Scientific Computing TU Berlin Winter 2021/22 © Jürgen Fuhrmann
Notebook 16

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using LinearAlgebra
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Regular Splittings

Definiton

- $A = M - N$ is a regular splitting if
  - M is nonsingular
  - M^{-1}, N are nonnegative, i.e.~have nonnegative entries

- Regard the iteration $u_{k+1} = M^{-1}Nu_k + M^{-1}b$.
  - $B = I - M^{-1}A = M^{-1}N$ is a nonnegative matrix.

Theorem: Assume A is nonsingular, $A^{-1} \geq 0$, and $A = M - N$ is a regular splitting. Then $\rho(M^{-1}N) < 1$.

Proof: Let $B = M^{-1}N$. Then $A = M(I - B)$, therefore $I - B$ is nonsingular.

In addition

$$A^{-1}N = (M(I - M^{-1}N))^{-1}N = (I - M^{-1}N)^{-1}M^{-1}N = (I - B)^{-1}B$$

By Perron-Frobenius (for general matrices), $\rho(B)$ is an eigenvalue with a nonnegative eigenvector $\vec{x}$. Thus,

$$0 \leq A^{-1}N\vec{x} = \frac{\rho(B)}{1 - \rho(B)}\vec{x}$$

Therefore $0 \leq \rho(B) \leq 1$.\ Assume that $\rho(B) = 1$. Then there exists $\vec{x} \neq \mathbf{0}$ such that $B\vec{x} = \vec{x}$. Consequently, $(I - B)\vec{x} = \mathbf{0}$, contradicting the nonsingularity of $I - B$. Therefore, $\rho(B) < 1$. □

Convergence rate comparison

Corollary: $\rho(M^{-1}N) = \frac{\tau}{1+\tau}$ where $\tau = \rho(A^{-1}N)$.

Proof: Rearrange $\tau = \frac{\rho(B)}{1-\rho(B)}$ □

Corollary: Let $A^{-1} \geq 0$, $A = M_1 - N_1$ and $A = M_2 - N_2$ be regular splittings.

If $N_2 \geq N_1$, then $1 > \rho(M_2^{-1}N_2) \geq \rho(M_1^{-1}N_1)$.

Proof: $\tau_2 = \rho(A^{-1}N_2) \geq \rho(A^{-1}N_1) = \tau_1$

But $\frac{\tau}{1+\tau}$ is strictly increasing. □

- Let $A^{-1} \geq 0$, $A = D - E - F$, $D > 0$ diagonal, $E, F \geq 0$ upper resp. lower triangular parts.
- Jacobi: $M_J = D$, $N_J = E + F$. $M_J^{-1} > 0 \Rightarrow$ regular splitting
- Gauss-Seidel: $M_{GS} = D - E$, $N_{GS} = F \geq 0$. Show $M_{GS}^{-1} \geq 0$:

$$M_{GS} = \begin{pmatrix} d_{11} & -e_{12} & -e_{13} & \dots & -e_{1n} \\ & d_{22} & -e_{23} & \dots & -e_{2n} \\ & & \ddots & \ddots & \vdots \\ & & & d_{n-1,n-1} & -e_{n-1,n} \\ & & & \dots & d_{nn} \end{pmatrix}$$

Elimination steps for $M_{GS}v = r$:

$$v_n = \frac{r_n}{d_{nn}}, \qquad v_{n-1} = \frac{r_n + e_{n-1,n}v_n}{d_{n-1,n-1}} \dots$$

All coefficients are nonnegative $\Rightarrow M_{GS} - N_{GS}$: regular splitting

- $N_{GS} \leq N_J \Rightarrow \rho(M_{GS}^{-1}N_{GS}) \leq \rho(M_J^{-1}N_J)$

M-Matrices

Definition Let A be an $n \times n$ real matrix. A is called M-Matrix if

- (i) $a_{ij} \leq 0$ for $i \neq j$
- (ii) A is nonsingular
- (iii) $A^{-1} \geq 0$

Corollary: If A is an M-Matrix, then $A^{-1} > 0 \Leftrightarrow A$ is irreducible.

Proof: See Varga. □

Theorem: If A is an M-matrix, then its diagonal $D_A > 0$ is positive.

Proof: Let $C = A^{-1} \geq 0$. The $AC = I$ and $(AC)_{ii} = 1$.

$$\sum_{k=1}^{n} a_{ik}c_{ki} = 1$$

$$a_{ii}c_{ii} = 1 - \sum_{k=1,k\neq i}^{n} a_{ik}c_{ki} \geq 1$$

The last inequality is due to $c_{ki} \geq 0$ and $a_{ik} < 0$ for $k \neq i$. As $a_{ii}c_{ii} \geq 1$, neither factor can be 0. So $c_{ii} > 0$ and $a_{ii} > 0$.

Theorem: (Saad, Th. 1.31) Assume

- (i) $a_{ij} \leq 0$ for $i \neq j$
- (ii) $a_{ii} > 0$

Then A is an M-Matrix if and only if $\rho(I - D^{-1}A) < 1$.

Proof: See Saad. □

Main practical M-Matrix criterion

Corollary: Let A be sdd or idd. Assume that $a_{ii} > 0$ and $a_{ij} \leq 0$ for $i \neq j$. Then A is an M-Matrix.

Proof: We know that A is nonsingular, but we have to show $A^{-1} \geq 0$.

- Let $B = I - D^{-1}A$. Then $\rho(B) < 1$, therefore $I - B$ is nonsingular.
- We have for $k > 0$:

$$I - B^{k+1} = (I - B)(I + B + B^2 + \cdots + B^k)$$
$$(I - B)^{-1}(I - B^{k+1}) = (I + B + B^2 + \cdots + B^k)$$

The left hand side for $k \to \infty$ converges to $(I - B)^{-1}$, therefore

$$(I - B)^{-1} = \sum_{k=0}^{\infty} B^k$$

As $B \geq 0$, we have $(I - B)^{-1} = A^{-1}D \geq 0$. As $D > 0$ we must have $A^{-1} \geq 0$. □

M-Matrix comparison criterion

Theorem(Saad, Th. 1.33): Let A, B $n \times n$ matrices such that

- (i) $A \leq B$
- (ii) $b_{ij} \leq 0$ for $i \neq j$.

Then, if A is an M-Matrix, so is B.

Proof: From M-property of A and $A \leq B$ we have $0 < D_A \leq D_B$. We have $D_B - B \geq 0$ and

$$\begin{aligned} D_A - A &\geq D_B - B \\ I - D_A^{-1}A &\geq D_A^{-1}(D_B - B) \\ &\geq D_B^{-1}(D_B - B) \\ &\geq I - D_B^{-1}B =: G \geq 0 \end{aligned}$$

Perron-Frobenius $\Rightarrow \rho(G) = \rho(I - D_B^{-1}B) \leq \rho(I - D_A^{-1}A) < 1 \backslash \Rightarrow I - G$ is nonsingular. From the proof of the M-matrix criterion, $D_B^{-1}B = (I - G)^{-1} = \sum_{k=0}^{\infty} G^k \geq 0$. As $D_B > 0$, we get $B \geq 0$.

□

Corollary: $A \leq M_{GS} \Rightarrow M_{GS}$ is an M-Matrix.

- Given some matrix, we now have some nice recipies to establish nonsingularity and iterative method convergence:
- **Check if the matrix is irreducible.**
  - This is mostly the case for elliptic and parabolic PDEs.
- **Check if the matrix is strictly or irreducibly diagonally dominant**.
  - If yes, it is in addition nonsingular.
- **Check if main diagonal entries are positive and off-diagonal entries are nonpositive.**
  - If yes, in addition, the matrix is an M-Matrix, its inverse is nonnegative, and elementary iterative methods converge.
- These critera do not depend on the symmetry of the matrix!

$$Au = \begin{pmatrix} \alpha+\frac{1}{h} & -\frac{1}{h} & & & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & & \\ & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \\ & & & & -\frac{1}{h} & \frac{1}{h}+\alpha \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-2} \\ u_{N-1} \\ u_N \end{pmatrix} = f = \begin{pmatrix} \alpha v_1 \\ hf_2 \\ hf_3 \\ \vdots \\ hf_{N-2} \\ hf_{N-1} \\ \alpha v_n \end{pmatrix}$$

- idd
- positive main diagonal entries, nonpositive off-diagonal entries

$\Rightarrow A$ is nonsingular, has the M-property, and we can e.g. apply the Jacobi and Gauss-Seidel iterative method to solve it (ok, in 1D we already know this is a bad idea . . .).

$\Rightarrow$ for $f \geq 0$ and $v \geq 0$ it follows that $u \geq 0.\backslash \equiv$ heating and positive environment temperatures cannot lead to negative temperatures in the interior.

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pyplot (generic function with 1 method)
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```
begin
    using PlutoUI
    using PyPlot
    using HypertextLiteral
    using Markdown
    using PlutoUI

    function pyplot(f;width=3,height=3)
    clf()
    f()
    fig=gcf()
    fig.set_size_inches(width,height)
    fig
    end
end
```

```
begin

  highlight(mdstring,color)= htl"""<blockquote style="padding: 10px; background-color: $(color);">$(mdstring)</blockquote>"""

  macro important_str(s)  :(highlight(Markdown.parse($s),"#ffcccc")) end
  macro definition_str(s) :(highlight(Markdown.parse($s),"#ccccff")) end
  macro statement_str(s)  :(highlight(Markdown.parse($s),"#ccffcc")) end


  html"""
  <style>
   h1{background-color:#dddddd;  padding: 10px;}
   h2{background-color:#e7e7e7;  padding: 10px;}
   h3{background-color:#eeeeee;  padding: 10px;}
   h4{background-color:#f7f7f7;  padding: 10px;}
  </style>
"""
end

```