Scientific Computing TU Berlin Winter 2021/22 © Jürgen Fuhrmann
Notebook 15

```
using LinearAlgebra
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Diagonally dominant matrices
Practical nonsingularity criterion
Perron-Frobenius Theorem
Application to the Jacobi method

Diagonally dominant matrices

Let $A = (a_{ij})$ be an $n \times n$ matrix.

- A is *diagonally dominant* if for $i = 1 \dots n$, $|a_{ii}| \geq \sum_{\substack{j=1\dots n\\ j\neq i}} |a_{ij}|$
- A is *strictly diagonally dominant* (sdd) if for $i = 1 \dots n$, $|a_{ii}| > \sum_{\substack{j=1\dots n\\ j\neq i}} |a_{ij}|$
- A is *irreducibly diagonally dominant* (idd) if
  1. A is irreducible
  2. A is diagonally dominant: for $i = 1 \dots n$, $|a_{ii}| \geq \sum_{\substack{j=1\dots n\\ j\neq i}} |a_{ij}|$
  3. for at least one r, $1 \leq r \leq n$, $|a_{rr}| > \sum_{\substack{j=1\dots n\\ j\neq r}} |a_{rj}|$

Practical nonsingularity criterion

Theorem (Varga, Th. 1.21): Let A be strictly diagonally dominant or irreducibly diagonally dominant. Then A is nonsingular.

If in addition, $a_{ii} > 0$ is real for $i = 1 \dots n$, then all real parts of the eigenvalues of A are positive:

$$\mathrm{Re}\lambda_i > 0, \quad i = 1 \dots n$$

Proof

- Assume A strictly diagonally dominant. Then the union of the Gershgorin disks does not contain 0 and $\lambda = 0$ cannot be an eigenvalue $\Rightarrow A$ is nonsingular.
- As for the real parts, the union of the disks is $\bigcup_{i=1\dots n} \{\mu \in \mathbb{C} : |\mu - a_{ii}| \leq \Lambda_i\}$ and $\mathrm{Re}\mu$ must be larger than zero if μ should be contained.

- Assume A irreducibly diagonally dominant. Then, if 0 is an eigenvalue, it sits on the boundary of one of the Gershgorin disks.

  By Taussky theorem, we have $|a_{ii}| = \Lambda_i$ for all $i = 1 \dots n$.

  This is a contradiction as by definition there is at least one i such that $|a_{ii}| > \Lambda_i$
- Assume $a_{ii} > 0$, real. All real parts of the eigenvalues must be ≥ 0.

  Therefore, if a real part is 0, it lies on the boundary of at least one disk.

  By Taussky theorem it must be contained at the same time in the boundary of all the disks and in the imaginary axis.

  This contradicts the fact that there is at least one disk which does not touch the imaginary axis as by definition there is at least one i such that $|a_{ii}| > \Lambda_i$

□

Theorem: If A is complex hermitian or real symmetric, sdd or idd, with positive diagonal entries, it is positive definite.

Proof: All eigenvalues of A are real, and due to the nonsingularity criterion, they must be positive, so A is positive definite. □

Application to the heat conduction matrix:

$$A = \begin{pmatrix} \alpha + \frac{1}{h} & -\frac{1}{h} & & & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & & \\ & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} \\ & & & & -\frac{1}{h} & \frac{1}{h} + \alpha \end{pmatrix}$$

- A is idd $\Rightarrow A$ is nonsingular
- $\mathrm{diag} A$ is positive real $\Rightarrow$ eigenvalues of A have positive real parts
- A is real, symmetric $\Rightarrow A$ is positive definite

Perron-Frobenius Theorem

Definition: A real n-vector $\vec{x}$ is

- positive ($\vec{x} > 0$) if all entries of $\vec{x}$ are positive
- nonnegative ($\vec{x} \geq 0$) if all entries of $\vec{x}$ are nonnegative

Definition: A real $n \times n$ matrix A is

- positive ($A > 0$) if all entries of A are positive
- nonnegative ($A \geq 0$) if all entries of A are nonnegative

Theorem (Varga, Th. 2.7) Let $A \geq 0$ be an irreducible $n \times n$ matrix. Then

- (i) A has a positive real eigenvalue equal to its spectral radius $\rho(A)$.
- (ii)]To $\rho(A)$ there corresponds a positive eigenvector $\vec{x} > 0$.
- (iii) $\rho(A)$ increases when any entry of A increases.
- (iv) $\rho(A)$ is a simple eigenvalue of A.

Proof: See Varga. □

Each $n \times n$ matrix can be brought to the normal form

$$PAP^T = \begin{pmatrix} R_{11} & R_{12} & \dots & R_{1m} \\ 0 & R_{22} & \dots & R_{2m} \\ \vdots & & \ddots & \\ 0 & 0 & \dots & R_{mm} \end{pmatrix}$$

where for $j = 1 \dots m$, either R_{jj} irreducible or $R_{jj} = (0)$.

Theorem (Varga, Th. 2.20) Let $A \geq 0$ be an $n \times n$ matrix. Then

- (i) A has a nonnegative real eigenvalue equal to its spectral radius $\rho(A)$. This eigenvalue is positive unless A is reducible and its normal form is strictly upper triangular
- (ii) To $\rho(A)$ there corresponds a nonzero eigenvector $\vec{x} \geq 0$.
- (iii) $\rho(A)$ does not decrease when any entry of A increases.

Proof: See Varga; $\sigma(A) = \bigcup_{j=1}^{m} \sigma(R_{jj})$, apply irreducible Perron-Frobenius to R_{jj}. □

Let us check this experimentally:

Julia conveniently returns positive matrices when calling rand()

```
N = 100
```

```
N=100
```

```
A0 = 100×100 Matrix{Float64}:
 0.820517   0.340232   0.769914  0.22492    …  0.631448   0.110252   0.16271
 0.508726   0.309175   0.494175  0.0689409     0.885589   0.323651   0.791291
 0.179755   0.280721   0.149346  0.0122112     0.423037   0.534891   0.251425
 0.333994   0.820904   0.98683   0.0136435     0.753926   0.0245198  0.161562
 0.894072   0.374506   0.327145  0.732889      0.581772   0.46325    0.946713
 0.687467   0.810695   0.395317  0.38666    …  0.108575   0.973517   0.05618
 0.747385   0.882254   0.422417  0.485958      0.120324   0.232926   0.785146
 ⋮                                          ⋱
 0.0346037  0.224155   0.338776  0.835514      0.678156   0.103897   0.717401
 0.414652   0.90565    0.891557  0.298527  …  0.629689   0.771796   0.340733
 0.968054   0.31567    0.44654   0.693277      0.787286   0.315305   0.310824
 0.251027   0.0893812  0.611264  0.858514      0.538906   0.309333   0.476897
 0.86239    0.20199    0.780093  0.281093      0.64704    0.0656345  0.659397
 0.886446   0.761422   0.713692  0.577544      0.0735567  0.905476   0.892443
```

```
A0=rand(N,N)
```

Add some value to one of the entries

0.0

```
@bind sval Slider(0:0.1:10, show_value=true)
```

```
begin
```

```
e=LinearAlgebra.eigen(A);
```

```
[49.9747+0.0im, 3.03833+0.0im, 2.65002+0.733004im, 2.65002-0.733004im, 2.46963+0.52346
```

```
reverse(e.values)
```

The index of the largest by absolute value eigenvalue is 100

The largest by absolute value eigenvalue is 49.97467282961314 + 0.0im

The corresponding eigenvector is real: true

A corresponding eigenvector is positive: true

Application to the Jacobi method

Theorem: Let A be sdd or idd, and D its diagonal. Then

$$\rho(|I - D^{-1}A|) < 1$$

Proof: Let $B = (b_{ij}) = I - D^{-1}A$. Then

$$b_{ij} = \begin{cases} 0, & i = j \\ -\frac{a_{ij}}{a_{ii}}, & i \neq j \end{cases}$$

If A is sdd, then for $i = 1 \dots n$,

$$\sum_{j=1\dots n} |b_{ij}| = \sum_{\substack{j=1\dots n \\ j\neq i}} |\frac{a_{ij}}{a_{ii}}| = \frac{\Lambda_i}{|a_{ii}|} < 1$$

Therefore, $\rho(|B|) < 1$.

If A is idd, then for $i = 1 \dots n$,

$$\sum_{j=1\dots n} |b_{ij}| = \sum_{\substack{j=1\dots n \\ j\neq i}} |\frac{a_{ij}}{a_{ii}}| = \frac{\Lambda_i}{|a_{ii}|} \leq 1$$

$$\sum_{j=1\dots n} |b_{rj}| = \frac{\Lambda_r}{|a_{rr}|} < 1 \text{ for at least one } r$$

Therefore, $\rho(|B|) <= 1$. Assume $\rho(|B|) = 1$. By Perron-Frobenius, 1 is an eigenvalue. As it is in the union of the Gershgorin disks, for some i,

$$|\lambda| = 1 \leq \frac{\Lambda_i}{|a_{ii}|} \leq 1$$

it must lie on the boundary of this union. By Taussky then one has for all i

$$|\lambda| = 1 \leq \frac{\Lambda_i}{|a_{ii}|} = 1$$

which contradicts the idd condition. $\square$

Corollary: Let A be sdd or idd, and D its diagonal. Assume that $a_{ii} > 0$ and $a_{ij} \leq 0$ for $i \neq j$. Then $\rho(I - D^{-1}A) < 1$, i.e.~the Jacobi method converges.

Proof In this case, $|B| = B \square$.

- Here, we made assumptions on the sign pattern and the diagonal dominance of the matrix. No additional information on the nonzero pattern or the symmetry has been used.
- Does this generalize to other iterative methods ?

```
begin
using HypertextLiteral
    using Markdown
    using PlutoUI
end
```

```
begin

    highlight(mdstring,color)= htl"""<blockquote style="padding: 10px; background-color: $(color);">$(mdstring)</blockquote>"""

    macro important_str(s)  :(highlight(Markdown.parse($s),"#ffcccc")) end
    macro definition_str(s) :(highlight(Markdown.parse($s),"#ccccff")) end
    macro statement_str(s)  :(highlight(Markdown.parse($s),"#ccffcc")) end


    html"""
    <style>
     h1{background-color:#dddddd;  padding: 10px;}
     h2{background-color:#e7e7e7;  padding: 10px;}
     h3{background-color:#eeeeee;  padding: 10px;}
     h4{background-color:#f7f7f7;  padding: 10px;}
    </style>
"""
end
```