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Slide lecture 3

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Diagonally dominant matrices

Definition Let $A = (a_{ij})$ be an $n \times n$ matrix.

- A is *diagonally dominant* if

  (i) for $i = 1 \ldots n$, $|a_{ii}| \geq \sum_{\substack{j=1\ldots n \\ j \neq i}} |a_{ij}|$

- A is *strictly diagonally dominant* (sdd) if

  (i) for $i = 1 \ldots n$, $|a_{ii}| > \sum_{\substack{j=1\ldots n \\ j \neq i}} |a_{ij}|$

- A is *irreducibly diagonally dominant* (idd) if

  (i) A is irreducible

  (ii) A is diagonally dominant –
  for $i = 1 \ldots n$, $|a_{ii}| \geq \sum_{\substack{j=1\ldots n \\ j \neq i}} |a_{ij}|$

  (iii) for at least one r, $1 \leq r \leq n$, $|a_{rr}| > \sum_{\substack{j=1\ldots n \\ j \neq r}} |a_{rj}|$

A very practical nonsingularity criterion

Theorem (Varga, Th. 1.21): Let A be strictly diagonally dominant or irreducibly diagonally dominant. Then A is nonsingular.

If in addition, $a_{ii} > 0$ is real for $i = 1 \dots n$, then all real parts of the eigenvalues of A are positive:

$$\mathrm{Re}\lambda_i > 0, \quad i = 1 \dots n$$

Proof:

- Assume A strictly diagonally dominant. Then the union of the Gershgorin disks does not contain 0 and $\lambda = 0$ cannot be an eigenvalue $\Rightarrow$ A is nonsingular.
- As for the real parts, the union of the disks is

  $$\bigcup_{i=1\dots n} \{\mu \in \mathbb{C} : |\mu - a_{ii}| \leq \Lambda_i\}$$

  and $\mathrm{Re}\mu$ must be larger than zero if μ should be contained.

A very practical nonsingularity criterion, proof I

- Assume A irreducibly diagonally dominant. Then, if 0 is an eigenvalue, it sits on the boundary of one of the Gershgorin disks.

  By Taussky theorem, we have $|a_{ii}| = \Lambda_i$ for all $i = 1 \dots n$.

  This is a contradiction as by definition there is at least one i such that $|a_{ii}| > \Lambda_i$

- Assume $a_{ii} > 0$, real. All real parts of the eigenvalues must be ≥ 0.

  Therefore, if a real part is 0, it lies on the boundary of at least one disk.

  By Taussky theorem it must be contained at the same time in the boundary of all the disks and in the imaginary axis.

  This contradicts the fact that there is at least one disk which does not touch the imaginary axis as by definition there is at least one i such that $|a_{ii}| > \Lambda_i$ $\square$

Corollary

Theorem: If A is complex hermitian or real symmetric, sdd or idd, with positive diagonal entries, it is positive definite.

Proof: All eigenvalues of A are real, and due to the nonsingularity criterion, they must be positive, so A is positive definite.

□

Heat conduction matrix

$$A = \begin{pmatrix} \alpha + \frac{1}{h} & -\frac{1}{h} & & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & \\ & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \\ & & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} \\ & & & & -\frac{1}{h} & \frac{1}{h} + \alpha \end{pmatrix}$$

- A is idd $\Rightarrow$ A is nonsingular
- $\mathrm{diag} A$ is positive real $\Rightarrow$ eigenvalues of A have positive real parts
- A is real, symmetric $\Rightarrow$ A is positive definite

Definition: A real n-vector $\vec{x}$ is

- positive ($\vec{x} > 0$) if all entries of $\vec{x}$ are positive
- nonnegative ($\vec{x} \geq 0$) if all entries of $\vec{x}$ are nonnegative

Definition: A real $n \times n$ matrix A is

- positive ($A > 0$) if all entries of A are positive
- nonnegative ($A \geq 0$) if all entries of A are nonnegative

Theorem(Varga, Th. 2.7) Let $A \geq 0$ be an irreducible $n \times n$ matrix. Then

(i) A has a positive real eigenvalue equal to its spectral radius $\rho(A)$.
(ii) To $\rho(A)$ there corresponds a positive eigenvector $\vec{x} > 0$.
(iii) $\rho(A)$ increases when any entry of A increases.
(iv) $\rho(A)$ is a simple eigenvalue of A.

Proof: See Varga. □

Perron-Frobenius for general nonnegative matrices

Each $n \times n$ matrix can be brought to the normal form

$$PAP^T = \begin{pmatrix} R_{11} & R_{12} & \dots & R_{1m} \\ 0 & R_{22} & \dots & R_{2m} \\ \vdots & & \ddots & \\ 0 & 0 & \dots & R_{mm} \end{pmatrix}$$

where for $j = 1 \dots m$, either R_{jj} irreducible or $R_{jj} = (0)$.

Theorem(Varga, Th. 2.20) Let $A \geq 0$ be an $n \times n$ matrix. Then

(i) A has a nonnegative eigenvalue equal to its spectral radius $\rho(A)$. This eigenvalue is positive unless A is reducible and its normal form is strictly upper triangular

(ii) To $\rho(A)$ there corresponds a nonzero eigenvector $\vec{x} \geq 0$.

(iii) $\rho(A)$ does not decrease when any entry of A increases.

Proof: See Varga; $\sigma(A) = \bigcup_{j=1}^{m} \sigma(R_{jj})$, apply irreducible Perron-Frobenius to R_{jj}.

□

Theorem: Let A be sdd or idd, and D its diagonal. Then

$$\rho(|I - D^{-1}A|) < 1$$

Proof: Let $B = (b_{ij}) = I - D^{-1}A$. Then

$$b_{ij} = \begin{cases} 0, & i = j \\ -\frac{a_{ij}}{a_{ii}}, & i \neq j \end{cases}$$

If A is sdd, then for $i = 1 \dots n$,

$$\sum_{j=1\dots n} |b_{ij}| = \sum_{\substack{j=1\dots n \\ j\neq i}} |\frac{a_{ij}}{a_{ii}}| = \frac{\Lambda_i}{|a_{ii}|} < 1$$

Therefore, $\rho(|B|) < 1$.

Theorem on Jacobi matrix II

If A is idd, then for $i = 1 \dots n$,

$$\sum_{j=1\dots n} |b_{ij}| = \sum_{\substack{j=1\dots n\\ j\neq i}} |\frac{a_{ij}}{a_{ii}}| = \frac{\Lambda_i}{|a_{ii}|} \leq 1$$

$$\sum_{j=1\dots n} |b_{rj}| = \frac{\Lambda_r}{|a_{rr}|} < 1 \text{ for at least one } r$$

Therefore, $\rho(|B|) <= 1$. Assume $\rho(|B|) = 1$. By Perron-Frobenius, 1 is an eigenvalue. As it is in the union of the Gershgorin disks, for some i,

$$|\lambda| = 1 \leq \frac{\Lambda_i}{|a_{ii}|} \leq 1$$

it must lie on the boundary of this union. By Taussky then one has for all i

$$|\lambda| = 1 \leq \frac{\Lambda_i}{|a_{ii}|} = 1$$

which contradicts the idd condition. □

Corollary: Let A be sdd or idd, and D its diagonal. Assume that $a_{ii} > 0$ and $a_{ij} \leq 0$ for $i \neq j$. Then $\rho(I - D^{-1}A) < 1$, i.e. the Jacobi method converges.

Proof In this case, $|B| = B$ $\square$.

- Here, we made assumptions on the sign pattern and the diagonal dominance of the matrix. No additional information on the nonzero pattern or the symmetry has been used.
- Does this generalize to other iterative methods ?

Regular splittings

- $A = M - N$ is a regular splitting if
  - M is nonsingular
  - M^{-1}, N are nonnegative, i.e. have nonnegative entries
- Regard the iteration $u_{k+1} = M^{-1}Nu_k + M^{-1}b$.
- $B = I - M^{-1}A = M^{-1}N$ is a nonnegative matrix.

Convergence theorem for regular splitting

Theorem: Assume A is nonsingular, $A^{-1} \geq 0$, and $A = M - N$ is a regular splitting. Then $\rho(M^{-1}N) < 1$.

Proof: Let $B = M^{-1}N$. Then $A = M(I - B)$, therefore $I - B$ is nonsingular.

In addition

$$A^{-1}N = (M(I - M^{-1}N))^{-1}N = (I - M^{-1}N)^{-1}M^{-1}N = (I - B)^{-1}B$$

By Perron-Frobenius (for general matrices), $\rho(B)$ is an eigenvalue with a nonnegative eigenvector $\vec{x}$. Thus,

$$0 \leq A^{-1}N\vec{x} = \frac{\rho(B)}{1 - \rho(B)}\vec{x}$$

Therefore $0 \leq \rho(B) \leq 1$.
Assume that $\rho(B) = 1$. Then there exists $\vec{x} \neq \mathbf{0}$ such that $B\vec{x} = \vec{x}$.
Consequently, $(I - B)\vec{x} = \mathbf{0}$, contradicting the nonsingularity of $I - B$. Therefore, $\rho(B) < 1$. $\square$

Convergence rate comparison

Corollary: $\rho(M^{-1}N) = \frac{\tau}{1+\tau}$ where $\tau = \rho(A^{-1}N)$.

Proof: Rearrange $\tau = \frac{\rho(B)}{1-\rho(B)}$ $\square$

Corollary: Let $A^{-1} \geq 0$, $A = M_1 - N_1$ and $A = M_2 - N_2$ be regular splittings.

If $N_2 \geq N_1$, then $1 > \rho(M_2^{-1}N_2) \geq \rho(M_1^{-1}N_1)$.

Proof: $\tau_2 = \rho(A^{-1}N_2) \geq \rho(A^{-1}N_1) = \tau_1$

But $\frac{\tau}{1+\tau}$ is strictly increasing. $\square$

Convergence rate comparison II

- Let $A^{-1} \geq 0$, $A = D - E - F$, $D > 0$ diagonal, $E, F \geq 0$ upper resp. lower triangular parts.
- Jacobi: $M_J = D$, $N_J = E + F$. $M_J^{-1} > 0 \Rightarrow$ regular splitting
- Gauss-Seidel: $M_{GS} = D - E$, $N_{GS} = F \geq 0$. Show $M_{GS}^{-1} \geq 0$:

$$M_{GS} = \begin{pmatrix} d_{11} & -e_{12} & -e_{13} & \dots & -e_{1n} \\ & d_{22} & -e_{23} & \dots & -e_{2n} \\ & & \ddots & \ddots & \vdots \\ & & & d_{n-1,n-1} & -e_{n-1,n} \\ & & & \dots & d_{nn} \end{pmatrix}$$

  Elimination steps for $M_{GS} v = r$:

$$v_n = \frac{r_n}{d_{nn}}, \qquad v_{n-1} = \frac{r_n + e_{n-1,n} v_n}{d_{n-1,n-1}} \dots$$

  All coefficients are nonnegative $\Rightarrow M_{GS} - N_{GS}$: regular splitting

- $N_{GS} \leq N_J \Rightarrow \rho(M_{GS}^{-1} N_{GS}) \leq \rho(M_J^{-1} N_J)$

M-Matrix definition

Definition Let A be an $n \times n$ real matrix. A is called M-Matrix if

(i) $a_{ij} \leq 0$ for $i \neq j$

(ii) A is nonsingular

(iii) $A^{-1} \geq 0$

Corollary: If A is an M-Matrix, then $A^{-1} > 0 \Leftrightarrow A$ is irreducible.

Proof: See Varga. □

M-Matrix main diagonal

Theorem: If A is an M-matrix, then its diagonal $D_A > 0$ is positive.

Proof: Let $C = A^{-1} \geq 0$. The $AC = I$ and $(AC)_{ii} = 1$.

$$\sum_{k=1}^{n} a_{ik} c_{ki} = 1$$

$$a_{ii} c_{ii} = 1 - \sum_{k=1, k\neq i}^{n} a_{ik} c_{ki} \geq 1$$

The last inequality is due to $c_{ki} \geq 0$ and $a_{ik} < 0$ for $k \neq i$. As $a_{ii} c_{ii} \geq 1$, neither factor can be 0. So $c_{ii} > 0$ and $a_{ii} > 0$.

Theorem: (Saad, Th. 1.31) Assume

(i) $a_{ij} \leq 0$ for $i \neq j$

(ii) $a_{ii} > 0$

Then A is an M-Matrix if and only if $\rho(I - D^{-1}A) < 1$.

Proof: See Saad. □

Main practical M-Matrix criterion

Corollary: Let A be sdd or idd. Assume that $a_{ii} > 0$ and $a_{ij} \leq 0$ for $i \neq j$. Then A is an M-Matrix.

Proof: We know that A is nonsingular, but we have to show $A^{-1} \geq 0$.

- Let $B = I - D^{-1}A$. Then $\rho(B) < 1$, therefore $I - B$ is nonsingular.
- We have for $k > 0$:

$$\begin{aligned} I - B^{k+1} &= (I - B)(I + B + B^2 + \cdots + B^k) \\ (I - B)^{-1}(I - B^{k+1}) &= (I + B + B^2 + \cdots + B^k) \end{aligned}$$

The left hand side for $k \to \infty$ converges to $(I - B)^{-1}$, therefore

$$(I - B)^{-1} = \sum_{k=0}^{\infty} B^k$$

As $B \geq 0$, we have $(I - B)^{-1} = A^{-1}D \geq 0$. As $D > 0$ we must have $A^{-1} \geq 0$. □

M-Matrix comparison criterion

Theorem(Saad, Th. 1.33): Let A, B $n \times n$ matrices such that

(i) $A \leq B$

(ii) $b_{ij} \leq 0$ for $i \neq j$.

Then, if A is an M-Matrix, so is B.

Proof: From M-property of A and $A \leq B$ we have $0 < D_A \leq D_B$. We have $D_B - B \geq 0$ and

$$\begin{aligned} D_A - A &\geq D_B - B \\ I - D_A^{-1} A &\geq D_A^{-1}(D_B - B) \\ &\geq D_B^{-1}(D_B - B) \\ &\geq I - D_B^{-1} B =: G \geq 0 \end{aligned}$$

Perron-Frobenius $\Rightarrow \rho(G) = \rho(I - D_B^{-1}B) \leq \rho(I - D_A^{-1}A) < 1$
$\Rightarrow I - G$ is nonsingular. From the proof of the M-matrix criterion,
$D_B^{-1}B = (I - G)^{-1} = \sum_{k=0}^{\infty} G^k \geq 0$. As $D_B > 0$, we get $B \geq 0$.

□

Corollary $A \leq M_{GS} \Rightarrow M_{GS}$ is an M-Matrix.

Intermediate Summary

- Given some matrix, we now have some nice recipies to establish nonsingularity and iterative method convergence:
- **Check if the matrix is irreducible.**
  This is mostly the case for elliptic and parabolic PDEs.
- **Check if the matrix is strictly or irreducibly diagonally dominant**.
  If yes, it is in addition nonsingular.
- **Check if main diagonal entries are positive and off-diagonal entries are nonpositive.**
  If yes, in addition, the matrix is an M-Matrix, its inverse is nonnegative, and elementary iterative methods converge.
- These critera do not depend on the symmetry of the matrix!

Example: 1D finite difference matrix:

$$Au = \begin{pmatrix} \alpha + \frac{1}{h} & -\frac{1}{h} & & & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & & \\ & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} \\ & & & & -\frac{1}{h} & \frac{1}{h} + \alpha \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-2} \\ u_{N-1} \\ u_N \end{pmatrix} = f = \begin{pmatrix} \alpha v_1 \\ hf_2 \\ hf_3 \\ \vdots \\ hf_{N-2} \\ hf_{N-1} \\ \alpha v_n \end{pmatrix}$$

- idd
- positive main diagonal entries, nonpositive off-diagonal entries

$\Rightarrow$ A is nonsingular, has the M-property, and we can e.g. apply the Jacobi and Gauss-Seidel iterative method to solve it (ok, in 1D we already know this is a bad idea ...).

$\Rightarrow$ for $f \geq 0$ and $v \geq 0$ it follows that $u \geq 0$.
$\equiv$ heating and positive environment temperatures cannot lead to negative temperatures in the interior.