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Lecture 18

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The Galerkin method II

- Let $V$ be a Hilbert space. Let $a : V \times V \to \mathbb{R}$ be a self-adjoint bilinear form, and $f$ a linear functional on $V$. Assume $a$ is coercive with coercivity constant $\alpha$, and continuity constant $\gamma$.

- Continuous problem: search $u \in V$ such that

  $$a(u, v) = f(v) \quad \forall v \in V$$

- Let $V_h \subset V$ be a finite dimensional subspace of $V$

- “Discrete” problem $\equiv$ Galerkin approximation: Search $u_h \in V_h$ such that

  $$a(u_h, v_h) = f(v_h) \quad \forall v_h \in V_h$$

By Lax-Milgram, this problem has a unique solution as well.
Céa’s lemma

- What is the connection between $u$ and $u_h$?
- Let $v_h \in V_h$ be arbitrary. Then

$$\alpha \|u - u_h\|^2 \leq a(u - u_h, u - u_h) \quad \text{(Coercivity)}$$

$$= a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h)$$

$$= a(u - u_h, u - v_h) \quad \text{(Galerkin Orthogonality)}$$

$$\leq \gamma \|u - u_h\| \cdot \|u - v_h\| \quad \text{(Boundedness)}$$

- As a result

$$\|u - u_h\| \leq \frac{\gamma}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|$$

- Up to a constant, the error of the Galerkin approximation is the error of the best approximation of the solution in the subspace $V_h$. 
From the Galerkin method to the matrix equation

- Let $\phi_1 \ldots \phi_n$ be a set of basis functions of $V_h$.
- Then, we have the representation $u_h = \sum_{j=1}^{n} u_j \phi_j$
- In order to search $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h) \quad \forall v_h \in V_h$$

it is actually sufficient to require

$$a(u_h, \phi_i) = f(\phi_i) \quad (i = 1 \ldots n)$$

$$a \left( \sum_{j=1}^{n} u_j \phi_j, \phi_i \right) = f(\phi_i) \quad (i = 1 \ldots n)$$

$$\sum_{j=1}^{n} a(\phi_j, \phi_i) u_j = f(\phi_i) \quad (i = 1 \ldots n)$$

$$AU = F$$

with $A = (a_{ij})$, $a_{ij} = a(\phi_i, \phi_j)$, $F = (f_i)$, $f_i = F(\phi_i)$, $U = (u_i)$.
- Matrix dimension is $n \times n$. Matrix sparsity?
Obtaining a finite dimensional subspace

- Let $\Omega = (a, b) \subset \mathbb{R}^1$
- Let $a(u, v) = \int_a^b \delta \vec{\nabla} u \vec{\nabla} vd\vec{x} + \alpha u(a)v(a) + \alpha u(b)v(b)$
- Calculus 101 provides a finite dimensional subspace: the space of sin/cos functions up to a certain frequency $\Rightarrow$ spectral method
- Ansatz functions have global support $\Rightarrow$ full $n \times n$ matrix
- OTOH: rather fast convergence for smooth data
- Generalization to higher dimensions possible
- Big problem in irregular domains: we need the eigenfunction basis of some operator...
- Spectral methods are successful in cases where one has regular geometry structures and smooth/constant coefficients – e.g. “Spectral Einstein Code”
The finite element idea I

- Choose basis functions with local support. \(\Rightarrow\) only integrals of basis function pairs with overlapping support contribute to matrix.

- Linear finite elements in \(\Omega = (a, b) \subset \mathbb{R}^1:\)

- Partition \(a = x_1 \leq x_2 \leq \cdots \leq x_n = b\)

- Basis functions (for \(i = 1 \ldots n\))

\[
\phi_i(x) = \begin{cases} 
\frac{x-x_{i-1}}{x_i-x_{i-1}}, & i > 1, x \in (x_{i-1}, x_i) \\
\frac{x_{i+1}-x}{x_{i+1}-x_i}, & i < n, x \in (x_i, x_{i+1}) \\
0, & \text{else}
\end{cases}
\]
Any function \( u_h \in V_h = \text{span}\{\phi_1 \ldots \phi_n\} \) is piecewise linear, and the coefficients in the representation \( u_h = \sum_{i=1}^{n} u_i \phi_i \) are the values \( u_h(x_i) \).

Fortunately, we are working with a weak formulation, and weak derivatives are well defined (and coincide with the classical derivatives where the basis functions are smooth).

Let \( \phi_i \), \( \phi_j \) be two basis functions, regard

\[
s_{ij} = \int_{a}^{b} \vec{\nabla} \phi_i \cdot \vec{\nabla} \phi_j \, dx
\]

We have \( \text{supp} \phi_i \cap \text{supp} \phi_j = \emptyset \) unless \( i = j \), \( i + 1 = j \) or \( i - 1 = j \).

Therefore \( s_{ij} = 0 \) unless \( i = j \), \( i + 1 = j \) or \( i - 1 = j \).
FE matrix elements for 1D heat equation II

- Let \( j = i + 1 \). Then \( \text{supp } \phi_i \cap \text{supp } \phi_j = (x_i, x_{i+1}) \), \( \phi'_i = -\frac{1}{h} \), \( \phi'_j = \frac{1}{h} \) where \( h = x_{i+1} - x_i \)

\[
\int_a^b \vec{\nabla} \phi_i \vec{\nabla} \phi_j d\vec{x} = \int_{x_i}^{x_{i+1}} \phi'_i \phi'_j d\vec{x} = -\int_{x_i}^{x_{i+1}} \frac{1}{h^2} d\vec{x} = -\frac{1}{h}
\]

- Similarly, for \( j = i - 1 \): \( \int_a^b \vec{\nabla} \phi_i \vec{\nabla} \phi_j d\vec{x} = -\frac{1}{h} \)
- For \( 1 < i < N \):

\[
\int_a^b \vec{\nabla} \phi_i \vec{\nabla} \phi_i d\vec{x} = \int_{x_{i-1}}^{x_{i+1}} (\phi'_i)^2 d\vec{x} = \int_{x_{i-1}}^{x_{i+1}} \frac{1}{h^2} d\vec{x} = \frac{2}{h}
\]

- For \( i = 1 \) or \( i = N \): \( \int_a^b \vec{\nabla} \phi_i \vec{\nabla} \phi_i d\vec{x} = \frac{1}{h} \)
- For the right hand side, calculate vector elements \( f_i = \int_a^b f(x) \phi_i dx \) using a quadrature rule.
Adding the boundary integrals yields

\[ A = \begin{pmatrix} \alpha + \frac{1}{h} & -\frac{1}{h} & \cdot & \cdot & \cdot \\ -\frac{1}{h} & \frac{2}{h} & \frac{1}{h} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -\frac{1}{h} & \frac{2}{h} & \frac{1}{h} & -\frac{1}{h} & \cdot \\ -\frac{1}{h} & \frac{2}{h} & \frac{1}{h} & -\frac{1}{h} & -\frac{1}{h} + \alpha \end{pmatrix} \]

... the same matrix as for the finite difference and finite volume methods
Simplices

- Let \( \{\vec{a}_1, \ldots, \vec{a}_{d+1}\} \subset \mathbb{R}^d \) such that the \( d \) vectors \( \vec{a}_2 - \vec{a}_1, \ldots, \vec{a}_{d+1} - \vec{a}_1 \) are linearly independent. Then the convex hull \( K \) of \( \vec{a}_1, \ldots, \vec{a}_{d+1} \) is called a simplex, and \( \vec{a}_1, \ldots, \vec{a}_{d+1} \) are called vertices of the simplex.

- **Unit simplex**: \( \vec{a}_1 = (0, \ldots, 0), \vec{a}_1 = (0,1, \ldots, 0) \ldots, \vec{a}_{d+1} = (0, \ldots, 0,1) \).

\[
K = \left\{ \vec{x} \in \mathbb{R}^d : x_i \geq 0 \ (i = 1 \ldots d) \text{ and } \sum_{i=1}^d x_i \leq 1 \right\}
\]

- A general simplex can be defined as an image of the unit simplex under some affine transformation

- \( F_i \): face of \( K \) opposite to \( \vec{a}_i \)

- \( \vec{n}_i \): outward normal to \( F_i \)
Simplex characteristics

- Diameter of $K$: $h_K = \max_{\vec{x}_1, \vec{x}_2 \in K} ||\vec{x}_1 - \vec{x}_2||$
  $\equiv$ length of longest edge if $K$

- $\rho_K$: diameter of largest ball that can be inscribed into $K$

- $\sigma_K = \frac{h_K}{\rho_K}$: local shape regularity measure
  - $\sigma_K = 2\sqrt{3}$ for equilateral triangle
  - $\sigma_K \to \infty$ if largest angle approaches $\pi$. 
Barycentric coordinates

Definition: Let \( K \subset \mathbb{R}^d \) be a \( d \)-simplex given by the points \( \vec{a}_1 \ldots \vec{a}_{d+1} \). Let \( \Lambda(\vec{x}) = (\lambda_1(\vec{x}) \ldots \lambda_{d+1}(\vec{x})) \) be a vector such that for all \( \vec{x} \in \mathbb{R}^d \)

\[
\sum_{j=1}^{d+1} \vec{a}_j \lambda_j(\vec{x}) = \vec{x}, \quad \sum_{j=1}^{d+1} \lambda_j(\vec{x}) = 1
\]

This vector is called the vector of barycentric coordinates of \( \vec{x} \) with respect to \( K \).
Lemma The barycentric coordinates of a given point is well defined and unique. Moreover, for the simplex edges \( \vec{a}_i \), one has

\[
\lambda_j(\vec{a}_i) = \delta_{ij}
\]

Proof: The definition of \( \Lambda \) given by a \((d + 1) \times (d + 1)\) system of equations with the matrix

\[
M = \begin{pmatrix}
a_{1,1} & a_{2,1} & \cdots & a_{d+1,1} \\
a_{1,2} & a_{2,2} & \cdots & a_{d+1,2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1,d} & a_{2,d} & \cdots & a_{d+1,d} \\
1 & 1 & \cdots & 1
\end{pmatrix}
\]

Subtracting the first column from the others gives

\[
M' = \begin{pmatrix}
a_{1,1} & a_{2,1} - a_{1,1} & \cdots & a_{d+1,1} - a_{1,1} \\
a_{1,2} & a_{2,2} - a_{1,2} & \cdots & a_{d+1,2} - a_{1,2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1,d} & a_{2,d} - a_{1,d} & \cdots & a_{d+1,d} - a_{1,d} \\
1 & 0 & \cdots & 0
\end{pmatrix}
\]
Barycentric coordinates III

\( \det M = \det M' \) is the determinant of the matrix whose columns are the edge vectors of \( K \) which are linearly independent.

For the simplex edges one has

\[
\sum_{j=1}^{d+1} \tilde{a}_j \lambda_j(\tilde{a}_i) = \tilde{a}_i
\]

which is fulfilled if \( \lambda_j(\tilde{a}_i) = 1 \) for \( i = j \) and \( \lambda_j(\tilde{a}_i) = 0 \) for \( i \neq j \). And we have uniqueness. \( \square \)

At the same time, the measure (area) is calculated as \( |K| = \frac{1}{d!} |\det M'| \).
Let $K_j(\vec{x})$ be the subsimplex of $K$ made of $\vec{x}$ and $\vec{a}_1 \ldots \vec{a}_{d+1}$ with $\vec{a}_j$ omitted.

Its measure $|K_j(\vec{x})|$ is established from its determinant and a linear function of the coordinates for $\vec{x}$.

One has $\frac{|K_j(\vec{a}_i)|}{|K|} = \delta_{ij}$ and therefore,

$$\lambda_j(\vec{x}) = \frac{|K_j(\vec{x})|}{|K|}$$

is the ratio of the measures of $K_j(\vec{x})$ and $K$. 

\[ \triangle \]
Conformal triangulations II

- $d = 1$: Each intersection $F = K_m \cap K_n$ is either empty or a common vertex
- $d = 2$: Each intersection $F = K_m \cap K_n$ is either empty or a common vertex or a common edge
- $d = 3$: Each intersection $F = K_m \cap K_n$ is either empty or a common vertex or a common edge or a common face
- Delaunay triangulations are conformal
Now we discuss a family of meshes $\mathcal{T}_h$ for $h \to 0$.

For given $\mathcal{T}_h$, assume that $h = \max_{K \in \mathcal{T}_h} h_K$.

A family of meshes is called shape regular if

$$\forall h, \forall K \in \mathcal{T}_h, \sigma_K = \frac{h_K}{\rho_K} \leq \sigma_0$$

In 1D, $\sigma_K = 1$

In 2D, $\sigma_K \leq \frac{2}{\sin \theta_K}$ where $\theta_K$ is the smallest angle.
Polynomial space $\mathbb{P}_k$

- Space of polynomials in $x_1 \ldots x_d$ of total degree $\leq k$ with real coefficients $\alpha_{i_1 \ldots i_d}$:

$$\mathbb{P}_k = \left\{ p(x) = \sum_{0 \leq i_1 \ldots i_d \leq k \atop i_1 + \ldots + i_d \leq k} \alpha_{i_1 \ldots i_d} x_1^{i_1} \ldots x_d^{i_d} \right\}$$

- Dimension:

$$\dim \mathbb{P}_k = \binom{d + k}{k} = \begin{cases} k + 1, & d = 1 \\ \frac{1}{2}(k + 1)(k + 2), & d = 2 \\ \frac{1}{6}(k + 1)(k + 2)(k + 3), & d = 3 \end{cases}$$

$$\dim \mathbb{P}_1 = d + 1$$

$$\dim \mathbb{P}_2 = \begin{cases} 3, & d = 1 \\ 6, & d = 2 \\ 10, & d = 3 \end{cases}$$
\( P_k \) simplex finite elements

- \( K \): simplex spanned by \( \vec{a}_1 \ldots \vec{a}_{d+1} \) in \( \mathbb{R}^d \)
- For \( 0 \leq i_1 \ldots i_{d+1} \leq k \), \( i_1 + \cdots + i_{d+1} = k \), let the set of nodes \( \Sigma = \{ \vec{\sigma}_1 \ldots \vec{\sigma}_s \} \) be defined by the points \( \vec{a}_{i_1 \ldots i_{d+1} k} \) with barycentric coordinates \( \left( \frac{i_1}{k} \ldots \frac{i_{d+1}}{k} \right) \).

\[
\begin{array}{ccc}
P_1 & P_2 & P_3 \\
\end{array}
\]

- \( s = \text{card } \Sigma = \dim P_K \Rightarrow \) there exists a basis \( \theta_1 \ldots \theta_s \) of \( P_k \) such that \( \theta_i(\vec{\sigma}_j) = \delta_{ij} \)
$P_1$ simplex finite elements

- $K$: simplex spanned by $a_1 \ldots a_{d+1}$ in $\mathbb{R}^d$
- $s = d + 1$
- Nodes $\equiv$ vertices
- Basis functions $\theta_1 \ldots \theta_{d+1} \equiv$ barycentric coordinates $\lambda_1 \ldots \lambda_{d+1}$
Global degrees of freedom

- Given a triangulation $\mathcal{T}_h$
- Let $\{\vec{a}_1 \ldots \vec{a}_N\} = \bigcup_{K \in \mathcal{T}_h} \{\vec{\sigma}_{K,1} \ldots \vec{\sigma}_{K,s}\}$ be the set of global degrees of freedom.
- Degree of freedom map
  
  $j : \mathcal{T}_h \times \{1 \ldots s\} \rightarrow \{1 \ldots N\}$
  
  $(K, m) \mapsto j(K, m)$ the global degree of freedom number
Lagrange finite element space

Given a triangulation $T_h$ of $\Omega$, define the spaces

$$P_h^k = \{ v_h \in C^0(\Omega) : v_h|_K \in \mathbb{P}_k \ \forall K \in T \} \subset H^1(\Omega)$$

$$P_{0,h}^k = \{ v_h \in P_h^k : v_h|_{\partial \Omega} = 0 \} \subset H^1_0(\Omega)$$

Global shape functions $\theta_1, \ldots, \theta_N \in P_h^k$ defined by

$$\phi_i|_K(\vec{a}_K, m) = \begin{cases} 
\delta_{mn} & \text{if } \exists n \in \{1 \ldots s\} : j(K, n) = i \\
0 & \text{otherwise}
\end{cases}$$

$$\{\phi_1, \ldots, \phi_N\}$$ is a basis of $P_h$, and $\gamma_1 \ldots \gamma_N$ is a basis of $L(P_h, \mathbb{R})$:

- $\{\phi_1, \ldots, \phi_N\}$ are linearly independent: if $\sum_{j=1}^{N} \alpha_j \phi_j = 0$ then evaluation at $\vec{a}_1 \ldots \vec{a}_N$ yields that $\alpha_1 \ldots \alpha_N = 0$.

- Let $v_h \in P_h$. Let $w_h = \sum_{j=1}^{N} v_h(\vec{a}_j) \phi_j$. Then for all $K \in T_h$, $v_h|_K$ and $w_h|_K$ coincide in the local nodes $\vec{a}_{K,1} \ldots \vec{a}_{K,2}$, $\Rightarrow v_h|_K = w_h|_K$. 
We have

<table>
<thead>
<tr>
<th>d</th>
<th>k</th>
<th>( N = \dim P_h^k )</th>
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<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>( N_v )</td>
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<tr>
<td>1</td>
<td>2</td>
<td>( N_v + N_{el} )</td>
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<tr>
<td>1</td>
<td>3</td>
<td>( N_v + 2N_{el} )</td>
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<tr>
<td>2</td>
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<td>( N_v + N_{ed} )</td>
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<td>2</td>
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<td>( N_v + 2N_{ed} + N_{el} )</td>
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<td>3</td>
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<td>( N_v + N_{ed} )</td>
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<td>3</td>
<td>3</td>
<td>( N_v + 2N_{ed} + N_f )</td>
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Local Lagrange interpolation operator

Let \( \{K, P, \Sigma\} \) be a finite element with shape function bases \( \{\theta_1 \ldots \theta_s\} \). Let \( V(K) = \mathbb{C}^0(K) \) and \( P \subset V(K) \).

local interpolation operator

\[
I_K : V(K) \to P
\]

\[
v \mapsto \sum_{i=1}^{s} v(\vec{\sigma}_i) \theta_i
\]

\( P \) is invariant under the action of \( I_K \), i.e. \( \forall p \in P, I_K(p) = p \):

Let \( p = \sum_{j=1}^{s} \alpha_j \theta_j \) Then,

\[
I_K(p) = \sum_{i=1}^{s} p(\vec{\sigma}_i) \theta_i = \sum_{i=1}^{s} \sum_{j=1}^{s} \alpha_j \theta_j(\vec{\sigma}_i) \theta_i
\]

\[
= \sum_{i=1}^{s} \sum_{j=1}^{s} \alpha_j \delta_{ij} \theta_i = \sum_{j=1}^{s} \alpha_j \theta_j
\]
Global Lagrange interpolation operator

Let $V_h = P_h^k$

\[ I_h : C^0(\bar{\Omega}_h) \rightarrow V_h \]

\[ v(\vec{x}) \mapsto v_h(\vec{x}) = \sum_{i=1}^{N} v(\vec{a}_i) \phi_i(\vec{x}) \]
Theorem: Let \( \{ \hat{K}, \hat{P}, \hat{\Sigma} \} \) be a finite element with associated normed vector space \( V(\hat{K}) \). Assume that

\[
\mathbb{P}_k \subset \hat{P} \subset H^2(\hat{K}) \subset V(\hat{K})
\]

Then there exists \( c > 0 \) such that for all \( m = 0 \ldots 2, \, K \in \mathcal{T}_h, \, \nu \in H^2(K) \):

\[
|\nu - I^1_K \nu|_{m,K} \leq ch^{2-m}K^{m}\sigma_K |\nu|_{2,K}.
\]

I.e. the local interpolation error can be estimated through \( h_K, \sigma_K \) and the norm of a higher derivative.
Local interpolation: special cases

- $m = 0$: $|v - I_K^1 v|_{0,K} \leq ch_K^2 |v|_{2,K}$
- $m = 1$: $|v - I_K^1 v|_{1,K} \leq ch_K \sigma_K |v|_{2,K}$
Global interpolation error estimate for Lagrangian finite elements, $k = 1$

- Assume $v \in H^2(\Omega)$, e.g. if problem coefficients are smooth and the domain is convex

\[
\|v - I_h^1 v\|_{0, \Omega} + h \|v - I_h^1 v\|_{1, \Omega} \leq ch^2 \|v\|_{2, \Omega} \\
|v - I_h^1 v|_{1, \Omega} \leq ch \|v\|_{2, \Omega}
\]

\[
\lim_{h \to 0} \left( \inf_{v_h \in V_h^1} |v - v_h|_{1, \Omega} \right) = 0
\]

- If $v \in H^2(\Omega)$ cannot be guaranteed, estimates become worse. Example: L-shaped domain.

- These results immediately can be applied in Cea’s lemma.
Search $u \in H^1_0(\Omega)$ such that

$$
\int_{\Omega} \delta \vec{\nabla} u \vec{\nabla} v \, d\vec{x} = \int_{\Omega} f v \, d\vec{x} \quad \forall v \in H^1_0(\Omega)
$$

Then, $\lim_{h \to 0} ||u - u_h||_{1,\Omega} = 0$. If $u \in H^2(\Omega)$ (e.g. on convex domains) then

$$
||u - u_h||_{1,\Omega} \leq ch ||u||_{2,\Omega}, \\
||u - u_h||_{0,\Omega} \leq ch^2 ||u||_{2,\Omega}
$$

Under certain conditions (convex domain, smooth coefficients) one also has

$$
||u - u_h||_{0,\Omega} \leq ch ||u||_{1,\Omega}
$$

("Aubin-Nitsche-Lemma")
$H^2$-Regularity

- $u \in H^2(\Omega)$ may be *not* fulfilled e.g.
  - if $\Omega$ has re-entrant corners
  - if on a smooth part of the domain, the boundary condition type changes
  - if problem coefficients ($\delta$) are discontinuous
- Situations differ as well between two and three space dimensions
- Delicate theory, ongoing research in functional analysis
- Consequence for simulations
  - Deterioration of convergence rate
  - Remedy: local refinement of the discretization mesh
    - using a priori information
    - using a posteriori error estimators + automatic refinement of discretization mesh
Search \( u \in V = H^1_0(\Omega) \) such that

\[
\int_{\Omega} \delta \vec{\nabla} u \vec{\nabla} v \, d\vec{x} = \int_{\Omega} f v \, d\vec{x} \quad \forall v \in H^1_0(\Omega)
\]

Then,

\[
a(u, v) := \int_{\Omega} \delta \vec{\nabla} u \vec{\nabla} v \, d\vec{x}
\]

is a self-adjoint bilinear form defined on the Hilbert space \( H^1_0(\Omega) \).
Galerkin ansatz

- Let $V_h \subset V$ be a finite dimensional subspace of $V$
- “Discrete” problem $\equiv$ Galerkin approximation:
  Search $u_h \in V_h$ such that
  \[
  a(u_h, v_h) = f(v_h) \quad \forall v_h \in V_h
  \]
- E.g. $V_h$ is the space of P1 Lagrange finite element approximations
Stiffness matrix for Laplace operator for P1 FEM

- Element-wise calculation:
  \[ a_{ij} = a(\phi_i, \phi_j) = \int_{\Omega} \vec{\nabla} \phi_i \vec{\nabla} \phi_j \, d\mathbf{x} = \int_{\Omega} \sum_{K \in T_h} \vec{\nabla} \phi_i|_K \vec{\nabla} \phi_j|_K \, d\mathbf{x} \]

- Standard assembly loop:
  \[
  \text{for } i, j = 1 \ldots N \text{ do} \\
  \quad \text{set } a_{ij} = 0 \\
  \text{end} \\
  \text{for } K \in T_h \text{ do} \\
  \quad \text{for } m,n=0\ldots d \text{ do} \\
  \quad \quad s_{mn} = \int_K \vec{\nabla} \lambda_m \vec{\nabla} \lambda_n \, d\mathbf{x} \\
  \quad \quad a_{j \text{dof}}(K,m),j \text{dof}(K,n) = a_{j \text{dof}}(K,m),j \text{dof}(K,n) + s_{mn} \\
  \quad \text{end} \\
  \text{end} \\

- Local stiffness matrix:
  \[ S_K = (s_K; m,n) = \int_K \vec{\nabla} \lambda_m \vec{\nabla} \lambda_n \, d\mathbf{x} \]
Local stiffness matrix calculation for P1 FEM

- $a_0 \ldots a_d$: vertices of the simplex $K$, $a \in K$.
- Barycentric coordinates: $\lambda_j(a) = \frac{|K_j(a)|}{|K|}$
- For indexing modulo $d+1$ we can write
  
  $|K| = \frac{1}{d!} \det (a_{j+1} - a_j, \ldots a_{j+d} - a_j)$
  
  $|K_j(a)| = \frac{1}{d!} \det (a_{j+1} - a, \ldots a_{j+d} - a)$

- From this information, we can calculate explicitly $\vec{\nabla} \lambda_j(x)$ (which are constant vectors due to linearity) and the corresponding entries of the local stiffness

  $$s_{ij} = \int_K \vec{\nabla} \lambda_i \cdot \vec{\nabla} \lambda_j \, d\vec{x}$$
Local stiffness matrix calculation for P1 FEM in 2D

- \( a_0 = (x_0, y_0) \ldots a_d = (x_2, y_2) \): vertices of the simplex \( K \), \( a = (x, y) \in K \).

- Barycentric coordinates: \( \lambda_j(x, y) = \frac{|K_j(x, y)|}{|K|} \)

- For indexing modulo \( d+1 \) we can write

\[
|K| = \frac{1}{2} \det \begin{pmatrix}
x_{j+1} - x_j & x_{j+2} - x_j \\
y_{j+1} - y_j & y_{j+2} - y_j
\end{pmatrix}
\]

\[
|K_j(x, y)| = \frac{1}{2} \det \begin{pmatrix}
x_{j+1} - x & x_{j+2} - x \\
y_{j+1} - y & y_{j+2} - y
\end{pmatrix}
\]

- Therefore, we have

\[
|K_j(x, y)| = \frac{1}{2} \left( (x_{j+1} - x)(y_{j+2} - y) - (x_{j+2} - x)(y_{j+1} - y) \right)
\]

\[
\partial_x |K_j(x, y)| = \frac{1}{2} \left( (y_{j+1} - y) - (y_{j+2} - y) \right) = \frac{1}{2} (y_{j+1} - y_{j+2})
\]

\[
\partial_y |K_j(x, y)| = \frac{1}{2} \left( (x_{j+2} - x) - (x_{j+1} - x) \right) = \frac{1}{2} (x_{j+2} - x_{j+1})
\]
Local stiffness matrix calculation for P1 FEM in 2D II

\[ s_{ij} = \int_{K} \nabla \lambda_i \nabla \lambda_j \, d\vec{x} = \frac{|K|}{4|K|^2} \left( y_{i+1} - y_{i+2}, x_{i+2} - x_{i+1} \right) \left( y_{j+1} - y_{j+2}, x_{j+2} - x_{j+1} \right) \]

So, let \( V = \begin{pmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{pmatrix} \)

Then

\[ x_1 - x_2 = V_{00} - V_{01} \]
\[ y_1 - y_2 = V_{10} - V_{11} \]

and

\[ 2|K| \nabla \lambda_0 = \begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \end{pmatrix} = \begin{pmatrix} V_{10} - V_{11} \\ V_{01} - V_{00} \end{pmatrix} \]
\[ 2|K| \nabla \lambda_1 = \begin{pmatrix} y_2 - y_0 \\ x_0 - x_2 \end{pmatrix} = \begin{pmatrix} V_{11} \\ -V_{01} \end{pmatrix} \]
\[ 2|K| \nabla \lambda_2 = \begin{pmatrix} y_0 - y_1 \\ x_1 - x_0 \end{pmatrix} = \begin{pmatrix} -V_{10} \\ V_{00} \end{pmatrix} \]
Degree of freedom map representation for P1 finite elements

- List of global nodes $a_0 \ldots a_N$: two dimensional array of coordinate values with $N$ rows and $d$ columns
- Local-global degree of freedom map: two-dimensional array $C$ of index values with $N_{el}$ rows and $d + 1$ columns such that $C(i, m) = j_{dof}(K_i, m)$.
- The mesh generator triangle generates this information directly
Practical realization of boundary conditions

- Robin boundary value problem

\[- \nabla \cdot \delta \vec{\nabla} u = f \quad \text{in } \Omega\]
\[\delta \vec{\nabla} u + \alpha (u - g) = 0 \quad \text{on } \partial \Omega\]

- Weak formulation: search \( u \in H^1(\Omega) \) such that

\[\int_\Omega \delta \vec{\nabla} u \vec{\nabla} v \, d\vec{x} + \int_{\partial \Omega} \alpha uv \, ds = \int_\Omega fv \, d\vec{x} + \int_{\partial \Omega} \alpha gv \, ds \quad \forall v \in H^1(\Omega)\]

- In 2D, for \( P^1 \) FEM, boundary integrals can be calculated by trapezoidal rule without sacrificing approximation order
- Use Dirichlet penalty method to handle Dirichlet boundary conditions
More complicated integrals

- Assume non-constant right hand side $f$, space dependent heat conduction coefficient $\delta$.
- Right hand side integrals

$$f_i = \int_K f(x)\lambda_i(x) \, d\vec{x}$$

- $P^1$ stiffness matrix elements

$$a_{ij} = \int_K \delta(x) \vec{\nabla}\lambda_i \, \vec{\nabla}\lambda_j \, d\vec{x}$$

- $P^k$ stiffness matrix elements created from higher order ansatz functions
Quadrature rules

- **Quadrature rule:**
  \[ \int_K g(x) \, d\vec{x} \approx |K| \sum_{l=1}^{l_q} \omega_l g(\xi_l) \]

- \( \xi_l \): nodes, Gauss points
- \( \omega_l \): weights
- The largest number \( k \) such that the quadrature is exact for polynomials of order \( k \) is called *order* \( k_q \) of the quadrature rule, i.e.
  \[ \forall k \leq k_q, \forall p \in \mathbb{P}^k, \int_K p(x) \, d\vec{x} = |K| \sum_{l=1}^{l_q} \omega_l p(\xi_l) \]

- Error estimate:
  \[ \forall \phi \in C^{k_q+1}(K), \left| \frac{1}{|K|} \int_K \phi(x) \, d\vec{x} - \sum_{l=1}^{l_q} \omega_l g(\xi_l) \right| \leq c h_K^{k_q+1} \sup_{x \in K, |\alpha| = k_q+1} |\partial^\alpha \phi(x)| \]
Some common quadrature rules

Nodes are characterized by the barycentric coordinates

<table>
<thead>
<tr>
<th>$d$</th>
<th>$k_q$</th>
<th>$l_q$</th>
<th>Nodes</th>
<th>Weights</th>
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<td>1</td>
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<td>$\frac{1}{2}, \frac{1}{2}$</td>
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<td>$(\frac{1}{2} + \frac{\sqrt{3}}{6}, \frac{1}{2} - \frac{\sqrt{3}}{6}), (\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} + \frac{\sqrt{3}}{6})$</td>
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<td>$\frac{8}{18}, \frac{5}{18}, \frac{5}{18}$</td>
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</tr>
<tr>
<td></td>
<td>1</td>
<td>3</td>
<td>$(1, 0, 0), (0, 1, 0), (0, 0, 1)$</td>
<td>$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$</td>
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<tr>
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<td>3</td>
<td>$(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})$</td>
<td>$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$</td>
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</tr>
<tr>
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<td>1</td>
<td>4</td>
<td>$(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$</td>
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<td>$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$</td>
</tr>
</tbody>
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“Variational crime”: instead of

\[ a(u_h, v_h) = f(v_h) \quad \forall v_h \in V_h \]

we solve

\[ a_h(u_h, v_h) = f_h(v_h) \quad \forall v_h \in V_h \]

where \( a_h, f_h \) are derived from their exact counterparts by quadrature.

For \( P^1 \) finite elements, zero order quadrature for volume integrals and first order quadrature for surface integrals is sufficient to keep the convergence order estimates stated before.

The rule of thumb for the volume quadrature is that the highest order terms must be evaluated exactly if the coefficients of the PDE are constant.
Practical realization of integrals

- Integral over barycentric coordinate function
  \[
  \int_K \lambda_i(x) \, d\vec{x} = \frac{1}{3} |K|
  \]

- Right hand side integrals. Assume \( f(x) \) is given as a piecewise linear function with given values in the nodes of the triangulation
  \[
  f_i = \int_K f(x) \lambda_i(x) \, d\vec{x} \approx \frac{1}{3} |K| (f(a_0) + f(a_1) + f(a_2))
  \]

- Integral over space dependent heat conduction coefficient: Assume \( \delta(x) \) is given as a piecewise linear function with given values in the nodes of the triangulation
  \[
  a_{ij} = \int_K \delta(x) \nabla \lambda_i \cdot \nabla \lambda_j \, d\vec{x} = \frac{1}{3} (\delta(a_0) + \delta(a_1) + \delta(a_2)) \int_K \nabla \lambda_i \cdot \nabla \lambda_j \, d\vec{x}
  \]
Homogeneous dirichlet boundary value problem

\[-\nabla \cdot \delta \nabla u = f \quad \text{in } \Omega \]
\[u|_{\partial \Omega} = 0\]

Lagrange degrees of freedom \(a_1 \ldots a_N\) corresponding to global basis functions \(\phi_1 \ldots \phi_N\) such that \(\phi_i|_{\partial \Omega} = 0\) aka \(\phi_i \in V_h \subset H_0^1(\Omega)\)

Stiffness matrix \(A = (a_{ij})\):

\[a_{ij} = a(\phi_i, \phi_j) = \int_{\Omega} \delta \nabla \phi_i \nabla \phi_j \, d\vec{x}\]

bilinear form \(a(\cdot, \cdot)\) is self-adjoint, therefore \(A\) is symmetric, positive definite

Condition number estimate for \(P^1\) finite elements on quasi-uniform triangulation:

\[\kappa(A) \leq ch^{-2}\]
The problem with Dirichlet boundary conditions

- Homogeneous Dirichlet BC $\Rightarrow$ include boundary condition into set of basis functions
- Inhomogeneous Dirichlet, may be only at a part of the boundary
  - Use exact approach from as in continuous formulation (with lifting $u_g$ etc) $\Rightarrow$ highly technical
  - Eliminate Dirichlet BC algebraically after building of the matrix, i.e. fix “known unknowns” at the Dirichlet boundary $\Rightarrow$ highly technical
  - Modify matrix such that equations at boundary exactly result in Dirichlet values $\Rightarrow$ loss of symmetry of the matrix
- Penalty method
Dirichlet BC: Algebraic manipulation

- Assume 1D situation with BC $u_1 = g$
- From integration in $H^1$ regardless of boundary values:

$$AU = \begin{pmatrix} \frac{1}{h} & -\frac{1}{h} & -\frac{1}{h} \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

- Fix $u_1$ and eliminate:

$$A'U = \begin{pmatrix} \frac{2}{h} & -\frac{1}{h} & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\vdots \end{pmatrix} = \begin{pmatrix} f_2 + \frac{1}{h}g \\ f_3 \\vdots \end{pmatrix}$$

- $A'$ becomes idd and stays symmetric
- operation is quite technical
Dirichlet BC: Modify boundary equations

- From integration in $H^1$ regardless of boundary values:

$$AU = \begin{pmatrix} \frac{1}{h} & -\frac{1}{h} & \frac{1}{h} \\ \frac{1}{h} & \frac{1}{h} & -\frac{1}{h} \\ -\frac{1}{h} & -\frac{1}{h} & \frac{1}{h} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

- Modify equation at boundary to exactly represent Dirichlet values

$$A'U = \begin{pmatrix} \frac{1}{h} & 0 & -\frac{1}{h} \\ -\frac{1}{h} & \frac{1}{h} & -\frac{1}{h} \\ -\frac{1}{h} & -\frac{1}{h} & \frac{1}{h} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{h}g \\ f_2 \\ f_3 \end{pmatrix}$$

- $A'$ becomes idd
- loses symmetry ⇒ problem e.g. with CG method
Dirichlet BC: Discrete penalty trick

- From integration in $H^1$ regardless of boundary values:
  
  \[
  AU = \begin{pmatrix}
    \frac{1}{h} & -\frac{1}{h} & & \\
    -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \\
    & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} \\
    & & & \ddots \ddots \ddots \\
  \end{pmatrix}
  \begin{pmatrix}
    u_1 \\
    u_2 \\
    u_3 \\
    \vdots \\
  \end{pmatrix}
  = \begin{pmatrix}
    f_1 \\
    f_2 \\
    f_3 \\
    \vdots \\
  \end{pmatrix}
  \]

- Add penalty terms
  
  \[
  A'U = \begin{pmatrix}
    \frac{1}{\epsilon} + \frac{1}{h} & -\frac{1}{h} & & \\
    -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \\
    & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} \\
    & & & \ddots \ddots \ddots \\
  \end{pmatrix}
  \begin{pmatrix}
    u_1 \\
    u_2 \\
    u_3 \\
    \vdots \\
  \end{pmatrix}
  = \begin{pmatrix}
    f_1 + \frac{1}{\epsilon}g \\
    f_2 \\
    f_3 \\
    \vdots \\
  \end{pmatrix}
  \]

- $A'$ becomes idd, keeps symmetry, and the realization is technically easy.
- If $\epsilon$ is small enough, $u_1 = g$ will be satisfied exactly within floating point accuracy.
- Iterative methods should be initialized with Dirichlet values.
- Works for nonlinear problems, finite volume methods
Dirichlet penalty trick, general formulation

- Dirichlet boundary value problem
  \[-\nabla \cdot \delta \vec{\nabla} u = f \quad \text{in } \Omega \]
  \[u|_{\partial \Omega} = g\]

- We discussed approximation of Dirichlet problem by Robin problem
- Practical realization uses discrete approach for Lagrange degrees of freedom \(a_1 \ldots a_N\) corresponding to global basis functions \(\phi_1 \ldots \phi_N\):
- Search \(u_h = \sum_{i=1}^{N} u_i \phi_i \in V_h = \text{span}\{\phi_1 \ldots \phi_N\}\) such that
  \[AU + \Pi U = F + \Pi G\]

where
- \(U = (u_1 \ldots u_N)\)
- \(A = (a_{ij})\): stiffness matrix with \(a_{ij} = \int_\Omega \delta \vec{\nabla} \phi_i \vec{\nabla} \phi_j \, d\vec{x}\)
- \(F = \int_\Omega f \vec{\nabla} \phi_i \, d\vec{x}\)
- \(G = (g_i)\) with \(g_i = \begin{cases} g(a_i), & a_i \in \partial \Omega \\ 0, & \text{else} \end{cases}\)
- \(\Pi = (\pi_{ij})\) is a diagonal matrix with \(\pi_{ij} = \begin{cases} \frac{1}{\varepsilon}, & i = j, a_i \in \partial \Omega \\ 0, & \text{else} \end{cases}\)
Row sums:

$$\sum_{j=1}^{N} a_{ij} = \sum_{j=1}^{N} \int_{\Omega} \nabla \phi_i \cdot \nabla \phi_j \, dx = \int_{\Omega} \nabla \phi_i \cdot \nabla \left( \sum_{j=1}^{N} \phi_j \right) \, dx$$

$$= \int_{\Omega} \nabla \phi_i \cdot \nabla (1) \, dx$$

$$= 0$$
Local stiffness matrix $S_K$

\[
s_{ij} = \int_K \vec{\nabla} \lambda_i \vec{\nabla} \lambda_j \ dx = \frac{|K|}{2|K|^2} \left( y_{i+1} - y_{i+2}, x_{i+2} - x_{i+1} \right) \left( y_{j+1} - y_{j+2}, x_{j+2} - x_{j+1} \right)
\]

- Main diagonal entries positive
- Local contributions from element stiffness matrices: Scalar products of vectors orthogonal to edges. These are nonpositive if the angle between the edges are $\leq 90^\circ$
  - *weakly acute triangulation*: all triangle angles are less than $\leq 90^\circ$
- In fact, for constant coefficients, in 2D, Delaunay is sufficient, as contributions from opposite angles compensate each other
- All row sums are zero $\Rightarrow A$ is singular
- Matrix becomes irreducibly diagonally dominant if we add at least one positive value to the main diagonal, e.g. from Dirichlet BC
Stationary linear reaction-diffusion

- Assume additional process in each REV which produces or destroys species depending on the amount of species present with given rate $r$.

Search function $u : \Omega \to \mathbb{R}$ such that

$$-\nabla \cdot \delta \vec{\nabla} u + ru = f \quad \text{in} \Omega$$

$$\delta \vec{\nabla} u \cdot \vec{n} + \alpha (u - g) = 0 \quad \text{on} \partial \Omega$$

- Coercivity guaranteed e.g. for $\alpha \geq 0$, $r > 0$ which means species destruction

FEM formulation: search $u_h \in V_h = \text{span}\{\phi_1 \ldots \phi_N\}$ such that

$$\int_{\Omega} \delta \vec{\nabla} u_h \vec{\nabla} v_h \, d\vec{x} + \int_{\Omega} ru_h v_h \, d\vec{x} + \int_{\partial \Omega} \alpha u_h v_h \, ds$$

"stiffness matrix" \quad "mass matrix" \quad "boundary mass matrix"

$$= \int_{\Omega} f v_h \, d\vec{x} + \int_{\partial \Omega} \alpha g v_h \, ds \quad \forall v_h \in V_h$$

- Coercivity $+$ symmetry $\Rightarrow$ positive definiteness
Mass matrix properties

- Mass matrix (for \( r = 1 \)): \( M = (m_{ij}) \):
  \[
m_{ij} = \int_{\Omega} \phi_i \phi_j \, dx
  \]

- Self-adjoint, coercive bilinear form \( \Rightarrow M \) is symmetric, positive definite

- For a family of quasi-uniform, shape-regular triangulations, for every eigenvalue \( \mu \) one has the estimate
  \[
c_1 h^d \leq \mu \leq c_2 h^d
  \]

  \( T \Rightarrow \) condition number \( \kappa(M) \) bounded by constant independent of \( h \):
  \[
  \kappa(M) \leq c
  \]

- How to see this? Let \( u_h = \sum_{i=1}^{N} U_i \phi_i \), and \( \mu \) an eigenvalue (positive, real!) Then
  \[
  \| u_h \|_0^2 = (U, MU)_{\mathbb{R}^N} = \mu (U, U)_{\mathbb{R}^N} = \mu \| U \|_{\mathbb{R}^N}^2
  \]

  From quasi-uniformity we obtain
  \[
c_1 h^d \| U \|_{\mathbb{R}^N}^2 \leq \| u_h \|_0^2 \leq c_2 h^d \| U \|_{\mathbb{R}^N}^2
  \]
For $P^1$-finite elements, all integrals $m_{ij} = \int_{\Omega} \phi_i \phi_j \, dx$ are zero or positive, so we get positive off diagonal elements.

No $M$-Property!
Mass matrix lumping (P1 FEM)

- Local mass matrix for P1 FEM on element $K$
  (calculated by 2nd order exact edge midpoint quadrature rule):

$$ M_K = |K| \begin{pmatrix} \frac{1}{6} & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{6} \end{pmatrix} $$

- Lumping: sum up off diagonal elements to main diagonal, set off diagonal entries to zero

$$ \tilde{M}_K = |K| \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} $$

- Interpretation as change of quadrature rule to first order exact vertex based quadrature rule

- Loss of accuracy, gain of stability
Effect of mass matrix lumping

- For P1 FEM on weakly acute or Delaunay triangulations, mass matrix lumping can guarantee M-Property of system matrix.

- Adding a mass matrix which is not lumped yields a positive definite matrix (due to coercivity) and thus nonsingularity, but *destroys* $M$-property unless the absolute values of its off diagonal entries are less than those of $A$, i.e. for small $r$.

- Same situation with Robin boundary conditions and boundary mass matrix.

- Introducing the Dirichlet Penalty trick at continuous level without mass lumping would be disastrous.
The values of a piecewise linear function $u = u(x)$ can be defined in dependence of the values $u_i = u(p_i)$ at the vertices of the simplex by the expression

$$u(x) = \sum_{k=0}^{n} u_k \lambda_k(x)$$

The gradient of this function is given by

$$(7) \quad \nabla u(x) = \sum_{k=0}^{n} u_k \nabla \lambda_k(x)$$

where the $\nabla \lambda_k$ can be determined from the equations

$$\delta_{ij} = \frac{\partial x_i}{\partial x_j} = \sum_{k=0}^{n} p_{ki} \frac{\partial \lambda_k}{\partial x_j}, \quad 0 = \sum_{k=0}^{n} \frac{\partial \lambda_k}{\partial x_j} \quad (i, j = 1, \ldots, n)$$

(cf. (3)), i.e. we have to solve the linear systems
\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
p_0 & p_1 & \cdots & p_n
\end{pmatrix}
\begin{pmatrix}
\lambda_0(0) & \frac{\partial \lambda_0}{\partial x_1} & \cdots & \frac{\partial \lambda_0}{\partial x_n} \\
\lambda_1(0) & \frac{\partial \lambda_1}{\partial x_1} & \cdots & \frac{\partial \lambda_1}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_n(0) & \frac{\partial \lambda_n}{\partial x_1} & \cdots & \frac{\partial \lambda_n}{\partial x_n}
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1
\end{pmatrix}.
\]

So we get
\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
p_0 & p_1 & \cdots & p_n
\end{pmatrix}
\begin{pmatrix}
\partial_i \lambda_0 \\
\partial_i \lambda_1 \\
\vdots \\
\partial_i \lambda_n
\end{pmatrix}
= e_i = \begin{pmatrix}
0 \\
\vdots \\
1 \\
0
\end{pmatrix},
\]

\[
\partial_i \lambda_0 + \partial_i \lambda_1 + \cdots + \partial_i \lambda_n = e_{i,0}
\]
\[
p_0 \partial_i \lambda_0 + p_1 \partial_i \lambda_1 + \cdots + p_n \partial_i \lambda_n = e_{i,(1\ldots n)}
\]

Comparing this with the determination of the normals of the faces (cf.
(2)) we get the identities

$$\nabla \lambda_i = g_i, \quad \lambda_i(0) = d_i \quad (i = 0, \ldots, n).$$

and therefore

$$\nabla u(x) = \sum_{i=0}^{n} g_i u_i = \begin{pmatrix} g_0 & g_1 & \cdots & g_n \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{pmatrix}$$

holds. That means that

$$\sum_{i=0}^{n} g_i u(p_i) = \sum_{i=0}^{n} \frac{1}{h_i} a_i u(p_i)$$

can be considered as a discrete gradient of a function $u$. 