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Lecture 20

Jürgen Fuhrmann

juergen.fuhrmann@wias-berlin.de
Convection-Diffusion problems
The convection - diffusion equation

Search function $u : \Omega \rightarrow \mathbb{R}$ such that

$$-\nabla \cdot (D \vec{\nabla} u - u \vec{v}) = f \quad \text{in } \Omega$$

$$u = g \quad \text{on } \Gamma$$

- $u(x)$: species concentration, temperature
- $\vec{j} = D \vec{\nabla} u - u \vec{v}$: species flux
- $D$: diffusion coefficient
- $\vec{v}(x)$: velocity of medium (e.g. fluid)
  - Given analytically
  - Solution of free flow problem (Navier-Stokes equation)
  - Flow in porous medium (Darcy equation): $\vec{v} = -\kappa \vec{\nabla} p$ where
    $$-\nabla \cdot (\kappa \vec{\nabla} p) = 0$$

- For constant density, the divergence condition $\nabla \cdot \vec{v} = 0$ holds.
Weak formulation

- Let \( u_g \in H^1(\Omega) \) a lifting of \( g \). Find \( u \in H^1(\Omega) \) such that

\[
 u = u_g + \phi
\]

\[
 \int_\Omega (D\vec{\nabla} \phi - \phi \vec{v}) \cdot \vec{w} \, d\vec{x} = \int_\Omega f w \, d\vec{x} + \int_\Omega (D\vec{\nabla} u_g - u_g \vec{v}) \cdot \vec{w} \quad \forall \, w \in H^1_0(\Omega)
\]

Is this bilinear form coercive? - Use Lax-Milgram, for it being true, is not necessary that the bilinear form is self-adjoint.

- It follows, that

\[
 \int_\Omega (D\vec{\nabla} u - u \vec{v}) \cdot \vec{w} \, d\vec{x} = \int_\Omega f w \, d\vec{x} \quad \forall \, w \in H^1_0(\Omega)
\]

- Green's theorem: If \( w = 0 \) on \( \partial \Omega \):

\[
 \int_\Omega \vec{v} \cdot \vec{\nabla} w \, d\vec{x} = - \int_\Omega w \nabla \cdot \vec{v} \, d\vec{x}
\]
Coercivity of bilinear form

Regard the convection contribution to the coercivity estimate:

\[- \int_{\Omega} u \vec{v} \cdot \vec{\nabla} u \, d\vec{x} = \int_{\Omega} u \vec{\nabla} \cdot (u \vec{v}) \, d\vec{x} \quad \text{Green’s theorem}\]

\[\int_{\Omega} u^2 \vec{\nabla} \cdot \vec{v} \, d\vec{x} + \int_{\Omega} u \vec{v} \cdot \vec{\nabla} u \, d\vec{x} = \int_{\Omega} u \vec{\nabla} \cdot (u \vec{v}) \, d\vec{x} \quad \text{Product rule}\]

\[\int_{\Omega} u^2 \vec{\nabla} \cdot \vec{v} \, d\vec{x} + 2 \int_{\Omega} u \vec{v} \cdot \vec{\nabla} u \, d\vec{x} = 0 \quad \text{Equation difference}\]

\[\int_{\Omega} u \vec{v} \cdot \vec{\nabla} u \, d\vec{x} = 0 \quad \text{Divergence condition} \, \vec{\nabla} \cdot \vec{v} = 0\]

Then

\[\int_{\Omega} (D \vec{\nabla} u - u \vec{v}) \cdot \vec{\nabla} u \, d\vec{x} = \int_{\Omega} D \vec{\nabla} u \cdot \vec{\nabla} u \, d\vec{x} \geq C \|u\|_{H^1_0(\Omega)}\]

One could allow for fixed sign of \(\nabla \cdot \vec{v}\).
Convection diffusion problem: maximum principle

- Let $f \leq 0, \nabla \cdot \vec{v} = 0$
- Let $g^\# = \sup_{\partial \Omega} g$.
- Let $w = (u - g^\#)^+ = \max\{u - g^\#, 0\} \in H^1_0(\Omega)$
- Consequently, $w \geq 0$
- As $\nabla \vec{u} = \nabla (u - g^\#)$ and $\nabla w = 0$ where $w \neq u - g^\#$, one has

$$0 \geq \int_{\Omega} fw \, d\vec{x} = \int_{\Omega} D(\nabla u - u\vec{v})\nabla w \, d\vec{x}$$

Variational identity

$$= \int_{\Omega} D(\nabla w - w\vec{v})\nabla w \, d\vec{x} - Dg^\# \int_{\Omega} \vec{v} \cdot \nabla w \, d\vec{x}$$

Replace $u$ by $w$

$$= \int_{\Omega} D(\nabla w - w\vec{v})\nabla w \, d\vec{x} + Dg^\# \int_{\Omega} w \nabla \cdot \vec{v} \, d\vec{x}$$

Green

$$\geq C||w||_{H^1_0(\Omega)}$$

Coercivity, $\nabla \cdot \vec{v} = 0$

- Therefore: $w = (u - g^\#)^- = 0$ and $u \leq g^\#
- Similar for minimum part
Mimimax for convection-diffusion

**Theorem:** If $\nabla \cdot \vec{v} = 0$, the weak solution of the inhomogeneous Dirichlet problem

$$-\nabla \cdot (D\vec{\nabla} u - uv) = f \quad \text{in } \Omega$$

$$u = g \quad \text{on } \partial \Omega$$

fulfills the global mimimax principle: it attains its maximum at the boundary if $f \leq 0$ and attains its minimum at the boundary if $f \geq 0$.

**Corollary:** If $f = 0$ then $u$ attains both its minimum and its maximum at the boundary.

**Corollary:** Local mimimax principle: This is true of any subdomain $\omega \subset \Omega$. 
Finite volumes for convection diffusion

\[-\nabla \cdot \vec{j} = 0 \quad \text{in } \Omega\]
\[\vec{j} \cdot \vec{n} + \alpha u = g \quad \text{on } \Gamma = \partial \Omega\]

- Integrate time discrete equation over control volume

\[0 = -\int_{\omega_k} \nabla \cdot \vec{j} d\omega = -\int_{\partial \omega_k} \vec{j} \cdot \vec{n} d\gamma\]

\[= -\sum_{l \in \mathcal{N}_k\sigma_{kl}} \int_{\gamma_k} \vec{j} \cdot \vec{n}_{kl} d\gamma - \int_{\gamma_k} \vec{j} \cdot \vec{n} d\gamma\]

\[\approx \sum_{l \in \mathcal{N}_k} \frac{|\sigma_{kl}|}{h_{kl}} g_{kl}(u_k, u_l) + |\gamma_k| \alpha u_k - |\gamma_k| g_k\]

- \(A = A_\Omega + A_\Gamma\)
Central Difference Flux Approximation

- $g_{kl}$ approximates normal convective-diffusive flux between control volumes $\omega_k, \omega_l$: $g_{kl}(u_k - u_l) \approx -(D \vec{\nabla} u - u \vec{v}) \cdot n_{kl}$

- Let $\sigma_{kl} = \omega_k \cap \omega_l$
  Let $v_{kl} = \frac{1}{|\sigma_{kl}|} \int_{\sigma_{kl}} \vec{v} \cdot \vec{n}_{kl} d\gamma$ approximate the normal velocity $\vec{v} \cdot \vec{n}_{kl}$

- Central difference flux:
  
  $$g_{kl}(u_k, u_l) = D(u_k - u_l) + h_{kl} \frac{1}{2} (u_k + u_l)v_{kl}$$

  $$= (D + \frac{1}{2} h_{kl} v_{kl}) u_k - (D - \frac{1}{2} h_{kl} v_{kl}) u_l$$

- if $v_{kl}$ is large compared to $h_{kl}$, the corresponding matrix (off-diagonal) entry may become positive

- Non-positive off-diagonal entries only guaranteed for $h \rightarrow 0$!

- If all off-diagonal entries are non-positive, we can prove the discrete maximum principle
Simple upwind flux discretization

- Force correct sign of convective flux approximation by replacing central difference flux approximation $h_{kl} \frac{1}{2}(u_k + u_l)v_{kl}$ by
  
  $$
  \begin{cases}
  h_{kl}u_kv_{kl}, & v_{kl} < 0 \\
  h_{kl}u_lv_{kl}, & v_{kl} > 0
  \end{cases}
  = h_{kl} \frac{1}{2}(u_k + u_l)v_{kl} + \frac{1}{2}h_{kl}|v_{kl}|(u_k - u_l)
  $$

- Artificial Diffusion $\tilde{D}$

- Upwind flux:

  $$
  g_{kl}(u_k, u_l) = D(u_k - u_l) + \begin{cases}
  h_{kl}u_kv_{kl}, & v_{kl} > 0 \\
  h_{kl}u_lv_{kl}, & v_{kl} < 0
  \end{cases}
  $$

  $$
  = (D + \tilde{D})(u_k - u_l) + h_{kl} \frac{1}{2}(u_k + u_l)v_{kl}
  $$

- M-Property guaranteed unconditionally!
- Artificial diffusion introduces error: second order approximation replaced by first order approximation
Exponential fitting flux I

- Project equation onto edge $x_K x_L$ of length $h = h_{kl}$, let $v = -v_{kl}$, integrate once

\[
\begin{align*}
  u' - uv &= j \\
  u|_0 &= u_k \\
  u|_h &= u_l
\end{align*}
\]

- Linear ODE

- Solution of the homogeneous problem:

\[
\begin{align*}
  u' - uv &= 0 \\
  u'/u &= v \\
  \ln u &= u_0 + vx \\
  u &= K \exp(vx)
\end{align*}
\]
Exponential fitting II

Solution of the inhomogeneous problem: set \( K = K(x) \):

\[
K' \exp(vx) + vK \exp(vx) - vK \exp(vx) = -j
\]

\[
K' = -j \exp(-vx)
\]

\[
K = K_0 + \frac{1}{v}j \exp(-vx)
\]

Therefore,

\[
u = K_0 \exp(vx) + \frac{1}{v}j
\]

\[
u_k = K_0 + \frac{1}{v}j
\]

\[
u_l = K_0 \exp(vh) + \frac{1}{v}j
\]
Use boundary conditions

\[ K_0 = \frac{u_k - u_l}{1 - \exp(vh)} \]

\[ u_k = \frac{u_k - u_l}{1 - \exp(vh)} + \frac{1}{v} j \]

\[ j = \frac{v}{\exp(vh) - 1} (u_k - u_l) + vu_k \]

\[ = v \left( \frac{1}{\exp(vh) - 1} + 1 \right) u_k - \frac{v}{\exp(vh) - 1} u_l \]

\[ = v \left( \frac{\exp(vh)}{\exp(vh) - 1} \right) u_k - \frac{v}{\exp(vh) - 1} u_l \]

\[ = \frac{-v}{\exp(-vh) - 1} u_k - \frac{v}{\exp(vh) - 1} u_l \]

\[ = \frac{B(-vh)u_k - B(vh)u_l}{h} \]

where \( B(\xi) = \frac{\xi}{\exp(\xi) - 1} \): Bernoulli function
Exponential fitting IV

- General case: \( Du' - uv = D(u' - u \frac{v}{D}) \)

- Upwind flux:

\[
g_{kl}(u_k, u_l) = D(B(-\frac{v_{kl} h_{kl}}{D})u_k - B(\frac{v_{kl} h_{kl}}{D})u_l)
\]

- Allen+Southwell 1955
- Scharfetter+Gummel 1969
- Ilin 1969
- Chang+Cooper 1970
- Guaranteed sign pattern, \( M \) property!
Exponential fitting: Artificial diffusion

- Difference of exponential fitting scheme and central scheme

- Use: \( B(-x) = B(x) + x \) \(\Rightarrow\)

\[
B(x) + \frac{1}{2}x = B(-x) - \frac{1}{2}x = B(|x|) + \frac{1}{2}|x|
\]

\[D_{\text{art}}(u_k - u_l) = D(B\left(-\frac{\nu h}{D}\right)u_k - B\left(\frac{\nu h}{D}\right)u_l) - D(u_k - u_l) + h\frac{1}{2}(u_k + u_l)\nu\]

\[= D\left(\frac{-\nu h}{2D} + B\left(-\frac{\nu h}{D}\right)\right)u_k - D\left(\frac{\nu h}{2D} + B\left(\frac{\nu h}{D}\right)u_l\right) - D(u_k - u_l)\]

\[= D \left(\frac{1}{2} \frac{\nu h}{D} + B\left(\frac{\nu h}{D}\right) - 1\right)(u_k - u_l)\]

- Further, for \( x > 0 \):

\[
\frac{1}{2}x \geq \frac{1}{2}x + B(x) - 1 \geq 0
\]

- Therefore

\[
\frac{|\nu h|}{2} \geq D_{\text{art}} \geq 0
\]
Comparison of artificial diffusion functions $\frac{1}{2} |x|$ (upwind) and $\frac{1}{2} |x| + B(|x|) - 1$ (exp. fitting)
Convection-Diffusion test problem, $N=20$

- $\Omega = (0, 1), \ -\nabla \cdot (D \vec{\nabla} u + uv) = 0, \ u(0) = 0, \ u(1) = 1$
- $V = 1, \ D = 0.01$

- Exponential fitting: sharp boundary layer, for this problem it is exact
- Central differences: unphysical
- Upwind: larger boundary layer
Convection-Diffusion test problem, $N=40$

- $\Omega = (0, 1)$, $-\nabla \cdot (D\vec{\nabla} u + uv) = 0$, $u(0) = 0$, $u(1) = 1$
- $V = 1$, $D = 0.01$

Exponential fitting: sharp boundary layer, for this problem it is exact
Central differences: unphysical, but less “wiggles”
Upwind: larger boundary layer
Convection-Diffusion test problem, \( N = 80 \)

- \( \Omega = (0, 1), -\nabla \cdot (D \nabla u + uv) = 0, u(0) = 0, u(1) = 1 \)
- \( V = 1, D = 0.01 \)

- Exponential fitting: sharp boundary layer, for this problem it is exact
- Central differences: grid is fine enough to yield M-Matrix property, good approximation of boundary layer due to higher convergence order
- Upwind: “smearing” of boundary layer
1D convection diffusion summary

- Upwinding and exponential fitting unconditionally yield the $M$-property of the discretization matrix.
- Exponential fitting for this case (zero right hand side, 1D) yields exact solution. It is anyway “less diffusive” as artificial diffusion is optimized.
- Central scheme has higher convergence order than upwind (and exponential fitting) but on coarse grid it may lead to unphysical oscillations.
- For 2/3D problems, sufficiently fine grids to stabilize central scheme may be prohibitively expensive.
- Local grid refinement may help to offset artificial diffusion.
Discrete minimax principle

- \( Au = f \)
- \( A \): matrix from diffusion or convection-diffusion
- \( A \) irreducibly diagonally dominant, positive main diagonal entries, negative off diagonal entries

\[
a_{ii} u_i = \sum_{j \neq i} -a_{ij} u_j + f_i
\]

\[
u_i = \sum_{j \neq i, a_{ij} \neq 0} -\frac{a_{ij}}{a_{ii}} u_j + f_i
\]

- For interior points, \( a_{ii} = -\sum_{j \neq i} a_{ij} \)
- Assume \( i \) is interior point. Assume \( f_i \geq 0 \) ⇒

\[
u_i \geq \min_{j \neq i, a_{ij} \neq 0} u_j \sum_{j \neq i, a_{ij} \neq 0} -\frac{a_{ij}}{a_{ii}} = \min_{j \neq i, a_{ij} \neq 0} u_j
\]

- Assume \( i \) is interior point. Assume \( f_i \leq 0 \) ⇒

\[
u_i \leq \max_{j \neq i, a_{ij} \neq 0} u_j \sum_{j \neq i, a_{ij} \neq 0} -\frac{a_{ij}}{a_{ii}} = \max_{j \neq i, a_{ij} \neq 0} u_j
\]
P1 finite elements, Voronoi finite volumes: matrix graph \equiv triangulation of domain

The set \( \{j \neq i, a_{ij} \neq 0\} \) is exactly the set of neighbor nodes

Solution in point \( x_i \) estimated by solution in neighborhood

The estimate can be propagated to the boundary of the domain
Discussion of discrete minimax principle II

- Minimax principle + positivity/nonnegativity of solutions can be seen as an important qualitative property of the physical process.
- Along with good approximation quality, its preservation in the discretization process may be necessary.
- Guaranteed for irreducibly diagonally dominant matrices.
- Nonnegativity for nonnegative right hand sides guaranteed by $M$-Property.
- Finite volume method may be preferred as it can guarantee these properties for boundary conforming Delaunay grids.
Convection-diffusion and finite elements

Search function $u : \Omega \to \mathbb{R}$ such that

$$-\vec{\nabla}(\cdot D\vec{\nabla}u - u\vec{v}) = f \quad \text{in } \Omega$$
$$u = g \quad \text{on } \partial\Omega$$

- Assume $\nu$ is divergence-free, i.e. $\nabla \cdot \nu = 0$.
- Then the main part of the equation can be reformulated as

$$-\vec{\nabla}(\cdot D\vec{\nabla}u) + \vec{v} \cdot \vec{\nabla}u = 0 \quad \text{in } \Omega$$

yielding a weak formulation: find $u \in H^1(\Omega)$ such that $u - g \in H^1_0(\Omega)$ and $\forall w \in H^1_0(\Omega)$,

$$\int_{\Omega} D\vec{\nabla}u \cdot \vec{\nabla}w \, dx + \int_{\Omega} \vec{v} \cdot \vec{\nabla}u \, w \, dx = \int_{\Omega} fw \, dx$$

- Galerkin formulation: find $u_h \in V_h$ with bc. such that $\forall w_h \in V_h$

$$\int_{\Omega} D\vec{\nabla}u_h \cdot \vec{\nabla}w_h \, dx + \int_{\Omega} \vec{v} \cdot \vec{\nabla}u_h \, w_h \, dx = \int_{\Omega} fw_h \, dx$$
Galerkin ansatz has similar problems as central difference ansatz in the finite volume/finite difference case ⇒ stabilization?

Most popular: streamline upwind Petrov-Galerkin (SUPG)

\[
\int_{\Omega} \mathbf{D} \nabla u_h \cdot \nabla w_h \, dx + \int_{\Omega} \mathbf{v} \cdot \nabla u_h \, w_h \, dx + S(u_h, w_h) = \int_{\Omega} f w_h \, dx
\]

with

\[
S(u_h, w_h) = \sum_{K} \int_{K} (\mathbf{v} \cdot (\mathbf{D} \nabla u_h - u_h \mathbf{v}) - f) \delta_K \mathbf{v} \cdot w_h \, dx
\]

where \( \delta_K = \frac{h_K^{\nu}}{2|\mathbf{v}|} \xi \left( \frac{|\mathbf{v}| h_K^{\nu}}{D} \right) \) with \( \xi(\alpha) = coth(\alpha) - \frac{1}{\alpha} \) and \( h_K^{\nu} \) is the size of element \( K \) in the direction of \( \mathbf{v} \).

Comparison paper:


- if it is necessary to compute solutions without spurious oscillations: use FVM, taking care on the construction of an appropriate grid might be essential for reducing the smearing of the layers,
- if sharpness and position of layers are important and spurious oscillations can be tolerated: often the SUPG method is a good choice.

Topic of ongoing research
Transient problems
Time dependent Robin boundary value problem

- Choose final time $T > 0$. Regard functions $(x, t) \rightarrow \mathbb{R}$.

  \[ \partial_t u - \nabla \cdot D\nabla u = f \quad \text{in } \Omega \times [0, T] \]

  \[ D\nabla u \cdot \vec{n} + \alpha u = g \quad \text{on } \partial\Omega \times [0, T] \]

  \[ u(x, 0) = u_0(x) \quad \text{in } \Omega \]

- This is an initial boundary value problem

- This problem has a weak formulation in the Sobolev space $L^2([0, T], H^1(\Omega))$, which then allows for a Galerkin approximation in a corresponding subspace

- We will proceed in a simpler manner: first, perform a finite difference discretization in time, then perform a finite element (finite volume) discretization in space.

  - *Rothe method*: first discretize in time, then in space
  - *Method of lines*: first discretize in space, get a huge ODE system, then apply perform discretization
Time discretization

- Choose time discretization points $0 = t^0 < t^1 \cdots < t^N = T$
- Let $\tau^n = t^n - t^{n-1}$
  For $i = 1 \ldots N$, solve

$$\frac{u^n - u^{n-1}}{\tau^n} - \nabla \cdot D\nabla u^\theta = f \quad \text{in } \Omega \times [0, T]$$

$$D\nabla u^\theta \cdot \vec{n} + \alpha u^\theta = g \quad \text{on } \partial\Omega \times [0, T]$$

where $u^\theta = \theta u^n + (1 - \theta) u^{n-1}$

- $\theta = 1$: backward (implicit) Euler method
  Solve PDE problem in each timestep. First order accuracy in time.

- $\theta = \frac{1}{2}$: Crank-Nicolson scheme
  Solve PDE problem in each timestep. Second order accuracy in time.

- $\theta = 0$: forward (explicit) Euler method
  First order accurate in time. This does not involve the solution of a PDE problem $\Rightarrow$ Cheap? What do we have to pay for this?
Finite volumes for time dependent hom. Neumann problem

Search function $u : \Omega \times [0, T] \rightarrow \mathbb{R}$ such that $u(x, 0) = u_0(x)$ and

$$\partial_t u - \nabla \cdot D\nabla u = 0 \quad \text{in} \Omega \times [0, T]$$

$$D\nabla u \cdot \vec{n} = 0 \quad \text{on} \Gamma \times [0, T]$$

Given control volume $\omega_k$, integrate equation over space-time control volume $\omega_k \times (t^{n-1}, t^n)$, divide by $\tau^n$:

$$0 = \int_{\omega_k} \left( \frac{1}{\tau^n} (u^n - u^{n-1}) - \nabla \cdot D\nabla u^\theta \right) d\omega$$

$$= \frac{1}{\tau^n} \int_{\omega_k} (u^n - u^{n-1}) d\omega - \int_{\partial \omega_k} D\nabla u^\theta \cdot \vec{n}_k d\gamma$$

$$= -\sum_{l \in \mathcal{N}_k} \int_{\gamma_{kl}} D\nabla u^\theta \cdot \vec{n}_{kl} d\gamma - \int_{\gamma_k} D\nabla u^\theta \cdot \vec{n} d\gamma - \frac{1}{\tau^n} \int_{\omega_k} (u^n - u^{n-1}) d\omega$$

$$\approx \frac{|\omega_k|}{\tau^n} (u^n_k - u^{n-1}_k) + \sum_{l \in \mathcal{N}_k} \frac{|\sigma_{kl}|}{h_{kl}} (u^\theta_k - u^\theta_l) \rightarrow M$$

$$\rightarrow A$$
Matrix equation

Resulting matrix equation:

\[
\frac{1}{\tau^n} \left( Mu^n - Mu^{n-1} \right) + Au^\theta = 0
\]

\[
\frac{1}{\tau^n} Mu^n + \theta Au^n = \frac{1}{\tau^n} Mu^{n-1} + (\theta - 1)Au^{n-1}
\]

\[
u^n + \tau^n M^{-1} \theta Au^n = u^{n-1} + \tau^n M^{-1}(\theta - 1)Au^{n-1}
\]

- \( M = (m_{kl}) \), \( A = (a_{kl}) \) with

\[
a_{kl} = \begin{cases} 
\sum_{l' \in \mathcal{N}_k} D \frac{|\sigma_{kl'}|}{h_{kl'}} & l = k \\
-D \frac{\sigma_{kl}}{h_{kl}}, & l \in \mathcal{N}_k \\
0, & \text{else}
\end{cases}
\]

\[
m_{kl} = \begin{cases} 
|\omega_k| & l = k \\
0, & \text{else}
\end{cases}
\]

\[\Rightarrow \theta A + M \text{ is strictly diagonally dominant!}\]
**Lecture 20 Slide 32**

**A matrix norm estimate**

**Lemma:** Assume $A$ has positive main diagonal entries, nonpositive off-diagonal entries and row sum zero. Then, $\|(I + A)^{-1}\|_\infty \leq 1$

**Proof:** Assume that $\|(I + A)^{-1}\|_\infty > 1$. $I + A$ is a irreducible $M$-matrix, thus $(I + A)^{-1}$ has positive entries. Then for $\alpha_{ij}$ being the entries of $(I + A)^{-1}$,

$$\max_{i=1}^n \sum_{j=1}^n \alpha_{ij} > 1.$$

Let $k$ be a row where the maximum is reached. Let $e = (1 \ldots 1)^T$. Then for $v = (I + A)^{-1}e$ we have that $v > 0$, $v_k > 1$ and $v_k \geq v_j$ for all $j \neq k$. The $k$th equation of $e = (I + A)v$ then looks like

$$1 = v_k + v_k \sum_{j \neq k} |a_{kj}| - \sum_{j \neq k} |a_{kj}|v_j$$

$$\geq v_k + v_k \sum_{j \neq k} |a_{kj}| - \sum_{j \neq k} |a_{kj}|v_k$$

$$= v_k$$

$$> 1$$
Stability estimate

- Matrix equation again:
  \[ u^n + \tau^n M^{-1} \theta Au^n = u^{n-1} + \tau^n M^{-1} (\theta - 1) Au^{n-1} =: B^n u^{n-1} \]
  \[ u^n = (I + \tau^n M^{-1} \theta A)^{-1} B^n u^{n-1} \]

- From the lemma we have \( \| (I + \tau^n M^{-1} \theta A)^n \|_\infty \leq 1 \)
  \[ \Rightarrow \| u^n \|_\infty \leq \| B^n u^{n-1} \|_\infty. \]

- For the entries \( b_{kl}^n \) of \( B^n \), we have
  \[ b_{kl}^n = \begin{cases} 
  1 + \frac{\tau^n}{m_{kk}} (\theta - 1) a_{kk}, & k = l \\
  \frac{\tau^n}{m_{kk}} (\theta - 1) a_{kl}, & \text{else} 
  \end{cases} \]
  
  In any case, \( b_{kl} \geq 0 \) for \( k \neq l \).
  
  If \( b_{kk} \geq 0 \), one estimates \( \| B \|_\infty = \max_{k=1}^N \sum_{l=1}^N b_{kl} \).

- But
  \[ \sum_{l=1}^N b_{kl} = 1 + (\theta - 1) \frac{\tau^n}{m_{kk}} \left( a_{kk} + \sum_{l \in \mathcal{N}_k} a_{kl} \right) = 1 \quad \Rightarrow \| B \|_\infty = 1. \]
Stability conditions

- For a shape regular triangulation in $\mathbb{R}^d$, we can assume that $m_{kk} = |\omega_k| \sim h^d$, and $a_{kl} = \frac{|\sigma_{kl}|}{h_{kl}} \sim \frac{h^{d-1}}{h} = h^{d-2}$, thus $\frac{a_{kk}}{m_{kk}} \leq \frac{1}{Ch^2}$.

- $b_{kk} \geq 0$ gives

$$ (1 - \theta) \tau^n \frac{a_{kk}}{m_{kk}} \leq 1 $$

- A sufficient condition is that for some $C > 0$,

$$ (1 - \theta) \tau^n \frac{1}{Ch^2} \leq 1 $$

$$ (1 - \theta) \tau^n \leq Ch^2 $$

- Method stability:
  - Implicit Euler: $\theta = 1 \Rightarrow$ unconditional stability !
  - Explicit Euler: $\theta = 0 \Rightarrow$ CFL condition $\tau \leq Ch^2$
  - Crank-Nicolson: $\theta = \frac{1}{2} \Rightarrow$ CFL condition $\tau \leq 2Ch^2$

  Tradeoff stability vs. accuracy.
Stability discussion

- \( \tau \leq Ch^2 \) CFL == “Courant-Friedrichs-Levy”
- Explicit (forward) Euler method can be applied on very fast systems (GPU), with small time step comes a high accuracy in time.
- Implicit Euler: unconditional stability – helpful when stability is of utmost importance, and accuracy in time is less important
- For hyperbolic systems (pure convection without diffusion), the CFL conditions is \( \tau \leq Ch \), thus in this case explicit computations are ubiquitous
- Comparison for a fixed size of the time interval. Assume for implicit Euler, time accuracy is less important, and the number of time steps is independent of the size of the space discretization.

<table>
<thead>
<tr>
<th></th>
<th>1D</th>
<th>2D</th>
<th>3D</th>
</tr>
</thead>
<tbody>
<tr>
<td># unknowns</td>
<td>( N = O(h^{-1}) )</td>
<td>( N = O(h^{-2}) )</td>
<td>( N = O(h^{-3}) )</td>
</tr>
<tr>
<td># steps</td>
<td>( M = O(N^2) )</td>
<td>( M = O(N) )</td>
<td>( M = O(N^{2/3}) )</td>
</tr>
<tr>
<td>complexity</td>
<td>( M = O(N^3) )</td>
<td>( M = O(N^2) )</td>
<td>( M = O(N^{5/3}) )</td>
</tr>
</tbody>
</table>

Complexity

- 1D: \( M = O(N^3) \)
- 2D: \( M = O(N^2) \)
- 3D: \( M = O(N^{5/3}) \)