k+1}$$

For the translation invariance first note that $f \circ \Theta_j$ is local

$$\begin{array}{ccc} \mu^{2^+}_{\Lambda_n}(f) & = & \mu^{2^+}_{\Lambda_n - j;\beta,h}(f \circ \Theta_j) \\ \downarrow & & \downarrow \\ \langle f \rangle^+_{\beta,h} & = & \langle f \circ \Theta_j \rangle^+_{\beta,h} \end{array}$$

□