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Lecture 18

Jürgen Fuhrmann

juergen.fuhrmann@wias-berlin.de

The Galerkin method II

- Let V be a Hilbert space. Let $a : V \times V \to \mathbb{R}$ be a self-adjoint bilinear form, and f a linear functional on V. Assume a is coercive with coercivity constant α, and continuity constant γ.
- Continuous problem: search $u \in V$ such that

$$a(u, v) = f(v) \; \forall v \in V$$

- Let $V_h \subset V$ be a finite dimensional subspace of V
- "Discrete" problem $\equiv$ Galerkin approximation:
  Search $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h) \; \forall v_h \in V_h$$

By Lax-Milgram, this problem has a unique solution as well.

Céa's lemma

- What is the connection between u and u_h ?
- Let $v_h \in V_h$ be arbitrary. Then

$$\begin{aligned}
\alpha||u - u_h||^2 &\leq a(u - u_h, u - u_h) \quad \text{(Coercivity)} \\
&= a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h) \\
&= a(u - u_h, u - v_h) \quad \text{(Galerkin Orthogonality)} \\
&\leq \gamma||u - u_h|| \cdot ||u - v_h|| \quad \text{(Boundedness)}
\end{aligned}$$

- As a result

$$||u - u_h|| \leq \frac{\gamma}{\alpha} \inf_{v_h \in V_h} ||u - v_h||$$

- Up to a constant, the error of the Galerkin approximation is the error of the best approximation of the solution in the subspace V_h.

From the Galerkin method to the matrix eqation

- Let $\phi_1 \dots \phi_n$ be a set of basis functions of V_h.
- Then, we have the representation $u_h = \sum_{j=1}^n u_j \phi_j$
- In order to search $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h)\ \forall v_h \in V_h$$

it is actually sufficient to require

$$a(u_h, \phi_i) = f(\phi_i)\ (i = 1 \dots n)$$

$$a\left(\sum_{j=1}^n u_j \phi_j, \phi_i\right) = f(\phi_i)\ (i = 1 \dots n)$$

$$\sum_{j=1}^n a(\phi_j, \phi_i) u_j = f(\phi_i)\ (i = 1 \dots n)$$

$$AU = F$$

with $A = (a_{ij}), a_{ij} = a(\phi_i, \phi_j)$, $F = (f_i), f_i = F(\phi_i)$, $U = (u_i)$.
- Matrix dimension is $n \times n$. Matrix sparsity ?

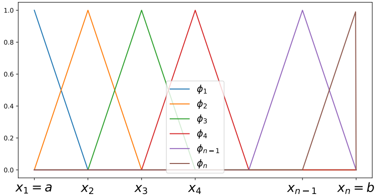
Obtaining a finite dimensional subspace

- Let $\Omega = (a, b) \subset \mathbb{R}^1$
- Let $a(u, v) = \int_a^b \delta \vec{\nabla} u \vec{\nabla} v d\vec{x} + \alpha u(a)v(a) + \alpha u(b)v(b)$
- Calculus 101 provides a finite dimensional subspace: the space of sin/cos functions up to a certain frequency $\Rightarrow$ *spectral method*
- Ansatz functions have global support $\Rightarrow$ full $n \times n$ matrix
- OTOH: rather fast convergence for smooth data
- Generalization to higher dimensions possible
- Big problem in irregular domains: we need the eigenfunction basis of some operator...
- Spectral methods are successful in cases where one has regular geometry structures and smooth/constant coefficients – e.g. "Spectral Einstein Code"

The finite element idea I

- Choose basis functions with local support. $\Rightarrow$ only integrals of basis function pairs with overlapping support contribute to matrix.
- Linear finite elements in $\Omega = (a, b) \subset \mathbb{R}^1$:
- Partition $a = x_1 \leq x_2 \leq \cdots \leq x_n = b$
- Basis functions (for $i = 1 \dots n$)

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}}, & i > 1, x \in (x_{i-1}, x_i) \\ \frac{x_{i+1} - x}{x_{i+1} - x_i}, & i < n, x \in (x_i, x_{i+1}) \\ 0, & \text{else} \end{cases}$$



FE matrix elements for 1D heat equation I

- Any function $u_h \in V_h = \mathrm{span}\{\phi_1 \dots \phi_n\}$ is piecewise linear, and the coefficients in the representation $u_h = \sum_{i=1}^{n} u_i \phi_i$ are the values $u_h(x_i)$.
- Fortunately, we are working with a weak formulation, and weak derivatives are well defined (and coincide with the classical derivatives where the basis functions are smooth)
- Let ϕ_i, ϕ_j be two basis functions, regard

$$s_{ij} = \int_a^b \vec{\nabla}\phi_i \cdot \vec{\nabla}\phi_j d\ x$$

- We have $\operatorname{supp}\phi_i \cap \operatorname{supp}\phi_j = \emptyset$ unless $i = j$, $i + 1 = j$ or $i - 1 = j$.
- Therefore $s_{ij} = 0$ unless $i = j$, $i + 1 = j$ or $i - 1 = j$.

FE matrix elements for 1D heat equation II

- Let $j = i + 1$. Then $\operatorname{supp} \phi_i \cap \operatorname{supp} \phi_j = (x_i, x_{i+1})$, $\phi_i' = -\frac{1}{h}$, $\phi_j' = frac1h$ where $h = x_{i+1} - x_i$

$$\int_a^b \vec{\nabla}\phi_i \vec{\nabla}\phi_j d\vec{x} = \int_{x_i}^{x_{i+1}} \phi_i' \phi_j' d\vec{x} = -\int_{x_i}^{x_{i+1}} \frac{1}{h^2} d\vec{x} = -\frac{1}{h}$$

- Similarly, for $j = i - 1$: $\int_a^b \vec{\nabla}\phi_i \vec{\nabla}\phi_j d\vec{x} = -\frac{1}{h}$
- For $1 < i < N$:

$$\int_a^b \vec{\nabla}\phi_i \vec{\nabla}\phi_i d\vec{x} = \int_{x_{i-1}}^{x_{i+1}} (\phi_i')^2 d\vec{x} = \int_{x_{i-1}}^{x_{i+1}} \frac{1}{h^2} d\vec{x} = \frac{2}{h}$$

- For $i = 1$ or $i = N$: $\int_a^b \vec{\nabla}\phi_i \vec{\nabla}\phi_i d\vec{x} = \frac{1}{h}$
- For the right hand side, calculate vector elements $f_i = \int_a^b f(x)\phi_i dx$ using a quadrature rule.

FE matrix elements for 1D heat equation III

Adding the boundary integrals yields

$$A = \begin{pmatrix} \alpha + \frac{1}{h} & -\frac{1}{h} & & & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & & \\ & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \\ & & & & -\frac{1}{h} & \frac{1}{h} + \alpha \end{pmatrix}$$

... the same matrix as for the finite difference and finite volume methods

Simplices

- Let $\{\vec{a}_1 \ldots \vec{a}_{d+1}\} \subset \mathbb{R}^d$ such that the d vectors $\vec{a}_2 - \vec{a}_1 \ldots \vec{a}_{d+1} - \vec{a}_1$ are linearly independent. Then the convex hull K of $\vec{a}_1 \ldots \vec{a}_{d+1}$ is called *simplex*, and $\vec{a}_1 \ldots \vec{a}_{d+1}$ are called *vertices* of the simplex.
- *Unit simplex*: $\vec{a}_1 = (0...0), \vec{a}_1 = (0, 1 \ldots 0) \ldots \vec{a}_{d+1} = (0 \ldots 0, 1)$.

$$K = \left\{ \vec{x} \in \mathbb{R}^d : x_i \geq 0 \ (i = 1 \ldots d) \text{ and } \sum_{i=1}^{d} x_i \leq 1 \right\}$$

- A general simplex can be defined as an image of the unit simplex under some affine transformation
- F_i: face of K opposite to $\vec{a}_i$
- $\vec{n}_i$: outward normal to F_i

Simplex characteristics

- Diameter of K: $h_K = \max_{\vec{x}_1, \vec{x}_2 \in K} ||\vec{x}_1 - \vec{x}_2||$
  $\equiv$ length of longest edge if K
- ρ_K diameter of largest ball that can be inscribed into K
- $\sigma_K = \frac{h_K}{\rho_K}$: local shape regularity measure
  - $\sigma_K = 2\sqrt{3}$ for equilateral triangle
  - $\sigma_K \to \infty$ if largest angle approaches π.

Barycentric coordinates

Definition: Let $K \subset \mathbb{R}^d$ be a d-simplex given by the points $\vec{a}_1 \dots \vec{a}_{d+1}$.
Let $\Lambda(x) = (\lambda_1(\vec{x}) \dots \lambda_{d+1}(\vec{x}))$ be a vector such that for all $\vec{x} \in \mathbb{R}^d$

$$\sum_{j=1}^{d+1} \vec{a}_j \lambda_j(\vec{x}) = \vec{x}, \qquad \sum_{j=1}^{d+1} \lambda_j(\vec{x}) = 1$$

This vector is called the vector of *barycentric coordinates* of $\vec{x}$ with respect to K.

Barycentric coordinates II

Lemma The barycentric coordinates of a given point is well defined and unique. Moreover, for the simplex edges $\vec{a}_i$, one has

$$\lambda_j(\vec{a}_i) = \delta_{ij}$$

Proof: The definition of Λ given by a $d+1 \times d+1$ system of equations with the matrix

$$M = \begin{pmatrix} a_{1,1} & a_{2,1} & \dots & a_{d+1,1} \\ a_{1,2} & a_{2,2} & \dots & a_{d+1,2} \\ \vdots & \vdots & & \vdots \\ a_{1,d} & a_{2,d} & \dots & a_{d+1,d} \\ 1 & 1 & \dots & 1 \end{pmatrix}$$

Subtracting the first column from the others gives

$$M' = \begin{pmatrix} a_{1,1} & a_{2,1} - a_{1,1} & \dots & a_{d+1,1} - a_{1,1} \\ a_{1,2} & a_{2,2} - a_{1,2} & \dots & a_{d+1,2} - a_{1,2} \\ \vdots & \vdots & & \vdots \\ a_{1,d} & a_{2,d} - a_{1,d} & \dots & a_{d+1,d} - a_{1,d} \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

Barycentric coordinates III

$\det M = \det M'$ is the determinant of the matrix whose columns are the edge vectors of K which are linearly independent.

For the simplex edges one has

$$\sum_{j=1}^{d+1} \vec{a}_j \lambda_j(\vec{a}_i) = \vec{a}_i$$

which is fulfilled if $\lambda_j(\vec{a}_i) = 1$ for $i = j$ and $\lambda_j(\vec{a}_i) = 0$ for $i \neq j$. And we have uniqueness. $\square$

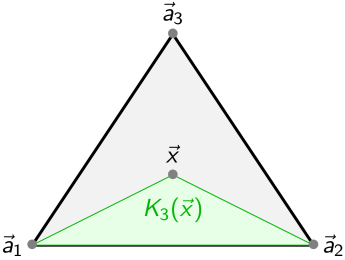
At the same time, the measure (area) is calculated as $|K| = \frac{1}{d!}|\det M'|$.

Barycentric coordinates IV

- Let $K_j(\vec{x})$ be the subsimplex of K made of $\vec{x}$ and $\vec{a}_1 \dots \vec{a}_{d+1}$ with $\vec{a}_j$ omitted.
- Its measure $|K_j(\vec{x})|$ is established from its determinant and a linear function of the coordinates for $\vec{x}$.
- One has $\frac{|K_j(\vec{a}_i)|}{|K|} = \delta_{ij}$ and therefore,

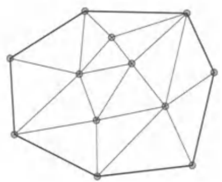
$$\lambda_j(\vec{x}) = \frac{|K_j(\vec{x})|}{|K|}$$

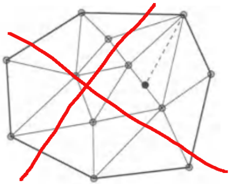
is the ratio of the measures of $K_j(\vec{x})$ and K.



Conformal triangulations II

- $d = 1$: Each intersection $F = K_m \cap K_n$ is either empty or a common vertex
- $d = 2$: Each intersection $F = K_m \cap K_n$ is either empty or a common vertex or a common edge





- $d = 3$: Each intersection $F = K_m \cap K_n$ is either empty or a common vertex or a common edge or a common face
- Delaunay triangulations are conformal

Shape regularity

- Now we discuss a family of meshes $\mathcal{T}_h$ for $h \to 0$.
- For given $\mathcal{T}_h$, assume that $h = \max_{K \in \mathcal{T}_h} h_K$
- A family of meshes is called *shape regular* if

$$\forall h, \forall K \in \mathcal{T}_h, \sigma_K = \frac{h_K}{\rho_K} \leq \sigma_0$$

- In 1D, $\sigma_K = 1$
- In 2D, $\sigma_K \leq \frac{2}{\sin \theta_K}$ where θ_K is the smallest angle

Polynomial space $\mathbb{P}_k$

- Space of polynomials in $x_1 \dots x_d$ of total degree $\leq k$ with real coefficients $\alpha_{i_1 \dots i_d}$:

$$\mathbb{P}_k = \left\{ p(x) = \sum_{\substack{0 \leq i_1 \dots i_d \leq k \\ i_1 + \dots + i_d \leq k}} \alpha_{i_1 \dots i_d} x_1^{i_1} \dots x_d^{i_d} \right\}$$

- Dimension:

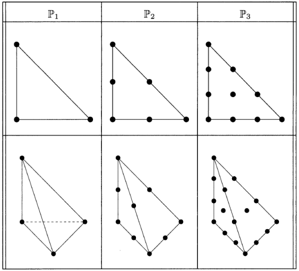
$$\dim \mathbb{P}_k = \binom{d+k}{k} = \begin{cases} k+1, & d=1 \\ \frac{1}{2}(k+1)(k+2), & d=2 \\ \frac{1}{6}(k+1)(k+2)(k+3), & d=3 \end{cases}$$

$$\dim \mathbb{P}_1 = d+1$$

$$\dim \mathbb{P}_2 = \begin{cases} 3, & d=1 \\ 6, & d=2 \\ 10, & d=3 \end{cases}$$

$\mathbb{P}_k$ simplex finite elements

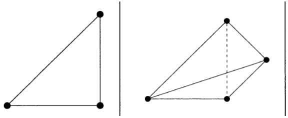
- K: simplex spanned by $\vec{a}_1 \dots \vec{a}_{d+1}$ in $\mathbb{R}^d$
- For $0 \leq i_1 \dots i_{d+1} \leq k$, $i_1 + \dots + i_{d+1} = k$, let the set of nodes $\Sigma = \{\vec{\sigma}_1 \dots \vec{\sigma}_s\}$ be defined by the points $\vec{a}_{i_1 \dots i_d;k}$ with barycentric coordinates $(\frac{i_1}{k} \dots \frac{i_{d+1}}{k})$.

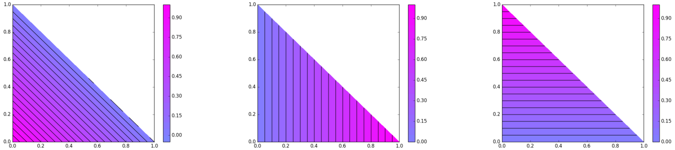


- $s = \text{card}\, \Sigma = \dim \mathbb{P}_K \Rightarrow$ there exists a basis $\theta_1 \dots \theta_s$ of $\mathbb{P}_k$ such that $\theta_i(\vec{\sigma}_j) = \delta_{ij}$

$\mathbb{P}_1$ simplex finite elements

- K: simplex spanned by $a_1 \dots a_{d+1}$ in $\mathbb{R}^d$
- $s = d + 1$
- Nodes $\equiv$ vertices
- Basis functions $\theta_1 \dots \theta_{d+1} \equiv$ barycentric coordinates $\lambda_1 \dots \lambda_{d+1}$





Global degrees of freedom

- Given a triangulation $\mathcal{T}_h$
- Let $\{\vec{a}_1 \dots \vec{a}_N\} = \bigcup_{K \in \mathcal{T}_h} \{\vec{\sigma}_{K,1} \dots \vec{\sigma}_{K,s}\}$ be the set of global degrees of freedom.
- Degree of freedom map

$$
\begin{aligned}
j : \mathcal{T}_h \times \{1 \dots s\} &\to \{1 \dots N\} \\
(K, m) &\mapsto j(K, m) \text{ the global degree of freedom number}
\end{aligned}
$$

Lagrange finite element space

- Given a triangulation $\mathcal{T}_h$ of Ω, define the spaces

$$P_h^k = \{v_h \in \mathbb{C}^0(\Omega) : v_h|_K \in \mathbb{P}_K \; \forall K \in \mathcal{T}_j\} \subset H^1(\Omega)$$
$$P_{0,h}^k = \{v_h \in P_h^k : v_h|_{\partial\Omega} = 0\} \subset H_0^1(\Omega)$$

- Global shape functions $\theta_1, \ldots, \theta_N \in P_h^k$ defined by

$$\phi_i|_K(\vec{a}_{K,m}) = \begin{cases} \delta_{mn} & \text{if } \exists n \in \{1 \ldots s\} : j(K,n) = i \\ 0 & \text{otherwise} \end{cases}$$

- $\{\phi_1, \ldots, \phi_N\}$ is a basis of P_h, and $\gamma_1 \ldots \gamma_N$ is a basis of $\mathcal{L}(P_h, \mathbb{R})$:
  - $\{\phi_1, \ldots, \phi_N\}$ are linearly independent: if $\sum_{j=1}^N \alpha_j \phi_j = 0$ then evaluation at $\vec{a}_1 \ldots \vec{a}_N$ yields that $\alpha_1 \ldots \alpha_N = 0$.
  - Let $v_h \in P_h$. Let $w_h = \sum_{j=1}^N v_h(\vec{a}_j)\phi_j$. Then for all $K \in \mathcal{T}_h$, $v_h|_K$ and $w_h|_K$ coincide in the local nodes $\vec{a}_{K,1} \ldots \vec{a}_{K,2}$, $\Rightarrow v_h|_K = w_h|_K$.

Finite element approximation space

We have

d	k	$N = \dim P_h^k$
1	1	N_v
1	2	$N_v + N_{el}$
1	3	$N_v + 2N_{el}$
2	1	N_v
2	2	$N_v + N_{ed}$
2	3	$N_v + 2N_{ed} + N_{el}$
3	1	N_v
3	2	$N_v + N_{ed}$
3	3	$N_v + 2N_{ed} + N_f$

Local Lagrange interpolation operator

- Let $\{K, P, \Sigma\}$ be a finite element with shape function bases $\{\theta_1 \dots \theta_s\}$. Let $V(K) = \mathbb{C}^0(K)$ and $P \subset V(K)$
- *local interpolation operator*

$$\begin{aligned}\mathcal{I}_K : V(K) &\to P\\ v &\mapsto \sum_{i=1}^{s} v(\vec{\sigma}_i)\theta_i\end{aligned}$$

- P is invariant under the action of $\mathcal{I}_K$, i.e. $\forall p \in P, \mathcal{I}_K(p) = p$:
  - Let $p = \sum_{j=1}^{s} \alpha_j \theta_j$ Then,

$$\begin{aligned}\mathcal{I}_K(p) &= \sum_{i=1}^{s} p(\vec{\sigma}_i)\theta_i = \sum_{i=1}^{s}\sum_{j=1}^{s} \alpha_j \theta_j(\vec{\sigma}_i)\theta_i\\ &= \sum_{i=1}^{s}\sum_{j=1}^{s} \alpha_j \delta_{ij}\theta_i = \sum_{j=1}^{s} \alpha_j \theta_j\end{aligned}$$

Global Lagrange interpolation operator

Let $V_h = P_h^k$

$$\mathcal{I}_h : \mathcal{C}^0(\bar{\Omega}_h) \to V_h$$

$$v(\vec{x}) \mapsto v_h(\vec{x}) = \sum_{i=1}^{N} v(\vec{a}_i)\phi_i(\vec{x})$$

Local interpolation error estimate I

Theorem: Let $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ be a finite element with associated normed vector space $V(\widehat{K})$. Assume that

$$\mathbb{P}_k \subset \widehat{P} \subset H^2(\widehat{K}) \subset V(\widehat{K})$$

Then there exists $c > 0$ such that for all $m = 0 \dots 2$, $K \in \mathcal{T}_h$, $v \in H^2(K)$:

$$|v - \mathcal{I}_K^1 v|_{m,K} \leq c h_K^{2-m} \sigma_K^m |v|_{2,K}.$$

I.e. the the local interpolation error can be estimated through h_K, σ_K and the norm of a higher derivative.

Local interpolation: special cases

- $m = 0$: $|v - \mathcal{I}_K^1 v|_{0,K} \leq c h_K^2 |v|_{2,K}$
- $m = 1$: $|v - \mathcal{I}_K^1 v|_{1,K} \leq c h_K \sigma_K |v|_{2,K}$

Global interpolation error estimate for Lagrangian finite elements, $k = 1$

- Assume $v \in H^2(\Omega)$, e.g. if problem coefficients are smooth and the domain is convex

$$
\begin{aligned}
||v - \mathcal{I}_h^1 v||_{0,\Omega} + h|v - \mathcal{I}_h^1 v|_{1,\Omega} &\leq ch^2|v|_{2,\Omega} \\
|v - \mathcal{I}_h^1 v|_{1,\Omega} &\leq ch|v|_{2,\Omega} \\
\lim_{h\to 0}\left(\inf_{v_h \in V_h^1} |v - v_h|_{1,\Omega}\right) &= 0
\end{aligned}
$$

- If $v \in H^2(\Omega)$ cannot be guaranteed, estimates become worse. Example: L-shaped domain.
- These results immediately can be applied in Cea's lemma.

Error estimates for homogeneous Dirichlet problem

- Search $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \delta \vec{\nabla} u \vec{\nabla} v \, d\vec{x} = \int_\Omega f v \, d\vec{x} \; \forall v \in H_0^1(\Omega) +$$

Then, $\lim_{h \to 0} ||u - u_h||_{1,\Omega} = 0$. If $u \in H^2(\Omega)$ (e.g. on convex domains) then

$$||u - u_h||_{1,\Omega} \leq ch|u|_{2,\Omega}$$
$$||u - u_h||_{0,\Omega} \leq ch^2|u|_{2,\Omega}$$

Under certain conditions (convex domain, smooth coefficients) one also has

$$||u - u_h||_{0,\Omega} \leq ch|u|_{1,\Omega}$$

("Aubin-Nitsche-Lemma")

H^2-Regularity

- $u \in H^2(\Omega)$ may be *not* fulfilled e.g.
  - if Ω has re-entrant corners
  - if on a smooth part of the domain, the boundary condition type changes
  - if problem coefficients (δ) are discontinuos
- Situations differ as well between two and three space dimensions
- Delicate theory, ongoing research in functional analysis
- Consequence for simuations
  - Deterioration of convergence rate
  - Remedy: local refinement of the discretization mesh
    - using a priori information
    - using a posteriori error estimators + automatic refinement of discretizatiom mesh

Weak formulation of homogeneous Dirichlet problem

- Search $u \in V = H_0^1(\Omega)$ such that

$$\int_\Omega \delta \vec{\nabla} u \vec{\nabla} v \, d\vec{x} = \int_\Omega f v \, d\vec{x} \; \forall v \in H_0^1(\Omega)$$

- Then,

$$a(u, v) := \int_\Omega \delta \vec{\nabla} u \vec{\nabla} v \, d\vec{x}$$

is a self-adjoint bilinear form defined on the Hilbert space $H_0^1(\Omega)$.

Galerkin ansatz

- Let $V_h \subset V$ be a finite dimensional subspace of V
- "Discrete" problem $\equiv$ Galerkin approximation:
  Search $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h) \; \forall v_h \in V_h$$

- E.g. V_h is the space of P1 Lagrange finite element approximations

Stiffness matrix for Laplace operator for P1 FEM

- Element-wise calculation:

$$a_{ij} = a(\phi_i, \phi_j) = \int_\Omega \vec{\nabla}\phi_i \vec{\nabla}\phi_j \; d\vec{x} = \int_\Omega \sum_{K\in\mathcal{T}_h} \vec{\nabla}\phi_i|_K \vec{\nabla}\phi_j|_K \; d\vec{x}$$

- Standard assembly loop:

for $i, j = 1 \dots N$ **do**
  set $a_{ij} = 0$
end
for $K \in \mathcal{T}_h$ **do**
  for *m,n=0...d* **do**

$$s_{mn} = \int_K \vec{\nabla}\lambda_m \vec{\nabla}\lambda_n \; d\vec{x}$$

$$a_{j_{dof}(K,m), j_{dof}(K,n)} = a_{j_{dof}(K,m), j_{dof}(K,n)} + s_{mn}$$

  end
end

- Local stiffness matrix:

$$S_K = (s_{K;m,n}) = \int_K \vec{\nabla}\lambda_m \vec{\nabla}\lambda_n \; d\vec{x}$$

Local stiffness matrix calculation for P1 FEM

- $a_0 \dots a_d$: vertices of the simplex K, $a \in K$.
- Barycentric coordinates: $\lambda_j(a) = \frac{|K_j(a)|}{|K|}$
- For indexing modulo d+1 we can write

$$|K| = \frac{1}{d!} \det \left(a_{j+1} - a_j, \dots a_{j+d} - a_j \right)$$
$$|K_j(a)| = \frac{1}{d!} \det \left(a_{j+1} - a, \dots a_{j+d} - a \right)$$

- From this information, we can calculate explicitely$\vec{\nabla}\lambda_j(x)$ (which are constant vectors due to linearity) and the corresponding entries of the local stiffness

$$s_{ij} = \int_K \vec{\nabla}\lambda_i \vec{\nabla}\lambda_j \; d\vec{x}$$

Local stiffness matrix calculation for P1 FEM in 2D

- $a_0 = (x_0, y_0) \dots a_d = (x_2, y_2)$: vertices of the simplex K, $a = (x, y) \in K$.
- Barycentric coordinates: $\lambda_j(x, y) = \frac{|K_j(x,y)|}{|K|}$
- For indexing modulo d+1 we can write

$$|K| = \frac{1}{2} \det \begin{pmatrix} x_{j+1} - x_j & x_{j+2} - x_j \\ y_{j+1} - y_j & y_{j+2} - y_j \end{pmatrix}$$
$$|K_j(x, y)| = \frac{1}{2} \det \begin{pmatrix} x_{j+1} - x & x_{j+2} - x \\ y_{j+1} - y & y_{j+2} - y \end{pmatrix}$$

- Therefore, we have

$$|K_j(x, y)| = \frac{1}{2} \left((x_{j+1} - x)(y_{j+2} - y) - (x_{j+2} - x)(y_{j+1} - y) \right)$$
$$\partial_x |K_j(x, y)| = \frac{1}{2} \left((y_{j+1} - y) - (y_{j+2} - y) \right) = \frac{1}{2}(y_{j+1} - y_{j+2})$$
$$\partial_y |K_j(x, y)| = \frac{1}{2} \left((x_{j+2} - x) - (x_{j+1} - x) \right) = \frac{1}{2}(x_{j+2} - x_{j+1})$$

Local stiffness matrix calculation for P1 FEM in 2D II

- $$s_{ij} = \int_K \vec\nabla\lambda_i \vec\nabla\lambda_j \; d\vec{x} = \frac{|K|}{4|K|^2} \left(y_{i+1} - y_{i+2}, x_{i+2} - x_{i+1}\right) \begin{pmatrix} y_{j+1} - y_{j+2} \\ x_{j+2} - x_{j+1} \end{pmatrix}$$
- So, let $V = \begin{pmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{pmatrix}$
- Then

$$x_1 - x_2 = V_{00} - V_{01}$$
$$y_1 - y_2 = V_{10} - V_{11}$$

and

$$2|K| \; \vec\nabla\lambda_0 = \begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \end{pmatrix} = \begin{pmatrix} V_{10} - V_{11} \\ V_{01} - V_{00} \end{pmatrix}$$
$$2|K| \; \vec\nabla\lambda_1 = \begin{pmatrix} y_2 - y_0 \\ x_0 - x_2 \end{pmatrix} = \begin{pmatrix} V_{11} \\ -V_{01} \end{pmatrix}$$
$$2|K| \; \vec\nabla\lambda_2 = \begin{pmatrix} y_0 - y_1 \\ x_1 - x_0 \end{pmatrix} = \begin{pmatrix} -V_{10} \\ V_{00} \end{pmatrix}$$

Degree of freedom map representation for P1 finite elements

- List of global nodes $a_0 \dots a_N$: two dimensional array of coordinate values with N rows and d columns
- Local-global degree of freedom map: two-dimensional array C of index values with N_{el} rows and $d+1$ columns such that $C(i,m) = j_{dof}(K_i, m)$.
- The mesh generator triangle generates this information directly

Practical realization of boundary conditions

- Robin boundary value problem

$$-\nabla \cdot \delta \vec{\nabla} u = f \quad \text{in } \Omega$$
$$\delta \vec{\nabla} u + \alpha(u - g) = 0 \quad \text{on } \partial\Omega$$

- Weak formulation: search $u \in H^1(\Omega)$ such that

$$\int_\Omega \delta \vec{\nabla} u \vec{\nabla} v \, d\vec{x} + \int_{\partial\Omega} \alpha uv \, ds = \int_\Omega fv \, d\vec{x} + \int_{\partial\Omega} \alpha gv \, ds \; \forall v \in H^1(\Omega)$$

- In 2D, for P^1 FEM, boundary integrals can be calculated by trapezoidal rule without sacrificing approximation order
- Use Dirichlet penalty method to handle Dirichlet boundary conditions

More complicated integrals

- Assume non-constant right hand side f, space dependent heat conduction coefficient δ.
- Right hand side integrals

$$f_i = \int_K f(x)\lambda_i(x)\, d\vec{x}$$

- P^1 stiffness matrix elements

$$a_{ij} = \int_K \delta(x)\, \vec{\nabla}\lambda_i\, \vec{\nabla}\lambda_j\, d\vec{x}$$

- P^k stiffness matrix elements created from higher order ansatz functions

Quadrature rules

- *Quadrature rule*:

$$\int_K g(x)\ d\vec{x} \approx |K| \sum_{l=1}^{l_q} \omega_l g(\xi_l)$$

- ξ_l: *nodes, Gauss points*
- ω_l: *weights*
- The largest number k such that the quadrature is exact for polynomials of order k is called *order* k_q of the quadrature rule, i.e.

$$\forall k \leq k_q, \forall p \in \mathbb{P}^k \int_K p(x)\ d\vec{x} = |K| \sum_{l=1}^{l_q} \omega_l p(\xi_l)$$

- Error estimate:

$$\forall \phi \in \mathcal{C}^{k_q+1}(K), \left| \frac{1}{|K|} \int_K \phi(x)\ d\vec{x} - \sum_{l=1}^{l_q} \omega_l g(\xi_l) \right|$$
$$\leq c h_K^{k_q+1} \sup_{x \in K, |\alpha| = k_q+1} |\partial^\alpha \phi(x)|$$

Some common quadrature rules

Nodes are characterized by the barycentric coordinates

d	k_q	l_q	Nodes	Weights
1	1	1	$(\frac{1}{2}, \frac{1}{2})$	1
	1	2	$(1,0),(0,1)$	$\frac{1}{2}, \frac{1}{2}$
	3	2	$(\frac{1}{2}+\frac{\sqrt{3}}{6}, \frac{1}{2}-\frac{\sqrt{3}}{6}),(\frac{1}{2}-\frac{\sqrt{3}}{6}, \frac{1}{2}+\frac{\sqrt{3}}{6})$	$\frac{1}{2}, \frac{1}{2}$
	5	3	$(\frac{1}{2},),(\frac{1}{2}+\sqrt{\frac{3}{20}}, \frac{1}{2}-\sqrt{\frac{3}{20}}),(\frac{1}{2}-\sqrt{\frac{3}{20}}, \frac{1}{2}+\sqrt{\frac{3}{20}})$	$\frac{8}{18}, \frac{5}{18}, \frac{5}{18}$
2	1	1	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	1
	1	3	$(1,0,0),(0,1,0),(0,0,1)$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$
	2	3	$(\frac{1}{2}, \frac{1}{2}, 0),(\frac{1}{2}, 0, \frac{1}{2}),(0, \frac{1}{2}, \frac{1}{2})$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$
	3	4	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}),(\frac{1}{5}, \frac{1}{5}, \frac{3}{5}),(\frac{1}{5}, \frac{3}{5}, \frac{1}{5}),(\frac{3}{5}, \frac{1}{5}, \frac{1}{5}),$	$-\frac{9}{16}, \frac{25}{48}, \frac{25}{48}, \frac{25}{48}$
3	1	1	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	1
	1	4	$(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$
	2	4	$(\frac{5-\sqrt{5}}{20}, \frac{5-\sqrt{5}}{20}, \frac{5-\sqrt{5}}{20}, \frac{5+3\sqrt{5}}{20})\ldots$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$

Matching of approximation order and quadrature order

- "Variational crime": instead of

$$a(u_h, v_h) = f(v_h)\ \forall v_h \in V_h$$

we solve

$$a_h(u_h, v_h) = f_h(v_h)\ \forall v_h \in V_h$$

where a_h, f_h are derived from their exact counterparts by quadrature

- For P^1 finite elements, zero order quadrature for volume integrals and first order quadrature for surface intergals is sufficient to keep the convergence order estimates stated before
- The rule of thumb for the volume quadrature is that the highest order terms must be evaluated exactly if the coefficients of the PDE are constant.

Practical realization of integrals

- Integral over barycentric coordinate function

$$\int_K \lambda_i(x)\ d\vec{x} = \frac{1}{3}|K|$$

- Right hand side integrals. Assume $f(x)$ is given as a piecewise linear function with given values in the nodes of the triangulation

$$f_i = \int_K f(x)\lambda_i(x)\ d\vec{x} \approx \frac{1}{3}|K|(f(a_0) + f(a_1) + f(a_2))$$

- Integral over space dependent heat conduction coefficient: Assume $\delta(x)$ is given as a piecewise linear function with given values in the nodes of the triangulation

$$a_{ij} = \int_K \delta(x)\ \vec{\nabla}\lambda_i\ \vec{\nabla}\lambda_j\ d\vec{x} = \frac{1}{3}(\delta(a_0)+\delta(a_1)+\delta(a_2))\int_K \vec{\nabla}\lambda_i\ \vec{\nabla}\lambda_j\ d\vec{x}$$

P1 FEM stiffness matrix condition number

- Homogeneous dirichlet boundary value problem

$$-\nabla \cdot \delta \vec{\nabla} u = f \quad \text{in } \Omega$$
$$u|_{\partial\Omega} = 0$$

- Lagrange degrees of freedom $a_1 \dots a_N$ corresponding to global basis functions $\phi_1 \dots \phi_N$ such that $\phi_i|_{\partial\Omega} = 0$ aka $\phi_i \in V_h \subset H_0^1(\Omega)$
- Stiffness matrix $A = (a_{ij})$:

$$a_{ij} = a(\phi_i, \phi_j) = \int_\Omega \delta \vec{\nabla} \phi_i \vec{\nabla} \phi_j \; d\vec{x}$$

- bilinear form $a(\cdot, \cdot)$ is self-adjoint, therefore A is symmetric, positive definite
- Condition number estimate for P^1 finite elements on quasi-uniform triangulation:

$$\kappa(A) \leq ch^{-2}$$

The problem with Dirichlet boundary conditions

- Homogeneous Dirichlet BC ⇒ include boundary condition into set of basis functions
- Inhomogeneous Dirichlet, may be only at a part of the boundary
  - Use exact approach from as in continous formulation (with lifting u_g etc) ⇒ highly technical
  - Eliminate Dirichlet BC algebraically after building of the matrix, i.e. fix "known unknowns" at the Dirichlet boundary ⇒ highly technical
  - Modifiy matrix such that equations at boundary exactly result in Dirichlet values ⇒ loss of symmetry of the matrix
  - Penalty method

Dirichlet BC: Algebraic manipulation

- Assume 1D situation with BC $u_1 = g$
- From integration in H^1 regardless of boundary values:

$$AU = \begin{pmatrix} \frac{1}{h} & -\frac{1}{h} & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \end{pmatrix}$$

- Fix u_1 and eliminate:

$$A'U = \begin{pmatrix} \frac{2}{h} & -\frac{1}{h} & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} f_2 + \frac{1}{h}g \\ f_3 \\ \vdots \end{pmatrix}$$

- A' becomes idd and stays symmetric
- operation is quite technical

Dirichlet BC: Modify boundary equations

- From integration in H^1 regardless of boundary values:

$$AU = \begin{pmatrix} \frac{1}{h} & -\frac{1}{h} & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \end{pmatrix}$$

- Modify equation at boundary to exactly represent Dirichlet values

$$A'U = \begin{pmatrix} \frac{1}{h} & 0 & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{1}{h}g \\ f_2 \\ f_3 \\ \vdots \end{pmatrix}$$

- A' becomes idd
- loses symmetry $\Rightarrow$ problem e.g. with CG method

Dirichlet BC: Discrete penalty trick

- From integration in H^1 regardless of boundary values:

$$AU = \begin{pmatrix} \frac{1}{h} & -\frac{1}{h} & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \end{pmatrix}$$

- Add penalty terms

$$A'U = \begin{pmatrix} \frac{1}{\varepsilon} + \frac{1}{h} & -\frac{1}{h} & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} f_1 + \frac{1}{\varepsilon} g \\ f_2 \\ f_3 \\ \vdots \end{pmatrix}$$

- A' becomes idd, keeps symmetry, and the realization is technically easy.
- If ε is small enough, $u_1 = g$ will be satisfied exactly within floating point accuracy.
- Iterative methods should be initialized with Dirichlet values.
- Works for nonlinear problems, finite volume methods

Dirichlet penalty trick, general formulation

- Dirichlet boundary value problem

$$-\nabla \cdot \delta \vec{\nabla} u = f \quad \text{in } \Omega$$
$$u|_{\partial\Omega} = g$$

- We discussed approximation of Dirichlet problem by Robin problem
- Practical realization uses discrete approach for Lagrange degrees of freedom $a_1 \dots a_N$ corresponding to global basis functions $\phi_1 \dots \phi_N$:
- Search $u_h = \sum_{i=1}^{N} u_i \phi_i \in V_h = \mathrm{span}\{\phi_1 \dots \phi_N\}$ such that

$$AU + \Pi U = F + \Pi G$$

where

- $U = (u_1 \dots u_N)$
- $A = (a_{ij})$: stiffness matrix with $a_{ij} = \int_\Omega \delta \vec{\nabla} \phi_i \vec{\nabla} \phi_j \, d\vec{x}$
- $F = \int_\Omega f \vec{\nabla} \phi_i \, d\vec{x}$
- $G = (g_i)$ with $g_i = \begin{cases} g(a_i), & a_i \in \partial\Omega \\ 0, & \text{else} \end{cases}$
- $\Pi = (\pi_{ij})$ is a diagonal matrix with $\pi_{ij} = \begin{cases} \frac{1}{\varepsilon}, & i = j, a_i \in \partial\Omega \\ 0, & \text{else} \end{cases}$

P1 FEM Stiffness matrix row sums

Row sums:

$$\begin{aligned}\sum_{j=1}^{N} a_{ij} &= \sum_{j=1}^{N} \int_\Omega \vec{\nabla}\phi_i \vec{\nabla}\phi_j \, dx = \int_\Omega \vec{\nabla}\phi_i \vec{\nabla}\left(\sum_{j=1}^{N} \phi_j\right) \, dx \\ &= \int_\Omega \vec{\nabla}\phi_i \vec{\nabla}(1) \, dx \\ &= 0\end{aligned}$$

P1 FEM stiffness matrix entry signs

Local stiffness matrix S_K

$$s_{ij} = \int_K \vec{\nabla}\lambda_i \vec{\nabla}\lambda_j \ dx = \frac{|K|}{2|K|^2} \left(y_{i+1} - y_{i+2}, x_{i+2} - x_{i+1}\right) \begin{pmatrix} y_{j+1} - y_{j+2} \\ x_{j+2} - x_{j+1} \end{pmatrix}$$

- Main diagonal entries positive
- Local contributions from element stiffness matrices: Scalar products of vectors orthogonal to edges. These are nonpositive if the angle between the edges are $\leq 90^\circ$
- *weakly acute triangulation*: all triangle angles are less than $\leq 90^\circ$
- In fact, for constant coefficients, in *2D*, Delaunay is sufficient, as contributions from opposite angles compensate each other
- All row sums are zero $\Rightarrow$ A is singular
- Matrix becomes irreducibly diagonally dominant if we add at least one positive value to the main diagonal, e.g. from Dirichlet BC

Stationary linear reaction-diffusion

- Assume additional process in each REV which produces or destroys species depending on the amount of species present with given rate r.

  Search function $u : \Omega \to \mathbb{R}$ such that

$$-\nabla \cdot \delta \vec{\nabla} u + ru = f \quad \text{in} \Omega$$

$$\delta \vec{\nabla} u \cdot \vec{n} + \alpha(u - g) = 0 \quad \text{on} \partial\Omega$$

- Coercivity guaranteed e.g. for $\alpha \geq 0$, $r > 0$ which means species destruction FEM formulation: search $u_h \in V_h = \mathrm{span}\{\phi_1 \dots \phi_N\}$ such that

$$\underbrace{\int_\Omega \delta \vec{\nabla} u_h \vec{\nabla} v_h \, d\vec{x}}_{\text{"stiffness matrix"}} + \underbrace{\int_\Omega r u_h v_h d\vec{x}}_{\text{"mass matrix"}} + \underbrace{\int_{\partial\Omega} \alpha u_h v_h \, ds}_{\text{"boundary mass matrix"}}$$

$$= \int_\Omega f v_h \, d\vec{x} + \int_{\partial\Omega} \alpha g v_h \, ds \; \forall v_h \in V_h$$

- Coercivity + symmetry $\Rightarrow$ positive definiteness

Mass matrix properties

- Mass matrix (for $r = 1$): $M = (m_{ij})$:

$$m_{ij} = \int_\Omega \phi_i \phi_j \, dx$$

- Self-adjoint, coercive bilinear form $\Rightarrow$ M is symmetric, positiv definite
- For a family of quasi-uniform, shape-regular triangulations, for every eigenvalue μ one has the estimate

$$c_1 h^d \leq \mu \leq c_2 h^d$$

T $\Rightarrow$ condition number $\kappa(M)$ bounded by constant independent of h:

$$\kappa(M) \leq c$$

- How to see this ? Let $u_h = \sum_{i=1}^{N} U_i \phi_i$, and μ an eigenvalue (positive,real!) Then

$$||u_h||_0^2 = (U, MU)_{\mathbb{R}^N} = \mu (U, U)_{\mathbb{R}^N} = \mu ||U||_{\mathbb{R}^N}^2$$

From quasi-uniformity we obtain

$$c_1 h^d ||U||_{\mathbb{R}^N}^2 \leq ||u_h||_0^2 \leq c_2 h^d ||U||_{\mathbb{R}^N}^2$$

Mass matrix M-Property (P1 FEM) ?

- For P^1-finite elements, all integrals $m_{ij} = \int_\Omega \phi_i \phi_j \, dx$ are zero or positive, so we get positive off diagonal elements.
- No M-Property!

Mass matrix lumping (P1 FEM)

- Local mass matrix for P1 FEM on element K
  (calculated by 2nd order exact edge midpoint quadrature rule):

$$M_K = |K| \begin{pmatrix} \frac{1}{6} & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{6} \end{pmatrix}$$

- Lumping: sum up off diagonal elements to main diagonal, set off diagonal entries to zero

$$\tilde{M}_K = |K| \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

- Interpretation as change of quadrature rule to first order exact vertex based quadrature rule
- Loss of accuracy, gain of stability

Effect of mass matrix lumping

- For P1 FEM on weakly acute or Delaunay triangulations, mass matrix lumping can guarantee M-Property of system matrix
- Adding a mass matrix which is not lumped yields a positive definite matrix (due to coercivity) and thus nonsingularity, but *destroys* M-property unless the absolute values of its off diagonal entries are less than those of A, i.e. for small r.
- Same situation witb Robin boundary conditions and boundary mass matrix
- Introducing the Dirichlet Penalty trick at continuous level without mass lumping would be disastrous.

The values of a piecewise linear function $u = u(\boldsymbol{x})$ can be defined in dependence of the values $u_i = u(\boldsymbol{p}_i)$ at the vertices of the simplex by the expression

$$u(\boldsymbol{x}) = \sum_{k=0}^{n} u_k \lambda_k(\boldsymbol{x})$$

The gradient of this function is given by

$$(7) \qquad \nabla u(\boldsymbol{x}) = \sum_{k=0}^{n} u_k \nabla \lambda_k(\boldsymbol{x})$$

where the $\nabla \lambda_k$ can be determinated from the equations

$$\delta_{ij} = \frac{\partial x_i}{\partial x_j} = \sum_{k=0}^{n} p_{ki} \frac{\partial \lambda_k}{\partial x_j}, \quad 0 = \sum_{k=0}^{n} \frac{\partial \lambda_k}{\partial x_j} \quad (i, j = 1, \ldots, n)$$

(cf. (3)), i.e. we have to solve the linear systems

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ \boldsymbol{p}_0 & \boldsymbol{p}_1 & \dots & \boldsymbol{p}_n \end{pmatrix} \begin{pmatrix} \lambda_0(\mathbf{0}) & \frac{\partial \lambda_0}{\partial x_1} & \dots & \frac{\partial \lambda_0}{\partial x_n} \\ \lambda_1(\mathbf{0}) & \frac{\partial \lambda_1}{\partial x_1} & \dots & \frac{\partial \lambda_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n(\mathbf{0}) & \frac{\partial \lambda_n}{\partial x_1} & \dots & \frac{\partial \lambda_n}{\partial x_n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

So we get

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ p_0 & p_1 & \dots & p_n \end{pmatrix} \begin{pmatrix} \partial_i \lambda_0 \\ \partial_i \lambda_1 \\ \vdots \\ \partial_i \lambda_n \end{pmatrix} = e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

$$\partial_i \lambda_0 + \partial_i \lambda_1 + \dots + \partial_i \lambda_n = e_{i,0}$$
$$p_0 \partial_i \lambda_0 + p_1 \partial_i \lambda_1 + \dots + p_n \partial_i \lambda_n = e_{i,(1\dots n)}$$

Comparing this with the determination of the normals of the faces (cf.

(2)) we get the identities

$$\nabla\lambda_i = \boldsymbol{g}_i, \quad \lambda_i(\mathbf{0}) = d_i \quad (i = 0, \ldots, n).$$

and therefore

$$\nabla u(\boldsymbol{x}) = \sum_{i=0}^{n} \boldsymbol{g}_i u_i = \begin{pmatrix} \boldsymbol{g}_0 & \boldsymbol{g}_1 & \ldots & \boldsymbol{g}_n \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{pmatrix}$$

holds. That means that

$$\sum_{i=0}^{n} \boldsymbol{g}_i u(\boldsymbol{p}_i) = \sum_{i=0}^{n} \frac{1}{h_i} \boldsymbol{a}_i u(\boldsymbol{p}_i)$$

can be considered as a discrete gradient of a function u.