Spin Systems & Phase Transitions

Recall: 1) Interested in large systems ($1cm^3$ iron $\approx 10^{23}$ atoms)

2) Mathematically clearest description of macroscopic system via infinite systems, where macroscopic observables become deterministic (ergodicity)

thermodynamic limit : $\Lambda \Uparrow \mathbb{Z}^d$

$\Lambda_n \Uparrow \mathbb{Z}^d$ with 1) $\Lambda_n \subset \Lambda_{n+1}$ increasing

2) $\bigcup_{n\geq 1} \Lambda_n = \mathbb{Z}^d$ cofinal, invasive

3) $\lim_{n\uparrow\infty} \frac{|\partial \Lambda_n|}{|\Lambda_n|} = 0$ boundary control

Ex: $B_n = \{-n, \dots n\}^d$

6 Pressure

Def (Pressure in finite volumes)
The pressure in $\Lambda \subset\subset \mathbb{Z}^d$ with boundary condition $\eta \in \Omega$ is defined by

$$\psi_\Lambda^\eta(\beta, h) = \frac{1}{|\Lambda|} \log Z^\eta_{\Lambda;\beta,h}$$

Remarks: 1) Term pressure associated to lattice gas model, equivalent to Ising model

2) Observe the relation to the

Cummulant generating function

$$C_X(t) = \log \mathbb{E}[e^{tX}];$$

cummulants : $c_r(X) := \frac{d^r}{dt^r} C_X(t)\Big|_{t=0}$

determi the distribution of X whenever the moments do.

3) $\psi^{\emptyset}_{\Lambda}(\beta, h) = \psi^{\emptyset}_{\Lambda}(\beta, -h)$ also true for $\rho \leftarrow \emptyset$

$\psi^{\eta^+}_{\Lambda}(\beta, h) = \psi^{\eta^-}_{\Lambda}(\beta, -h)$

Lemma (Finite-volume pressure is convex)
$\forall \Lambda \subset\subset \mathbb{Z}^d$ and boundary condition η
$$(\beta, h) \mapsto \psi^{\eta}_{\Lambda}(\beta, h) \text{ is convex}$$

<u>Proof:</u> Let $\alpha \in [0,1]$, by Hölder's inequality
$\left[\mu[XY] \le \mu[X^p]^{\frac{1}{p}} \mu[Y^q]^{\frac{1}{q}} \quad \forall \frac{1}{p} + \frac{1}{q} = 1 \right]$

$$\psi^{\eta}_{\Lambda}(\alpha\beta_1 + (1-\alpha)\beta_2, \alpha h_1 + (1-\alpha) h_2) =$$

$$\frac{1}{|\Lambda|} \log \sum_{\omega \in \Omega^{\eta}_{\Lambda}} e^{-\alpha H_{\Lambda, \beta_1, h_1}(\omega) - (1-\alpha) H_{\Lambda, \beta_2, h_2}(\omega)}$$

$$\underset{p = \frac{1}{\alpha}}{\overset{H.}{\le}} \frac{1}{|\Lambda|} \log \left[\left(\sum_{\omega \in \Omega^{\eta}_{\Lambda}} e^{-H_{\Lambda; \beta_1, h_1}(\omega)} \right)^{\alpha} \left(\sum_{\omega \in \Omega^{\eta}_{\Lambda}} e^{-H_{\Lambda; \beta_2, h_2}(\omega)} \right)^{1-\alpha} \right]$$

$$= \alpha\, \psi^{\eta}_{\Lambda}(\beta_1,h_1) + (1-\alpha)\, \psi^{\eta}_{\Lambda}(\beta_2,h_2) \qquad \square$$

Thm (Pressure)

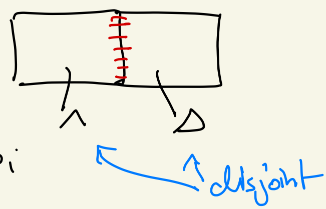
For all boundary conditions $\eta\in\Omega$

$$\psi(\beta,h) = \lim_{\Lambda\Uparrow\mathbb{Z}^d} \psi^{\eta}_{\Lambda}(\beta,h)$$

exists and is independent of η and $\Lambda \Uparrow \mathbb{Z}^d$.

<u>Proof</u>: We do the proof only for $\Lambda = [a_1,b_1]\times\dots\times[a_d,b_d]\subset\mathbb{Z}^d$
(See Friedli & Velenik p 88 for general proof)

Observe that $\forall\, \omega\in\Omega^{\eta}_{\Lambda\cup\Delta}$



$$H_{\Lambda\cup\Delta;\beta,h}(\omega) = -\beta \sum_{\{i,j\}\in\mathcal{E}^b_{\Lambda\cup\Delta}} \omega_i\omega_j - h\sum_{i\in\Lambda\cup\Delta}\omega_i$$

$$\geq H_{\Lambda;\beta,h}\underset{\Omega^{\eta}_{\Lambda}}{\underset{\cap}{(\omega)}} + H_{\Delta;\beta,h}\underset{\Omega^{\eta}_{\Delta}}{\underset{\cap}{(\omega)}} - 2\beta\big(|\partial\Lambda| + |\partial\Delta|\big)$$

$$\Rightarrow \quad \underbrace{\log Z^{\eta}_{\Lambda\cup\Delta}}_{a(\Lambda\cup\Delta)} \leq \log Z^{\eta}_{\Lambda} + \log Z^{\eta}_{\Delta} + 2\beta\big(|\partial\Lambda|+|\partial\Delta|\big)$$

$$a(\Lambda\cup\Delta) \leq a(\Lambda) + a(\Delta) \qquad \text{subadditive}$$

1) $\lim \frac{1}{|\Lambda|}\, c\,|\partial\Lambda| \to 0$

2) Version of Fekete's lemma $(a_{n+m}\leq a_n + a_m)$

$$\Rightarrow \quad \lim_{n\uparrow\infty} \frac{1}{|B_n|}\, a(B_n) = \inf_{\Lambda \text{ parallelepipeds}} \frac{1}{|\Lambda|}\, a(\Lambda)$$

Finally:

$$- \beta|\delta\Lambda| + \log Z_\Lambda^{\phi} \leq \log Z_\Lambda^{2} \leq \log Z_\Lambda^{\phi} + \beta|\delta\Lambda|$$

$\Rightarrow$ van Hove $\frac{|\delta\Lambda|}{|\Lambda|} \to 0$, boundary conditions are irrelevant

$\square$

7 Magnetization

Recall $m_\Lambda = \frac{1}{|\Lambda|} M_\Lambda$ the magnetization density

and define

$$m^2_\Lambda(\beta,h) = \mu^2_{\Lambda;\beta,h}(m_\Lambda) .$$

Observe

$$m^2_\Lambda(\beta,h) = \frac{\delta}{\delta h} \psi^2_\Lambda(\beta,h)$$

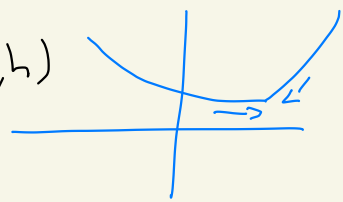
$$\left[\log \mathbb{E}[e^{x+hy}] \rightsquigarrow \frac{1}{\mathbb{E}[e^{x+hy}]} \mathbb{E}[y e^{x+hy}] \right]$$

How is the situation in the infinite volume?

$$\lim_{\Lambda \Uparrow \mathbb{Z}^d} \frac{\delta}{\delta h} \psi^2_\Lambda(\beta,h) \overset{?}{=} \frac{\delta}{\delta h} \lim_{\Lambda \Uparrow \mathbb{Z}^d} \psi^2_\Lambda(\beta,h)$$

By convexity the left- and right-sided derivatives

$$\frac{\delta}{\delta h^-} \psi(\beta,h) \; ; \; \frac{\delta}{\delta h^+} \psi(\beta,h)$$



always exist.

$$B_\beta := \{ h \in \mathbb{R} : h \mapsto \psi(\beta,h) \text{ not differentiable at } h \} \quad (\text{countable})$$

Cor (Properties of the magnetization)

1) $\forall h \notin \mathcal{B}_\beta$: $\quad m(\beta,h) := \lim_{\Lambda \Uparrow \mathbb{Z}^d} m_\Lambda^\eta(\beta,h)$

exists and is independent of $\Lambda \Uparrow \mathbb{Z}^d$, $\eta \in \Omega$.

Moreover $$m(\beta,h) = \frac{\delta}{\delta h} \psi(\beta,h)$$

2) $h \mapsto m(\beta,h)$ is non-decreasing and continuous on $\mathbb{R} \setminus \mathcal{B}_\beta$

3) It is discontinous on $\mathcal{B}_\beta$ with
$\lim_{h' \downarrow h} m(\beta,h') = \frac{\delta}{\delta h^+} \psi(\beta,h)$; $\lim_{h' \uparrow h} m(\beta,h') = \frac{\delta}{\delta h^-} \psi(\beta,h)$

The sponateous magnetization $m^*(\beta) = \lim_{h \downarrow 0} m(\beta,h)$ is well-defined.

Proof: 1) For $h \notin \mathcal{B}_\beta$

$$\frac{\delta}{\delta h} \psi(\beta,h) = \frac{\delta}{\delta h} \lim_{\Lambda \Uparrow \mathbb{Z}^d} \psi_\Lambda^\phi(\beta,h)$$

$$= \lim_{\Lambda \Uparrow \mathbb{Z}^d} \frac{\delta}{\delta h} \psi_\Lambda^\phi(\beta,h) = \lim_{\Lambda \Uparrow \mathbb{Z}^d} m_\Lambda^\phi(\beta,h)$$

follows from convexity (monotone convergence of diff. quotients)

2) Monotinicty and continuity follow from convexity
($\delta^+ f$ is right continuous, $\delta^- f$ is left continuous)

3) Discontinuity also follows from the right and left continuity of $\delta^+\psi$, $\delta^-\psi$

Def (Phase transition)

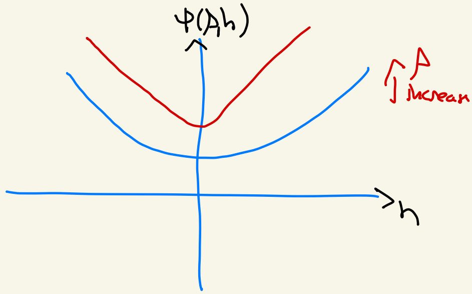
The pressure ψ exibits a first-order phase transition at (β, h) if $h \mapsto \psi(\beta, h)$ fails to be differentiable at (β, h).

8 One-dimensional Ising model

Thm (Pressure in $d=1$)

$\forall \beta \geq 0$ and $h \in \mathbb{R}$

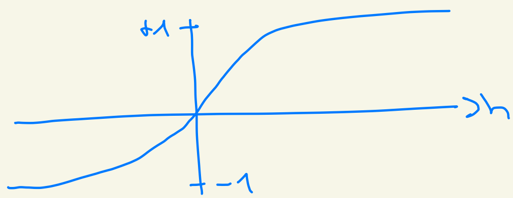
$$\psi(\beta,h) = \log\left\{ e^{\beta}\cosh(h) + \sqrt{e^{2\beta}\cosh^2(h) - 2\sinh(2\beta)} \right\}$$



Remark: 1) $\psi(\beta,h)$ differentiable, even analytic

$\Rightarrow$ $m(\beta,h)$ analytic

2) $m^*(\beta) = 0$ by symmetry $\forall \beta \geq 0$



Proof: We choose periodic boundary conditions and tori $V_n = \{0, \ldots, n-1\}$.

Then
$$Z^{per}_{V_n;\beta,h} = \sum_{\omega \in \Omega_{V_n}} e^{-H^{per}_{V_n;\beta,h}(\omega)}$$

$$= \sum_{\omega_0 \in \{\pm 1\}} \cdots \sum_{\omega_{n-1} \in \{\pm 1\}} \prod_{i=0}^{n-1} \underbrace{e^{\beta \omega_i \omega_j + h\omega_i}}_{a_{\omega_i \omega_j}}$$

Then matrix (a_{ij})

$$Z^{per}_{V_n;\beta,h} = \sum_{\omega_0 = \pm 1} (A^n)_{\omega_0,\omega_0} = \mathrm{Tr}(A^n)$$

$$\begin{pmatrix} e^{\beta+h} & e^{-\beta+h} \\ e^{-\beta-h} & e^{\beta-h} \end{pmatrix}$$

1) Compute eigenvalues

2) Laplace principle $\lim_{n\uparrow\infty} \frac{1}{|V_n|} \log Z^{per}_{V_n;\beta,h} = \lim_{n\uparrow\infty} \frac{1}{n}\left(\log \lambda_2^n + \lambda_1^n\right)$

See e.g. Lemma 1.2.15 [Large Deviations; Dembo & Zeitouni]

$$= \max\{\log \lambda_2, \log \lambda_1\}$$

use large eigenvalue. $\square$

Thm (d=1, large-deviations for the magnetization)

Let $0<\beta<\infty$ and $\varepsilon>0$, then there exists $c=c(\beta,\varepsilon)$ such that

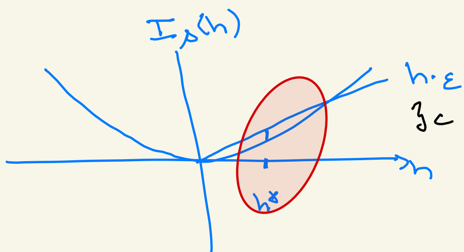
$$\mu^{\Omega}_{\Lambda_n;\beta,0}\left(m_\Lambda \notin (-\varepsilon,\varepsilon)\right) \le e^{-c|\Lambda_n|}$$

<u>Proof:</u> Using the exponential Markov inequality $\forall h\ge 0$

$$\mu^{\Omega}_{\Lambda_n,\beta,0}\left(m_{\Lambda_n}\ge\varepsilon\right) \le e^{-h\varepsilon|\Lambda_n|} \underbrace{\mu^{\Omega}_{\Lambda_n;\beta,0}\left(e^{h m_{\Lambda_n}|\Lambda_n|}\right)}_{\dfrac{Z^{\Omega}_{\Lambda_n;\beta,h}}{Z^{\Omega}_{\Lambda_n;\beta,0}}}$$

$$\Rightarrow \lim_{n\uparrow\infty}\frac{1}{|\Lambda_n|}\log \mu^{\Omega}_{\Lambda_n;\beta,0}\left(m_{\Lambda_n}\ge\varepsilon\right) \le -h\varepsilon + \underbrace{\left(\psi(\beta,h)-\psi(\beta,0)\right)}_{I_\beta(h)}$$

(LD rate function)



$$\le -\underbrace{\sup_{h\ge0}\left\{h\varepsilon - I_\beta(h)\right\}}$$

Legendre transform of I_β

Note $I_\beta(0)=0$ $\quad I_\beta'(0)=0$

$I_\beta'(h)\to 1$ for $h\uparrow\infty$.

$\square$