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Uniform exponential decay of the free energy for Voronoi finite volume discretized reaction-diffusion systems

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Contents

Model equations

X_i	species, $i = 1, \dots, m$	$\bar{u}_i$	reference densities
v_i	chemical potentials	$u_i = \bar{u}_i \mathrm{e}^{v_i}$	densities

- reversible reactions

$$\alpha_1 X_1 + \cdots + \alpha_m X_m \rightleftharpoons \beta_1 X_1 + \cdots + \beta_m X_m, \quad (\alpha, \beta) \in \mathscr{R} \subset \mathbb{Z}_+^m \times \mathbb{Z}_+^m$$

reaction rates $\quad k_{\alpha\beta}(\mathrm{e}^{v\cdot\alpha} - \mathrm{e}^{v\cdot\beta}), \qquad v = (v_1, \dots, v_m)$

generation rate of species X_i $\quad R_i := \sum_{(\alpha,\beta)\in\mathscr{R}} k_{\alpha\beta}(\mathrm{e}^{v\cdot\alpha} - \mathrm{e}^{v\cdot\beta})(\beta_i - \alpha_i)$

- flux density $\quad j_i = -u_i \mu_i \nabla v_i, \quad i = 1, \dots, m,$

continuity equations

$$\frac{\partial u_i}{\partial t} + \nabla \cdot j_i = R_i \text{ in } \mathbb{R}_+ \times \Omega, \quad \nu \cdot j_i = 0 \text{ on } \mathbb{R}_+ \times \Gamma,$$

$$u_i(0) = U_i \text{ in } \Omega, \quad i = 1, \dots, m.$$

Weak formulation

$$v = (v_1, \ldots, v_m) \in V = H^1(\Omega;\mathbb{R}^m), \quad u = (u_1, \ldots, u_m) \in V^*$$

Operators $A, E : V \cap L^\infty(\Omega, \mathbb{R}^m) \to V^*$

$$\langle A v, \bar v \rangle_V := \int_\Omega \Big\{ \sum_{i=1}^m \bar u_i \mathrm{e}^{v_i} \mu_i \nabla v_i \cdot \nabla \bar v_i + \sum_{(\alpha,\beta)\in\mathscr{R}} k_{\alpha\beta} \Big(\mathrm{e}^{\alpha\cdot v} - \mathrm{e}^{\beta\cdot v} \Big)(\alpha - \beta)\cdot \bar v \Big\} \mathrm{d}x$$

$$\langle E v, \bar v \rangle_V := \int_\Omega \sum_{i=1}^m \bar u_i \mathrm{e}^{v_i} \bar v_i \, \mathrm{d}x, \quad \bar v \in V$$

Problem (P)

$$\left.\begin{array}{l} u'(t) + Av(t) = 0,\ u(t) = Ev(t) \text{ f.a.a. } t \in \mathbb{R}_+,\ u(0) = U, \\ u \in H^1_{\mathrm{loc}}(\mathbb{R}_+; V^*),\ v \in L^2_{\mathrm{loc}}(\mathbb{R}_+; V) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\infty(\Omega)^m). \end{array}\right\} \qquad \text{(P)}$$

Energy functionals

Dissipation rate

$$D(v) = \int_\Omega \sum_{i=1}^m \bar{u}_i \mathrm{e}^{v_i} \mu_i \nabla v_i \cdot \nabla v_i \,\mathrm{d}x$$
$$+ \int_\Omega \sum_{(\alpha,\beta)\in\mathscr{R}} k_{\alpha\beta}(e^{v\cdot\alpha} - e^{v\cdot\beta})(\alpha-\beta)\cdot v \,\mathrm{d}x \geq 0 \quad \forall v \in V \cap L^\infty(\Omega,\mathbb{R}^m)$$

Free energy

$$F(u) = \int_\Omega \sum_{i=1}^m \Big(u_i(\ln \frac{u_i}{\bar{u}_i} - 1) + \bar{u}_i\Big)\,\mathrm{d}x$$

Stoichiometric subspace

$$\mathscr{S} = \mathrm{span}\{\alpha - \beta : (\alpha,\beta) \in \mathscr{R}\}$$

- Invariants: (u,v) solution to (P) $\implies$

$$u(t) - U \in \mathscr{U} := \Big\{u \in V^* : (\langle u_1, 1\rangle, \dots, \langle u_m, 1\rangle) \in \mathscr{S}\Big\} \quad \forall t > 0.$$

- Thermodynamic equilibrium: There exists a unique solution (u^*, v^*) to

$$Av^* = 0, \quad u^* = Ev^*, \quad u^* - U \in \mathscr{U}. \tag{S}$$

  Moreover, $\nabla v^* = 0$ and $v^* \in \mathscr{S}^\perp$.

- Monotonous decay of the free energy: Let (u,v) be a solution to (P). Then

$$F(u(t_2)) \leq F(u(t_1)) \leq F(U) \quad \text{for } t_2 \geq t_1 \geq 0.$$

- Exponential decay of the free energy: Let (u,v) be a solution to (P) and let (u^*, v^*) be the thermodynamic equilibrium. Then there exists a constant $\lambda > 0$ such that

$$F(u(t)) - F(u^*) \leq \mathrm{e}^{-\lambda t}(F(U) - F(u^*)) \quad \forall t \geq 0.$$

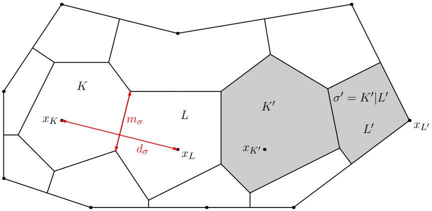
Voronoi finite volume discretization

$\Omega \subset \mathbb{R}^N$, $N \leq 3$, open, bounded, polyhedral domain

$\mathscr{M} = (\mathscr{P}, \mathscr{T}, \mathscr{E})$

- set $\mathscr{P}$ of grid points in $\bar{\Omega}$, $\mathscr{P}$ is assumed to be boundary conforming Delaunay
- family $\mathscr{T}$ of control volumes,
- family $\mathscr{E}$ of faces of the Voronoi boxes

$\mathscr{E}_{int} \subset \mathscr{E}$, $\mathscr{E}_K \subset \mathscr{E}$, $M = \#\mathscr{P}$



For $x_K \in \mathscr{P}$ the Voronoi box K is given by

$$K = \{x \in \Omega : |x - x_K| < |x - x_L| \quad \forall x_L \in \mathscr{P},\ x_L \neq x_K\}, \quad K \in \mathscr{T}.$$

m_σ = $(n-1)$-dimensional measure of Voronoi face σ,
d_σ = $|x_K - x_L|$ for $\sigma = K|L \in \mathscr{E}_{int}$

Discretization scheme

u_i^K Mass of species X_i in the box K $\qquad \vec{u} = (u_i^K)_{K\in\mathscr{T}, i=1,\dots,m} \in \mathbb{R}^{Mm}$

v_{iK} chemical potential of the species X_i in x_K $\qquad \vec{v} = (v_{iK})_{K\in\mathscr{T}, i=1,\dots,m} \in \mathbb{R}^{Mm}$

Voronoi finite volume - implicit Euler - discretization

$$\left.\begin{aligned} \frac{u_i^K(t_n)-u_i^K(t_{n-1})}{t_n-t_{n-1}} - \sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{int}} Y_i^\sigma Z_i^\sigma(t_n)(v_{iL}(t_n)-v_{iK}(t_n))\frac{m_\sigma}{d_\sigma} = R_i^K(t_n), \\ u_i^K(t_n) = \bar{u}_{iK}\mathrm{e}^{v_{iK}(t_n)}\mathrm{mes}(K), \quad n\geq 1, \\ u_i^K(0) = \int_K U_i\,\mathrm{d}x, \quad K\in\mathscr{T},\ i=1,\dots,m, \end{aligned}\right\} \qquad (\mathrm{P}_{\mathscr{M}})$$

$Z_i^\sigma = \frac{1}{2}(\mathrm{e}^{v_{iK}}+\mathrm{e}^{v_{iL}})$, $\sigma = K|L$, Y_i^σ suitable average of $\bar{u}_i\mu_i$ corresponding to face σ

$\bar{u}_{iK}, k_{\alpha\beta K}$ average of $\bar{u}_i, k_{\alpha\beta}$ on the box K,

$$R_i^K(t_n) = \sum_{(\alpha,\beta)\in\mathscr{R}} k_{\alpha\beta K}(\mathrm{e}^{\alpha\cdot v_K(t_n)} - \mathrm{e}^{\beta\cdot v_K(t_n)})(\beta_i-\alpha_i)\,\mathrm{mes}(K)$$

- Invariants: $(\vec{u},\vec{v}) \in (\mathbb{R}^{Mm})^2$ solution to the discretized problem $(\mathrm{P}_{\mathscr{M}}) \Longrightarrow$

$$\vec{u}(t_n) - \vec{U} \in \Big\{ \vec{u} \in \mathbb{R}^{Mm} : \big(\sum_{K\in\mathscr{T}} u_1^K, \ldots, \sum_{K\in\mathscr{T}} u_m^K \big) \in \mathscr{S} \Big\} \quad \forall n \in \mathbb{N}$$

- Stationary solutions: There is exactly one stationary solution $(\vec{u}^*,\vec{v}^*)$ to $(\mathrm{P}_{\mathscr{M}})$ fulfilling the above invariance property.
  $(\vec{u}^*,\vec{v}^*)$ is a discrete thermodynamic equilibrium.

- The scheme is dissipative and the discrete free energy

$$\hat{F}(\vec{u}) = \sum_{i=1}^{m} \sum_{K\in\mathscr{T}} \Big(u_i^K v_{iK} - u_i^K + \bar{u}_{iK}\mathrm{mes}(K) \Big)$$

  decays monotonously and exponentially to its equilibrium value.

Remarks

Results are also valid for (Gärtner/G. '09, G. '08)

- anisotropic diffusion $\mathbf{S}_i$ positive definite, symmetric

  flux densities

$$j_i = -u_i \mathbf{S}_i \nabla v_i$$

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$$j_i = -\bar{u}_i g_i'(v_i - \bar{v}_i) \mathbf{S}_i \nabla v_i$$

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- charged species X_i with charge numbers q_i

  in addition Poisson equation for the electrostatic potential v_0

$$-\nabla \cdot (\mathbf{S}_0 \nabla v_0) = f + \sum_{i=1}^{m} q_i u_i \quad \text{on } \mathbb{R}_+ \times \Omega \qquad + \quad \text{BCs}$$

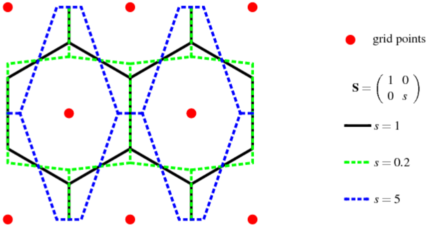
  flux densities

$$j_i = -\bar{u}_i g_i'(v_i - \bar{v}_i)\mathbf{S}_i \nabla (v_i + q_i v_0)$$

Remarks concerning anisotropies

Use for each species $X_i\,,\ i=1,\dots,m$, their own anisotropic Voronoi boxes

$$K^{(i)} = \{x \in \bar{\Omega} : (x-x_K)^T \mathbf{S}_i^{-1}(x-x_K) \le (x-x_L)^T \mathbf{S}_i^{-1}(x-x_L) \quad \forall x_L \in \mathscr{P}\}$$



- fluxes through faces $\sigma^{(i)}$ additionally contain the factor $|\mathbf{S}_i \nu^{\sigma^{(i)}}|$
- reaction terms have to be evaluated on all different intersection sets $K^{(1)}_{k_1} \cap K^{(2)}_{k_2} \cap \cdots \cap K^{(m)}_{k_m}$, $k_i \in \{1,\dots,M\}$

Formulation of ($P_{\mathscr{M}}$) by piecewise constant functions

Aim: Uniform exponential decay of the free energy for a class of Voronoi finite volume discretizations $\mathscr{M}$ of Ω with the properties

$$\operatorname{diam}(\sigma) \leq \kappa_1 d_\sigma \quad \forall \sigma \in \mathscr{E}_{int}, \qquad r_{K,out} \leq \kappa_2 r_{K,in} \quad \forall K \in \mathscr{T} \qquad \text{(Q)}$$

$X(\mathscr{M})$ = set of functions from Ω to $\mathbb{R}$, being constant on each $K \in \mathscr{T}$.

v_K = value of $\underline{v} \in X(\mathscr{M})$ on K.

Discrete H^1-seminorm for $\underline{v} \in X(\mathscr{M})$

$$|\underline{v}|^2_{1,\mathscr{M}} = \sum_{\sigma \in \mathscr{E}_{int}} |D_\sigma \underline{v}|^2 \frac{m_\sigma}{d_\sigma}, \quad \text{where } D_\sigma \underline{v} = |v_K - v_L| \text{ for } \sigma = K|L.$$

use functions

$\underline{v}_i$	$\underline{u}_i$	$\underline{\bar{u}}_i$	$\underline{k_{\alpha\beta}}$	$\underline{a}_i = \mathrm{e}^{\underline{v}_i/2}$	$\in X(\mathscr{M})$
v_{iK}	u_{iK}	$\bar{u}_{iK}$	$k_{\alpha\beta K}$	$\mathrm{e}^{v_{iK}/2}$	on K

where $u_{iK} := \dfrac{u_i^K}{\operatorname{mes}(K)}$

Formulation of ($P_{\mathscr{M}}$) by piecewise constant functions

$$E_{\mathscr{M}}:X(\mathscr{M})\to X(\mathscr{M}),\ E_{\mathscr{M}}(\underline{v})=(\bar{u}_i \mathrm{e}^{\underline{v}_i})_{i=1,\dots,m}, \qquad \underline{u}=E_{\mathscr{M}}(\underline{v})$$

Free energy: $F_{\mathscr{M}}:X(\mathscr{M})\to\mathbb{R}$,

$$F_{\mathscr{M}}(\underline{u})=\int_\Omega\sum_{i=1}^m(\underline{u}_i\underline{v}_i-\underline{u}_i+\bar{u}_i)\,\mathrm{d}x\neq F(\underline{u})$$

Dissipation rate:

$$D_{\mathscr{M}}(\underline{v})=\sum_{i=1}^m\sum_{\sigma\in\mathscr{E}_{int}}Y_i^\sigma Z_i^\sigma|D_\sigma\underline{v}_i|^2\frac{m_\sigma}{d_\sigma}+\int_\Omega\sum_{(\alpha,\beta)\in\mathscr{R}}\underline{k_{\alpha\beta}}\left[\mathrm{e}^{\alpha\cdot\underline{v}}-\mathrm{e}^{\beta\cdot\underline{v}}\right](\alpha-\beta)\cdot\underline{v}\,\mathrm{d}x$$

$$\geq c\sum_{i=1}^m|\underline{a}_i|^2_{1,\mathscr{M}}+c\int_\Omega\sum_{(\alpha,\beta)\in\mathscr{R}}\Big[\prod_{i=1}^m\underline{a}_i^{\alpha_i}-\prod_{i=1}^m\underline{a}_i^{\beta_i}\Big]^2\mathrm{d}x\geq 0, \qquad \underline{a}_i=\mathrm{e}^{\underline{v}_i/2}.$$

Invariants: $\underline{u}(t_n)-U\in\mathscr{U}\quad\forall n\in\mathbb{N}$

For the thermodynamic equilibrium $(\underline{u}^*,\underline{v}^*)$ to ($P_{\mathscr{M}}$) (resulting from $(\vec{u}^*,\vec{v}^*)$) we get

$$\underline{u}_i^*=\frac{\bar{u}_i}{\bar{u}_i}u_i^*,\quad i=1,\dots,m,\quad \underline{v}^*=v^*\in\mathscr{S}^\perp.$$

Theorem A. [Uniform estimate of the free energy by the dissipation rate]
Let a class of Voronoi finite volume discretizations fulfill (Q), moreover let $(\underline{u}^*, \underline{v}^*)$ be the thermodynamic equilibrium to ($P_{\mathscr{M}}$). Then, for all $\rho > 0$ there exists a constant $c_\rho > 0$ such that

$$F_{\mathscr{M}}(E_{\mathscr{M}}\underline{v}) - F_{\mathscr{M}}(\underline{u}^*) \leq c_\rho D_{\mathscr{M}}(\underline{v})$$

for all $\underline{v} \in \mathscr{N}_{\rho\mathscr{M}} := \Big\{ \underline{v} \in X(\mathscr{M}) : \; F_{\mathscr{M}}(E_{\mathscr{M}}\underline{v}) - F_{\mathscr{M}}(\underline{u}^*) \leq \rho, \, E_{\mathscr{M}}\underline{v} - U \in \mathscr{U} \Big\}$
uniformly for all Voronoi finite volume discretizations $\mathscr{M}$.

Indirect proof:
Suppose there is $\{\mathscr{M}_l\}$ with size$(\mathscr{M}_l) \to 0$ and corresponding $\underline{v}_l \in \mathscr{N}_{\rho\mathscr{M}_l}$ such that

$$F_{\mathscr{M}_l}(E_{\mathscr{M}_l}\underline{v}_l) - F_{\mathscr{M}_l}(\underline{u}_l^*) = C_l D_{\mathscr{M}_l}(\underline{v}_l) \quad \text{with} \quad C_l \to \infty,$$

construct a contradiction.

Formulation of ($P_{\mathscr{M}}$) by piecewise constant functions

Essential tool in this proof:

Theorem B. [Discrete Sobolev-Poincaré inequality] (G./Griepentrog' 10)

$$\|u - m_\Omega(u)\|_{L^q(\Omega)} \le c_q |u|_{1,\mathscr{M}} \quad \forall u \in X(\mathscr{M})$$

for $q \in [1, \infty)$ if $N = 2$, $q \in [1, \frac{2N}{N-2})$ if $N > 2$ uniformly for all $\mathscr{M}$ with the property (Q).

$$D_{\mathscr{M}_l}(\underline{v}_l) \to 0 \quad \Longrightarrow \quad |\underline{a}_{\mathscr{M}_l i}|_{1,\mathscr{M}_l} \to 0$$

$$\Longrightarrow \quad \underline{a}_{\mathscr{M}_l i} - m_\Omega(\underline{a}_{\mathscr{M}_l i}) \to 0 \text{ in } L^q(\Omega),$$

$$m_\Omega(\underline{a}_{\mathscr{M}_l i}) \to \mathrm{e}^{v_i^*/2}$$

guarantees the limit passage $l \to \infty$ in parts of the dissipation rate resulting from the reaction terms.

Theorem C. [Uniform exponential decay of the free energy]
We consider Voronoi finite volume discretizations $\mathscr{M}$ fulfilling the property (Q). Let $(\underline{u}, \underline{v})$ be a solution to $(P_{\mathscr{M}})$ and let $(\underline{u}^*, \underline{v}^*)$ be the thermodynamic equilibrium to $(P_{\mathscr{M}})$. Then

$$F_{\mathscr{M}}(\underline{u}(t_n)) - F_{\mathscr{M}}(\underline{u}^*) \leq \mathrm{e}^{-\lambda^* t_n}\big(F_{\mathscr{M}}(\underline{U}) - F_{\mathscr{M}}(\underline{u}^*)\big) \quad \forall n \geq 1$$

with a unified constant $\lambda^* > 0$.

Idea of the proof:

Discrete Brézis formula: For $n_2 > n_1 \geq 0$ and $\lambda \geq 0$ we obtain

$$\mathrm{e}^{-\lambda t_{n_2}}\Big(F_{\mathscr{M}}(\underline{u}(t_{n_2})) - F_{\mathscr{M}}(\underline{u}^*)\Big) - \mathrm{e}^{-\lambda t_{n_1}}\Big(F_{\mathscr{M}}(\underline{u}(t_{n_1})) - F_{\mathscr{M}}(\underline{u}^*)\Big)$$
$$\leq \sum_{r=n_1+1}^{n_2} \mathrm{e}^{\lambda t_{r-1}}(t_r - t_{r-1})\Big\{\mathrm{e}^{\lambda \bar{h}}\lambda\Big(F_{\mathscr{M}}(\underline{u}(t_r)) - F_{\mathscr{M}}(\underline{u}^*)\Big) - D_{\mathscr{M}}(\underline{v}(t_r))\Big\}.$$

Idea of the proof

For $n_2 > n_1 \geq 0$ and $\lambda \geq 0$ we obtain

$$\mathrm{e}^{-\lambda t_{n_2}}\Big(F_{\mathscr{M}}(\underline{u}(t_{n_2})) - F_{\mathscr{M}}(\underline{u}^*)\Big) - \mathrm{e}^{-\lambda t_{n_1}}\Big(F_{\mathscr{M}}(\underline{u}(t_{n_1})) - F_{\mathscr{M}}(\underline{u}^*)\Big)$$
$$\leq \sum_{r=n_1+1}^{n_2} \mathrm{e}^{\lambda t_{r-1}}(t_r - t_{r-1})\Big\{\mathrm{e}^{\lambda \bar{h}}\lambda\Big(F_{\mathscr{M}}(\underline{u}(t_r)) - F_{\mathscr{M}}(\underline{u}^*)\Big) - D_{\mathscr{M}}(\underline{v}(t_r))\Big\}.$$

- $\lambda = 0$:

$$F_{\mathscr{M}}(\underline{u}(t_{n_2})) + \sum_{r=n_1+1}^{n_2}(t_r - t_{r-1})D_{\mathscr{M}}(\underline{v}(t_r)) \leq F_{\mathscr{M}}(\underline{u}(t_{n_1})) \leq F_{\mathscr{M}}(\underline{U})$$

  $F_{\mathscr{M}}$ decays monotonously,
  $F_{\mathscr{M}}(\underline{U}) - F_{\mathscr{M}}(\underline{u}^*) \leq \rho$ for all $\mathscr{M}$ with (Q), $\quad E_{\mathscr{M}}\underline{v}(t_r) - U \in \mathscr{U}$
  $\Longrightarrow \underline{v}(t_r) \in \mathscr{N}_{\rho\mathscr{M}}$ for $r \geq 1$

- apply Theorem A, choose $\lambda > 0$ such that $\lambda \mathrm{e}^{\lambda\bar{h}} c_\rho < 1$
  $\Longrightarrow$ result follows uniformly for all $\mathscr{M}$ with (Q).

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Assumptions

$\Omega \subset \mathbb{R}^N$, $N = 2,3$, open, bounded polyhedral domain;
$\mu_i, \overline{u}_i, U_i \in L_+^\infty(\Omega)$, $\mu_i, \overline{u}_i \geq \delta > 0$, $q_i \in \mathbb{Z}$, $i = 1,\ldots,m$;
$\mathscr{R} \subset \mathbb{Z}_+^m \times \mathbb{Z}_+^m$ finite subset, $k_{\alpha\beta} \in L_+^\infty(\Omega)$, $k_{\alpha\beta} \geq \delta > 0$ for all $(\alpha,\beta) \in \mathscr{R}$.
If $N = 3$ then $\max_{(\alpha,\beta)\in\mathscr{R}} \max \left\{ \sum_{i=1}^m \alpha_i, \sum_{i=1}^m \beta_i \right\} \leq 3$;
(< 3 for uniform exponential decay of the free energy)

$\sum_{i=1}^m \langle U_i, \lambda_i \rangle > 0 \quad \forall \lambda \in \mathscr{S}_+^\perp \setminus \{0\}$ (Slater condition)

$\{b \in \mathbb{R}_+^m : b^\alpha = b^\beta \ \forall (\alpha,\beta) \in \mathscr{R},\ u - U \in \mathscr{U},\ u_i = \bar{u}_i b_i\} \cap \partial \mathbb{R}_+^m = \emptyset$;
(no false equilibria in the sense of Prigogine)

$\mathscr{M}$ Voronoi finite volume discretization,
$\mathscr{P}$ boundary conforming Delaunay grid, $\bar{h} = \sup_{n\in\mathbb{N}} (t_n - t_{n-1}) < \infty$.

Assumptions for uniform results

$\Omega \subset \mathbb{R}^N$ is star shaped with respect to some ball $B(y_0, R)$.

$$\rho(y) = \begin{cases} \exp\left\{-\frac{R^2}{R^2 - |y-y_0|^2}\right\} & \text{if } |y-y_0| < R, \\ 0 & \text{if } |y-y_0| \geq R. \end{cases}$$

$\rho^{\mathscr{M}} \in X(\mathscr{M})$: $\quad \rho_K^{\mathscr{M}}(x) = \min_{y \in \bar{K}} \rho(y) \quad$ for $x \in K$.

For all finite volume meshes $\mathscr{M}$ of Ω we suppose the following properties:

- $\int_\Omega \rho^{\mathscr{M}}(x)\, \mathrm{d}x \geq \kappa_0$,
- $\mathscr{E}_K \cap \mathscr{E}_{ext} \neq \emptyset$ implies $x_K \in \partial\Omega$,
- The geometric weights fulfill property (Q).

$\bar{\Omega} = \cup_{I \in \mathscr{I}} \bar{\Omega}_I$ finite disjoint union of subdomains s.t. discontinuities of $\bar{u}_i$, $i = 1, \ldots, m$, coincide with subdomain boundaries. $(N-1)$ dimensional measure of all internal subdomain boundaries is bounded. There exists some $\gamma \in (0, 1]$ such that $\bar{u}_i \in \mathscr{C}^{0,\gamma}(\Omega_I) := \{w|_{\Omega_I}, w \in C^{0,\gamma}(\mathbb{R}^N)\}$, $i = 1, \ldots, m$, $I \in \mathscr{I}$.