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Lecture 16

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Recap: Weak formulation

Weak formulation of homogeneous Dirichlet problem

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \lambda \vec{\nabla} u \cdot \vec{\nabla} v \, d\vec{x} = \int_\Omega f v \, d\vec{x} \; \forall v \in H_0^1(\Omega)$$

- Then,

$$a(u, v) := \int_\Omega \lambda \vec{\nabla} u \cdot \vec{\nabla} v \, d\vec{x}$$

  is a self-adjoint bilinear form defined on the Hilbert space $H_0^1(\Omega)$.

- It is bounded due to Cauchy-Schwarz:

$$|a(u, v)| = \lambda \cdot \left| \int_\Omega \vec{\nabla} u \cdot \vec{\nabla} v \, d\vec{x} \right| \leq \lambda ||u||_{H_0^1(\Omega)} \cdot ||v||_{H_0^1(\Omega)}$$

- $f(v) = \int_\Omega f v \, d\vec{x}$ is a linear functional on $H_0^1(\Omega)$. For Hilbert spaces V the dual space V' (the space of linear functionals) can be identified with the space itself.

The Lax-Milgram lemma

Theorem: Let V be a Hilbert space. Let $a : V \times V \to \mathbb{R}$ be a self-adjoint bilinear form, and f a linear functional on V. Assume a is coercive, i.e.

$$\exists \alpha > 0 : \forall u \in V, a(u,u) \geq \alpha ||u||_V^2.$$

Then the problem: find $u \in V$ such that

$$a(u,v) = f(v) \; \forall v \in V$$

admits one and only one solution with an a priori estimate

$$||u||_V \leq \frac{1}{\alpha} ||f||_{V'}$$

$\square$

Coercivity of weak formulation

Theorem: Assume $\lambda > 0$. Then the weak formulation of the heat conduction problem: search $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \lambda \vec{\nabla} u \cdot \vec{\nabla} v \, d\vec{x} = \int_\Omega fv \, d\vec{x} \; \forall v \in H_0^1(\Omega)$$

has an unique solution.

Proof: $a(u, v)$ is cocercive:

$$a(u, u) = \int_\Omega \lambda \vec{\nabla} u \cdot \vec{\nabla} u \, d\vec{x} = \lambda ||u||^2_{H_0^1(\Omega)}$$

□

Weak formulation of inhomogeneous Dirichlet problem

$$-\nabla \cdot \lambda \vec{\nabla} u = f \text{ in } \Omega$$
$$u = g \text{ on } \partial\Omega$$

If g is smooth enough, there exists a *lifting* $u_g \in H^1(\Omega)$ such that $u_g|_{\partial\Omega} = g$. Then, we can re-formulate:

$$-\nabla \cdot \lambda \vec{\nabla}(u - u_g) = f + \nabla \cdot \lambda \vec{\nabla} u_g \text{ in } \Omega$$
$$u - u_g = 0 \text{ on } \partial\Omega$$

Find $u \in H^1(\Omega)$ such that

$$u = u_g + \phi$$
$$\int_\Omega \lambda \vec{\nabla}\phi \cdot \vec{\nabla} v \, d\vec{x} = \int_\Omega f v \, d\vec{x} + \int_\Omega \lambda \vec{\nabla} u_g \cdot \vec{\nabla} v \quad \forall v \in H_0^1(\Omega)$$

Here, necessarily, $\phi \in H_0^1(\Omega)$ and we can apply the theory for the homogeneous Dirichlet problem.

Weak formulation of Robin problem II

Let

$$a^R(u,v) := \int_\Omega \lambda \vec{\nabla} u \cdot \vec{\nabla} v \, d\vec{x} + \int_{\partial\Omega} \alpha uv \, ds$$
$$f^R(v) := \int_\Omega fv \, d\vec{x} + \int_{\partial\Omega} gv \, ds$$

Find $u \in H^1(\Omega)$ such that

$$a^R(u,v) = f^R(v) \, \forall v \in H^1(\Omega)$$

If $\lambda > 0$ and $\alpha > 0$ then $a^R(u,v)$ is cocercive, and by Lax-Milgram we establish the existence of a weak solution

The Galerkin method I

- Weak formulations "live" in Hilbert spaces which essentially are infinite dimensional
- For computer representations we need finite dimensional approximations
- The Galerkin method and its modifications provide a general scheme for the derivation of finite dimensional appoximations
- Finite dimensional subspaces of Hilbert spaces are the spans of a set of basis functions, and are Hilbert spaces as well $\Rightarrow$ e.g. the Lax-Milgram lemma is valid there as well

The Galerkin method II

- Let V be a Hilbert space. Let $a : V \times V \to \mathbb{R}$ be a self-adjoint bilinear form, and f a linear functional on V. Assume a is coercive with coercivity constant α, and continuity constant γ.
- Continuous problem: search $u \in V$ such that

$$a(u, v) = f(v) \ \forall v \in V$$

- Let $V_h \subset V$ be a finite dimensional subspace of V
- "Discrete" problem $\equiv$ Galerkin approximation:
  Search $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h) \ \forall v_h \in V_h$$

By Lax-Milgram, this problem has a unique solution as well.

Céa's lemma

- What is the connection between u and u_h ?
- Let $v_h \in V_h$ be arbitrary. Then

$$\begin{aligned}
\alpha||u - u_h||^2 &\leq a(u - u_h, u - u_h) \quad \text{(Coercivity)} \\
&= a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h) \\
&= a(u - u_h, u - v_h) \quad \text{(Galerkin Orthogonality)} \\
&\leq \gamma||u - u_h|| \cdot ||u - v_h|| \quad \text{(Boundedness)}
\end{aligned}$$

- As a result

$$||u - u_h|| \leq \frac{\gamma}{\alpha} \inf_{v_h \in V_h} ||u - v_h||$$

- Up to a constant, the error of the Galerkin approximation is the error of the best approximation of the solution in the subspace V_h.

From the Galerkin method to the matrix eqation

- Let $\phi_1 \dots \phi_n$ be a set of basis functions of V_h.
- Then, we have the representation $u_h = \sum_{j=1}^n u_j\phi_j$
- In order to search $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h)\ \forall v_h \in V_h$$

  it is actually sufficient to require

$$\begin{aligned} a(u_h, \phi_i) &= f(\phi_i)\ (i = 1 \dots n) \\ a\left(\sum_{j=1}^n u_j\phi_j, \phi_i\right) &= f(\phi_i)\ (i = 1 \dots n) \\ \sum_{j=1}^n a(\phi_j, \phi_i)u_j &= f(\phi_i)\ (i = 1 \dots n) \\ AU &= F \end{aligned}$$

  with $A = (a_{ij}), a_{ij} = a(\phi_i, \phi_j),\ F = (f_i), f_i = F(\phi_i),\ U = (u_i)$.
- Matrix dimension is $n \times n$. Matrix sparsity ?

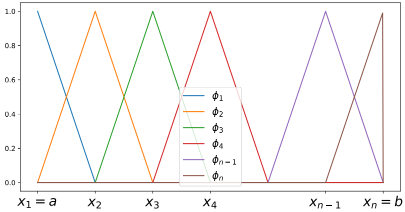
Obtaining a finite dimensional subspace

- Let $\Omega = (a, b) \subset \mathbb{R}^1$
- Let $a(u, v) = \int_a^b \delta \vec{\nabla} u \vec{\nabla} v d\vec{x} + \alpha u(a)v(a) + \alpha u(b)v(b)$
- Calculus 101 provides a finite dimensional subspace: the space of sin/cos functions up to a certain frequency $\Rightarrow$ *spectral method*
- Ansatz functions have global support $\Rightarrow$ full $n \times n$ matrix
- OTOH: rather fast convergence for smooth data
- Generalization to higher dimensions possible
- Big problem in irregular domains: we need the eigenfunction basis of some operator...
- Spectral methods are successful in cases where one has regular geometry structures and smooth/constant coefficients – e.g. "Spectral Einstein Code"

The finite element idea I

- Choose basis functions with local support. ⇒ only integrals of basis function pairs with overlapping support contribute to matrix.
- Linear finite elements in $\Omega = (a, b) \subset \mathbb{R}^1$:
- Partition $a = x_1 \leq x_2 \leq \cdots \leq x_n = b$
- Basis functions (for $i = 1 \dots n$)

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}}, & i > 1, x \in (x_{i-1}, x_i) \\ \frac{x_{i+1} - x}{x_{i+1} - x_i}, & i < n, x \in (x_i, x_{i+1}) \\ 0, & \text{else} \end{cases}$$



FE matrix elements for 1D heat equation I

- Any function $u_h \in V_h = \mathrm{span}\{\phi_1 \dots \phi_n\}$ is piecewise linear, and the coefficients in the representation $u_h = \sum_{i=1}^n u_i\phi_i$ are the values $u_h(x_i)$.
- Fortunately, we are working with a weak formulation, and weak derivatives are well defined (and coincide with the classical derivatives where the basis functions are smooth)
- Let ϕ_i, ϕ_j be two basis functions, regard

$$s_{ij} = \int_a^b \vec{\nabla}\phi_i \cdot \vec{\nabla}\phi_j d\ x$$

- We have $\mathrm{supp}\,\phi_i \cap \mathrm{supp}\,\phi_j = \emptyset$ unless $i = j$, $i + 1 = j$ or $i - 1 = j$.
- Therefore $s_{ij} = 0$ unless $i = j$, $i + 1 = j$ or $i - 1 = j$.

FE matrix elements for 1D heat equation II

- Let $j = i + 1$. Then $\operatorname{supp}\phi_i \cap \operatorname{supp}\phi_j = (x_i, x_{i+1})$, $\phi_i' = -\frac{1}{h}$, $\phi_j' = frac1h$ where $h = x_{i+1} - x_i$

$$\int_a^b \vec{\nabla}\phi_i \vec{\nabla}\phi_j d\vec{x} = \int_{x_i}^{x_{i+1}} \phi_i'\phi_j' d\vec{x} = -\int_{x_i}^{x_{i+1}} \frac{1}{h^2} d\vec{x} = -\frac{1}{h}$$

- Similarly, for $j = i - 1$: $\int_a^b \vec{\nabla}\phi_i \vec{\nabla}\phi_j d\vec{x} = -\frac{1}{h}$
- For $1 < i < N$:

$$\int_a^b \vec{\nabla}\phi_i \vec{\nabla}\phi_i d\vec{x} = \int_{x_{i-1}}^{x_{i+1}} (\phi_i')^2 d\vec{x} = \int_{x_{i-1}}^{x_{i+1}} \frac{1}{h^2} d\vec{x} = \frac{2}{h}$$

- For $i = 1$ or $i = N$: $\int_a^b \vec{\nabla}\phi_i \vec{\nabla}\phi_i d\vec{x} = \frac{1}{h}$
- For the right hand side, calculate vector elements $f_i = \int_a^b f(x)\phi_i dx$ using a quadrature rule.

FE matrix elements for 1D heat equation III

Adding the boundary integrals yields

$$A = \begin{pmatrix} \alpha + \frac{1}{h} & -\frac{1}{h} & & & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & & \\ & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \\ & & & & -\frac{1}{h} & \frac{1}{h} + \alpha \end{pmatrix}$$

... the same matrix as for the finite difference and finite volume methods

Simplices

- Let $\{\vec{a}_1 \ldots \vec{a}_{d+1}\} \subset \mathbb{R}^d$ such that the d vectors $\vec{a}_2 - \vec{a}_1 \ldots \vec{a}_{d+1} - \vec{a}_1$ are linearly independent. Then the convex hull K of $\vec{a}_1 \ldots \vec{a}_{d+1}$ is called *simplex*, and $\vec{a}_1 \ldots \vec{a}_{d+1}$ are called *vertices* of the simplex.
- *Unit simplex*: $\vec{a}_1 = (0...0), \vec{a}_1 = (0, 1 \ldots 0) \ldots \vec{a}_{d+1} = (0 \ldots 0, 1)$.

$$K = \left\{ \vec{x} \in \mathbb{R}^d : x_i \geq 0 \ (i = 1 \ldots d) \text{ and } \sum_{i=1}^{d} x_i \leq 1 \right\}$$

- A general simplex can be defined as an image of the unit simplex under some affine transformation
- F_i: face of K opposite to $\vec{a}_i$
- $\vec{n}_i$: outward normal to F_i

Simplex characteristics

- Diameter of K: $h_K = \max_{\vec{x}_1, \vec{x}_2 \in K} ||\vec{x}_1 - \vec{x}_2||$
  $\equiv$ length of longest edge if K
- ρ_K diameter of largest ball that can be inscribed into K
- $\sigma_K = \frac{h_K}{\rho_K}$: local shape regularity measure
  - $\sigma_K = 2\sqrt{3}$ for equilateral triangle
  - $\sigma_K \to \infty$ if largest angle approaches π.

Barycentric coordinates

Definition: Let $K \subset \mathbb{R}^d$ be a d-simplex given by the points $\vec{a}_1 \dots \vec{a}_{d+1}$.
Let $\Lambda(x) = (\lambda_1(\vec{x}) \dots \lambda_{d+1}(\vec{x}))$ be a vector such that for all $\vec{x} \in \mathbb{R}^d$

$$\sum_{j=1}^{d+1} \vec{a}_j \lambda_j(\vec{x}) = \vec{x}, \qquad \sum_{j=1}^{d+1} \lambda_j(\vec{x}) = 1$$

This vector is called the vector of *barycentric coordinates* of $\vec{x}$ with respect to K.

Barycentric coordinates II

Lemma The barycentric coordinates of a given point is well defined and unique. Moreover, for the simplex edges $\vec{a}_i$, one has

$$\lambda_j(\vec{a}_i) = \delta_{ij}$$

Proof: The definition of Λ given by a $d+1 \times d+1$ system of equations with the matrix

$$M = \begin{pmatrix} a_{1,1} & a_{2,1} & \dots & a_{d+1,1} \\ a_{1,2} & a_{2,2} & \dots & a_{d+1,2} \\ \vdots & \vdots & & \vdots \\ a_{1,d} & a_{2,d} & \dots & a_{d+1,d} \\ 1 & 1 & \dots & 1 \end{pmatrix}$$

Subtracting the first column from the others gives

$$M' = \begin{pmatrix} a_{1,1} & a_{2,1} - a_{1,1} & \dots & a_{d+1,1} - a_{1,1} \\ a_{1,2} & a_{2,2} - a_{1,2} & \dots & a_{d+1,2} - a_{1,2} \\ \vdots & \vdots & & \vdots \\ a_{1,d} & a_{2,d} - a_{1,d} & \dots & a_{d+1,d} - a_{1,d} \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

Barycentric coordinates III

$\det M = \det M'$ is the determinant of the matrix whose columns are the edge vectors of K which are linearly independent.

For the simplex edges one has

$$\sum_{j=1}^{d+1} \vec{a}_j \lambda_j(\vec{a}_i) = \vec{a}_i$$

which is fulfilled if $\lambda_j(\vec{a}_i) = 1$ for $i = j$ and $\lambda_j(\vec{a}_i) = 0$ for $i \neq j$. And we have uniqueness. $\square$

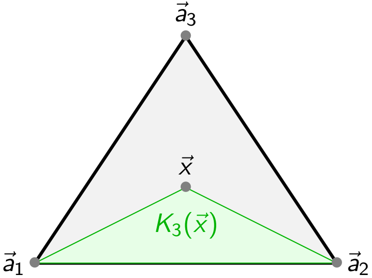
At the same time, the measure (area) is calculated as $|K| = \frac{1}{d!}|\det M'|$.

Barycentric coordinates IV

- Let $K_j(\vec{x})$ be the subsimplex of K made of $\vec{x}$ and $\vec{a}_1 \dots \vec{a}_{d+1}$ with $\vec{a}_j$ omitted.
- Its measure $|K_j(\vec{x})|$ is established from its determinant and a linear function of the coordiates for $\vec{x}$.
- One has $\frac{|K_j(\vec{a}_i)|}{|K|} = \delta_{ij}$ and therefore,

$$\lambda_j(\vec{x}) = \frac{|K_j(\vec{x})|}{|K|}$$

is the ratio of the measures of $K_j(\vec{x})$ and K.



Conformal triangulations

- Let $\mathcal{T}_h$ be a subdivision of the polygonal domain $\Omega \subset \mathbb{R}^d$ into non-intersecting compact simplices K_m, $m = 1 \dots n_e$:

$$\overline{\Omega} = \bigcup_{m=1}^{n_e} K_m$$

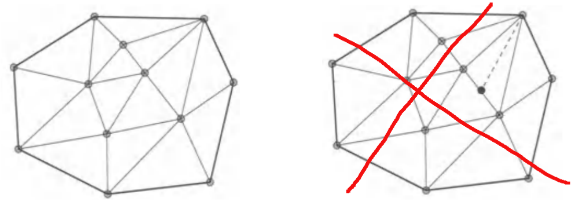
- Each simplex can be seen as the image of a affine transformation of a reference (e.g. unit) simplex $\widehat{K}$:

$$K_m = T_m(\widehat{K})$$

- We assume that it is conformal, i.e. if K_m, K_n have a $d-1$ dimensional intersection $F = K_m \cap K_n$, then there is a face $\widehat{F}$ of $\widehat{K}$ and renumberings of the vertices of K_n, K_m such that $F = T_m(\widehat{F}) = T_n(\widehat{F})$ and $T_m|_{\widehat{F}} = T_n|_{\widehat{F}}$

Conformal triangulations II

- $d = 1$: Each intersection $F = K_m \cap K_n$ is either empty or a common vertex
- $d = 2$: Each intersection $F = K_m \cap K_n$ is either empty or a common vertex or a common edge



- $d = 3$: Each intersection $F = K_m \cap K_n$ is either empty or a common vertex or a common edge or a common face
- Delaunay triangulations are conformal

Shape regularity

- Now we discuss a family of meshes $\mathcal{T}_h$ for $h \to 0$.
- For given $\mathcal{T}_h$, assume that $h = \max_{K \in \mathcal{T}_h} h_K$
- A family of meshes is called *shape regular* if

$$\forall h, \forall K \in \mathcal{T}_h, \sigma_K = \frac{h_K}{\rho_K} \leq \sigma_0$$

- In 1D, $\sigma_K = 1$
- In 2D, $\sigma_K \leq \frac{2}{\sin \theta_K}$ where θ_K is the smallest angle

Polynomial space $\mathbb{P}_k$

- Space of polynomials in $x_1 \dots x_d$ of total degree $\leq k$ with real coefficients $\alpha_{i_1 \dots i_d}$:

$$\mathbb{P}_k = \left\{ p(x) = \sum_{\substack{0 \leq i_1 \dots i_d \leq k \\ i_1 + \dots + i_d \leq k}} \alpha_{i_1 \dots i_d} x_1^{i_1} \dots x_d^{i_d} \right\}$$

- Dimension:

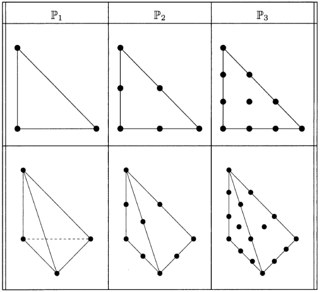
$$\dim \mathbb{P}_k = \binom{d+k}{k} = \begin{cases} k+1, & d=1 \\ \frac{1}{2}(k+1)(k+2), & d=2 \\ \frac{1}{6}(k+1)(k+2)(k+3), & d=3 \end{cases}$$

$$\dim \mathbb{P}_1 = d+1$$

$$\dim \mathbb{P}_2 = \begin{cases} 3, & d=1 \\ 6, & d=2 \\ 10, & d=3 \end{cases}$$

$\mathbb{P}_k$ simplex finite elements

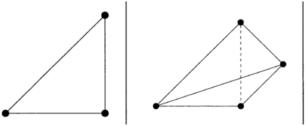
- K: simplex spanned by $\vec{a}_1 \dots \vec{a}_{d+1}$ in $\mathbb{R}^d$
- For $0 \leq i_1 \dots i_{d+1} \leq k$, $i_1 + \dots + i_{d+1} = k$, let the set of nodes $\Sigma = \{\vec{\sigma}_1 \dots \vec{\sigma}_s\}$ be defined by the points $\vec{a}_{i_1 \dots i_d; k}$ with barycentric coordinates $(\frac{i_1}{k} \dots \frac{i_{d+1}}{k})$.

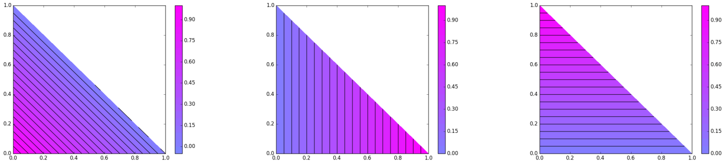


- $s = \operatorname{card} \Sigma = \dim \mathbb{P}_K \Rightarrow$ there exists a basis $\theta_1 \dots \theta_s$ of $\mathbb{P}_k$ such that $\theta_i(\vec{\sigma}_j) = \delta_{ij}$

$\mathbb{P}_1$ simplex finite elements

- K: simplex spanned by $a_1 \dots a_{d+1}$ in $\mathbb{R}^d$
- $s = d + 1$
- Nodes $\equiv$ vertices
- Basis functions $\theta_1 \dots \theta_{d+1} \equiv$ barycentric coordinates $\lambda_1 \dots \lambda_{d+1}$

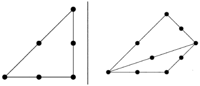


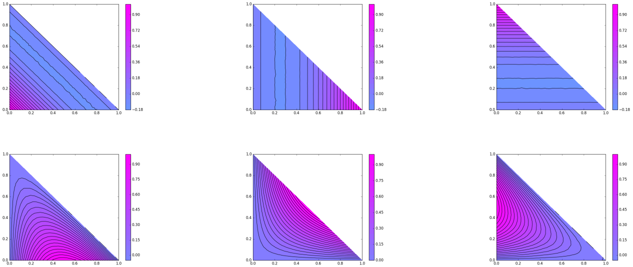


$\mathbb{P}_2$ simplex finite elements

- K: simplex spanned by $a_1 \dots a_{d+1}$ in $\mathbb{R}^d$
- Nodes $\equiv$ vertices + edge midpoints
- Basis functions:

  $\lambda_i(2\lambda_i - 1), (0 \leq i \leq d); \quad 4\lambda_i\lambda_j, \quad (0 \leq i < j \leq d) \quad$ ("edge bubbles")





Finite elements

- Finite element: triplet $\{K, P, \Sigma\}$ where
  - $K \subset \mathbb{R}^d$: compact, connected Lipschitz domain with non-empty interior
  - P: finite dimensional space of functions $p : K \to \mathbb{R}$
  - $\Sigma = \{\sigma_1 \dots \sigma_s\} \subset \mathcal{L}(P, \mathbb{R})$: set of linear forms defined on P called *local degrees of freedom* such that the mapping

    $$\begin{aligned} \Lambda_\Sigma : P &\to \mathbb{R}^s \\ p &\mapsto (\sigma_1(p) \dots \sigma_s(p)) \end{aligned}$$

    is bijective, i.e. Σ is a basis of $\mathcal{L}(P, \mathbb{R})$.
- Given a set of points $\{\vec{\sigma}_1 \dots \vec{\sigma}_s\} \subset K$, we use the symbols $\sigma_1 \dots \sigma_s$ to denote the evaluation of a function at the respective points. Elements with this type of degree of freedom functionals are called *Lagrangian*.

General finite elements

- Simplicial finite elements can be defined on triangulations of polygonal domains. During the course we will stick to this case.
- More general elements are possible: cuboids, but also prismatic elements etc.
- For vector PDEs, one can define finite elements for vector valued functions
- Different types of degrees of freedom (e.g. derivatives) are possible
- A curved domain Ω may be approximated by a polygonal domain Ω_h which is then triangulated. During the course, we will ignore this difference.
- Curved element geometries are possible. Isoparametric finite elements use the polynomial space to define a mapping of some polyghedral reference element to an element with curved boundary

Global degrees of freedom

- Given a triangulation $\mathcal{T}_h$
- Let $\{\vec{a}_1 \dots \vec{a}_N\} = \bigcup_{K \in \mathcal{T}_h} \{\vec{\sigma}_{K,1} \dots \vec{\sigma}_{K,s}\}$ be the set of global degrees of freedom.
- Degree of freedom map

$$
\begin{aligned}
j : \mathcal{T}_h \times \{1 \dots s\} &\to \{1 \dots N\} \\
(K, m) &\mapsto j(K, m) \text{ the global degree of freedom number}
\end{aligned}
$$

Lagrange finite element space

- Given a triangulation $\mathcal{T}_h$ of Ω, define the spaces

$$P_h^k = \{v_h \in \mathbb{C}^0(\Omega) : v_h|_K \in \mathbb{P}_K \ \forall K \in \mathcal{T}_j\} \subset H^1(\Omega)$$
$$P_{0,h}^k = \{v_h \in P_h^k : v_h|_{\partial\Omega} = 0\} \subset H_0^1(\Omega)$$

- Global shape functions $\theta_1, \ldots, \theta_N \in P_h^k$ defined by

$$\phi_i|_K(\vec{a}_{K,m}) = \begin{cases} \delta_{mn} & \text{if } \exists n \in \{1 \ldots s\} : j(K,n) = i \\ 0 & \text{otherwise} \end{cases}$$

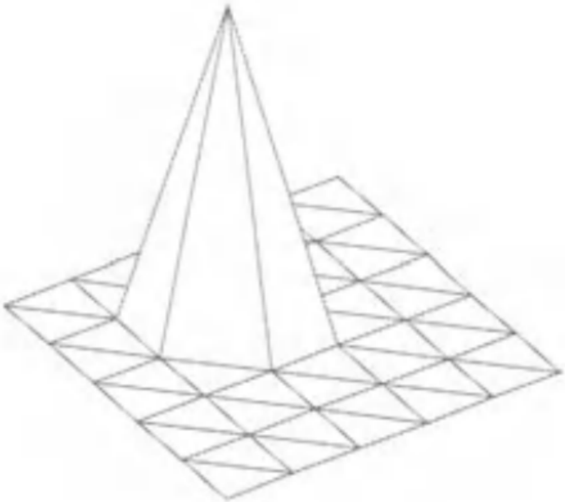
- $\{\phi_1, \ldots, \phi_N\}$ is a basis of P_h, and $\gamma_1 \ldots \gamma_N$ is a basis of $\mathcal{L}(P_h, \mathbb{R})$:
  - $\{\phi_1, \ldots, \phi_N\}$ are linearly independent: if $\sum_{j=1}^N \alpha_j \phi_j = 0$ then evaluation at $\vec{a}_1 \ldots \vec{a}_N$ yields that $\alpha_1 \ldots \alpha_N = 0$.
  - Let $v_h \in P_h$. Let $w_h = \sum_{j=1}^N v_h(\vec{a}_j)\phi_j$. Then for all $K \in \mathcal{T}_h$, $v_h|_K$ and $w_h|_K$ coincide in the local nodes $\vec{a}_{K,1} \ldots \vec{a}_{K,2}$, $\Rightarrow v_h|_K = w_h|_K$.

Finite element approximation space

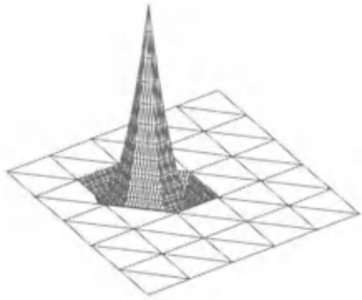
We have

d	k	$N = \dim P_h^k$
1	1	N_v
1	2	$N_v + N_{el}$
1	3	$N_v + 2N_{el}$
2	1	N_v
2	2	$N_v + N_{ed}$
2	3	$N_v + 2N_{ed} + N_{el}$
3	1	N_v
3	2	$N_v + N_{ed}$
3	3	$N_v + 2N_{ed} + N_f$

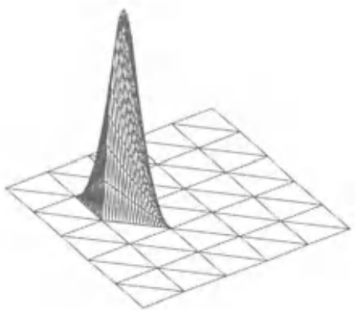
P^1 global shape functions



P^2 global shape functions



Node based



Edge based

Local Lagrange interpolation operator

- Let $\{K, P, \Sigma\}$ be a finite element with shape function bases $\{\theta_1 \dots \theta_s\}$. Let $V(K) = \mathbb{C}^0(K)$ and $P \subset V(K)$
- *local interpolation operator*

$$\mathcal{I}_K : V(K) \to P$$
$$v \mapsto \sum_{i=1}^{s} v(\vec{\sigma}_i)\theta_i$$

- P is invariant under the action of $\mathcal{I}_K$, i.e. $\forall p \in P, \mathcal{I}_K(p) = p$:
  - Let $p = \sum_{j=1}^{s} \alpha_j \theta_j$ Then,

$$\mathcal{I}_K(p) = \sum_{i=1}^{s} p(\vec{\sigma}_i)\theta_i = \sum_{i=1}^{s}\sum_{j=1}^{s} \alpha_j \theta_j(\vec{\sigma}_i)\theta_i$$
$$= \sum_{i=1}^{s}\sum_{j=1}^{s} \alpha_j \delta_{ij}\theta_i = \sum_{j=1}^{s} \alpha_j \theta_j$$

Global Lagrange interpolation operator

Let $V_h = P_h^k$

$$\mathcal{I}_h : \mathcal{C}^0(\bar{\Omega}_h) \to V_h$$
$$v(\vec{x}) \mapsto v_h(\vec{x}) = \sum_{i=1}^{N} v(\vec{a}_i)\phi_i(\vec{x})$$

Reference finite element

- Let $\{\widehat{P}, \widehat{K}, \widehat{\Sigma}\}$ be a fixed finite element
- Let T_K be some affine transformation and $K = T_K(\widehat{K})$
- There is a linear bijective mapping ψ_K between functions on K and functions on $\widehat{K}$:

$$\begin{aligned}\psi_K : V(K) &\to V(\widehat{K}) \\ f &\mapsto f \circ T_K\end{aligned}$$

- Let
  - $K = T_K(\widehat{K})$
  - $P_K = \{\psi_K^{-1}(\widehat{p}); \widehat{p} \in \widehat{P}\}$,
  - $\Sigma_K = \{\sigma_{K,i}, i = 1 \ldots s : \sigma_{K,i}(p) = \widehat{\sigma}_i(\psi_K(p))\}$

  Then $\{K, P_K, \Sigma_K\}$ is a finite element.
- This construction allows to develop theory for a reference element and to lift it later to an arbitrary element.

Commutativity of interpolation and reference mapping

- $\mathcal{I}_{\hat{K}} \circ \psi_K = \psi_K \circ \mathcal{I}_K$,
  i.e. the following diagram is commutative:

$$
\begin{array}{ccc}
V(K) & \xrightarrow{\psi_K} & V(\widehat{K}) \\
\big\downarrow \mathcal{I}_K & & \big\downarrow \mathcal{I}_{\hat{K}} \\
P_K & \xrightarrow{\psi_K} & P_{\widehat{K}}
\end{array}
$$

- $\equiv$ Interpolation and reference mapping are interchangeable

Affine transformation estimates I

Lemma T:

- $|\det J_K| = \frac{meas(K)}{meas(\widehat{K})}$
- $||J_K|| \leq \frac{h_K}{\rho_{\hat{K}}}$, $||J_K^{-1}|| \leq \frac{h_{\hat{K}}}{\rho_K}$
- $\Rightarrow ||J_K|| \cdot ||J_K^{-1}|| \leq c_{\hat{K}}\sigma_K$

Proof:

- $|\det J_K| = \frac{meas(K)}{meas(\widehat{K})}$: basic property of affine mappings
- Further:

$$||J_K|| = \sup_{\hat{x}\neq 0} \frac{||J_K\hat{x}||}{||\hat{x}||} = \frac{1}{\rho_{\hat{K}}} \sup_{||\hat{x}||=\rho_{\hat{K}}} ||J_K\hat{x}||$$

  Set $\hat{x} = \hat{x}_1 - \hat{x}_2$ with $\hat{x}_1, \hat{x}_2 \in \widehat{K}$. Then $J_K\hat{x} = T_K\hat{x}_1 - T_K\hat{x}_2$ and one can estimate $||J_K\hat{x}|| \leq h_K$.

- For $||J_K^{-1}||$ regard the inverse mapping $\square$

Estimate of derivatives under affine transformation

- For $w \in H^s(K)$ recall the H^s seminorm $|w|^2_{s,K} = \sum_{|\beta|=s} ||\partial^\beta w||^2_{L^2(K)}$
- We have

$$||v||^2_{L^2(K)} = \int_K |v(x)|^2 dx = \int_{\hat{K}} |v(T_K(\hat{x}))|^2 \det J_K d\hat{x}$$

- For the derivative

$$|v|^2_{H^1(K)} = \int_K ||\nabla v||^2(x) dx = \int_{\hat{K}} ||J_K^{-T} \vec{\hat{\nabla}} \hat{v}(\hat{x})||^2 \det J_K d\hat{x}$$

Estimate of derivatives under affine transformation II

Lemma D: Let $w \in H^s(K)$ and $\widehat{w} = w \circ T_K$. There exists a constant c such that

$$|\hat{w}|_{s,\hat{K}} \leq c||J_K||^s|\det J_K|^{-\frac{1}{2}}|w|_{s,K}$$
$$|w|_{s,K} \leq c||J_K^{-1}||^s|\det J_K|^{\frac{1}{2}}|\hat{w}|_{s,\hat{K}}$$

Proof: Let $|\alpha| = s$. By affinity and chain rule one obtains

$$||\partial^\alpha \hat{w}||_{L^2(\hat{K})} \leq c||J_K||^s \sum_{|\beta|=s} ||\partial^\beta w \circ T_K||_{L^2(K)}$$

Changing variables in the right hand side yields

$$||\partial^\alpha \hat{w}||_{L^2(\hat{K})} \leq c||J_K||^s|\det J_K|^{-\frac{1}{2}}|w|_{s,K}$$

Summation over α yields the first inequality. Regarding the inverse mapping yields the second estimate. □

Local interpolation error estimate I

Theorem: Let $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ be a finite element with associated normed vector space $V(\widehat{K})$. Assume that

$$\mathbb{P}_k \subset \widehat{P} \subset H^2(\widehat{K}) \subset V(\widehat{K})$$

Then there exists $c > 0$ such that for all $m = 0 \dots 2$, $K \in \mathcal{T}_h$, $v \in H^2(K)$:

$$|v - \mathcal{I}_K^1 v|_{m,K} \leq c h_K^{2-m} \sigma_K^m |v|_{2,K}.$$

I.e. the the local interpolation error can be estimated through h_K, σ_K and the norm of a higher derivative.

Local interpolation error estimate II

Draft of Proof

- Estimate on reference element $\hat{K}$ using deeper results from functional analysis:

$$|\hat{w} - \mathcal{I}^1_{\hat{K}}\hat{w}|_{m,\hat{K}} \leq c|\hat{w}|_{2,\hat{K}} \qquad (*)$$

  (From Poincare like inequality, e.g. for $v \in H^1_0(\Omega)$, $c||v||_{L^2} \leq ||\vec{\nabla} v||_{L^2}$: under certain circumstances, we can can estimate the norms of lower derivatives by those of the higher ones)
- Derive estimate on K from estimate on $\hat{K}$: Let $v \in H^2(K)$ and set $\hat{v} = v \circ T_K$. We know that $(\mathcal{I}^1_K v) \circ T_K = \mathcal{I}^1_{\hat{K}}\hat{v}$.

$$\begin{aligned}
|v - \mathcal{I}^1_K v|_{m,K} &\leq c||J_K^{-1}||^m |\det J_K|^{\frac{1}{2}} |\hat{v} - \mathcal{I}^1_{\hat{K}}\hat{v}|_{m,\hat{K}} && \text{(Lemma E)}\\
&\leq c||J_K^{-1}||^m |\det J_K|^{\frac{1}{2}} |\hat{v}|_{2,\hat{K}} && (*)\\
&\leq c||J_K^{-1}||^m ||J_K||^2 |v|_{2,K} && \text{(Lemma E)}\\
&= c(||J_K|| \cdot ||J_K^{-1}||)^m ||J_K||^{2-m} |v|_{2,K} &&\\
&\leq c h_K^{2-m} \sigma_K^m |v|_{2,K} && \text{(Lemma T)}
\end{aligned}$$

Local interpolation: special cases

- $m = 0$: $|v - \mathcal{I}_K^1 v|_{0,K} \leq ch_K^2 |v|_{2,K}$
- $m = 1$: $|v - \mathcal{I}_K^1 v|_{1,K} \leq ch_K \sigma_K |v|_{2,K}$

Global interpolation error estimate

Theorem Let Ω be polyhedral, and let $\mathcal{T}_h$ be a shape regular family of affine meshes. Then there exists c such that for all h, $v \in H^2(\Omega)$,

$$||v - \mathcal{I}_h^1 v||_{L^2(\Omega)} + \sum_{m=1}^{2} h^m \left(\sum_{K \in \mathcal{T}_h} |v - \mathcal{I}_h^1 v|_{m,K}^2 \right)^{\frac{1}{2}} \leq c h^2 |v|_{2,\Omega}$$

and

$$\lim_{h \to 0} \left(\inf_{v_h \in V_h^1} ||v - v_h||_{L^2(\Omega)} \right) = 0$$

Global interpolation error estimate for Lagrangian finite elements, $k = 1$

- Assume $v \in H^2(\Omega)$, e.g. if problem coefficients are smooth and the domain is convex

$$||v - \mathcal{I}_h^1 v||_{0,\Omega} + h|v - \mathcal{I}_h^1 v|_{1,\Omega} \leq ch^2|v|_{2,\Omega}$$

$$|v - \mathcal{I}_h^1 v|_{1,\Omega} \leq ch|v|_{2,\Omega}$$

$$\lim_{h\to 0}\left(\inf_{v_h \in V_h^1} |v - v_h|_{1,\Omega}\right) = 0$$

- If $v \in H^2(\Omega)$ cannot be guaranteed, estimates become worse. Example: L-shaped domain.
- These results immediately can be applied in Cea's lemma.

Error estimates for homogeneous Dirichlet problem

- Search $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \delta \vec{\nabla} u \vec{\nabla} v \, d\vec{x} = \int_\Omega fv \, d\vec{x} \; \forall v \in H_0^1(\Omega)+$$

Then, $\lim_{h\to 0} ||u - u_h||_{1,\Omega} = 0$. If $u \in H^2(\Omega)$ (e.g. on convex domains) then

$$||u - u_h||_{1,\Omega} \leq ch|u|_{2,\Omega}$$
$$||u - u_h||_{0,\Omega} \leq ch^2|u|_{2,\Omega}$$

Under certain conditions (convex domain, smooth coefficients) one also has

$$||u - u_h||_{0,\Omega} \leq ch|u|_{1,\Omega}$$

("Aubin-Nitsche-Lemma")

H^2-Regularity

- $u \in H^2(\Omega)$ may be *not* fulfilled e.g.
  - if Ω has re-entrant corners
  - if on a smooth part of the domain, the boundary condition type changes
  - if problem coefficients (δ) are discontinuos
- Situations differ as well between two and three space dimensions
- Delicate theory, ongoing research in functional analysis
- Consequence for simuations
  - Deterioration of convergence rate
  - Remedy: local refinement of the discretization mesh
    - using a priori information
    - using a posteriori error estimators + automatic refinement of discretizatiom mesh

Higher regularity

- If $u \in H^s(\Omega)$ for $s > 2$, convergence order estimates become even better for P^k finite elements of order $k > 1$.
- Depending on the regularity of the solution the combination of grid adaptation and higher oder ansatz functions may be successful