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Lecture 12

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Partial differential equations

Differential operators: notations

Given: domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3 \dots)$

- Dot product: for $\vec{x}, \vec{y} \in \mathbb{R}^d$, $\vec{x} \cdot \vec{y} = \sum_{i=1}^{d} x_i y_i$
- Bounded domain $\Omega \subset \mathbb{R}^d$, with piecewise smooth boundary
- Scalar function $u : \Omega \to \mathbb{R}$
- Vector function $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix} : \Omega \to \mathbb{R}^d$
- Partial derivative $\partial_i u = \frac{\partial u}{\partial x_i}$
- For a multiindex $\alpha = (\alpha_1 \dots \alpha_d)$, let
  - $|\alpha| = \alpha_1 + \dots + \alpha_d$
  - $\partial^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdot \dots \cdot \partial x_d^{\alpha_d}}$

Basic Differential operators

- *Gradient* of scalar function $u : \Omega \to \mathbb{R}$:

$$\mathrm{grad} = \vec{\nabla} = \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_d \end{pmatrix} : u \mapsto \vec{\nabla} u = \begin{pmatrix} \partial_1 u \\ \vdots \\ \partial_d u \end{pmatrix}$$

- *Divergence* of vector function $\vec{v} = \Omega \to \mathbb{R}^d$:

$$\mathrm{div} = \nabla \cdot : \vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix} \mapsto \nabla \cdot \ \vec{v} = \partial_1 v_1 + \cdots + \partial_d v_d$$

- *Laplace operator* of scalar function $u : \Omega \to \mathbb{R}$

$$\begin{aligned} \mathrm{div} \cdot \mathrm{grad} &= \nabla \cdot \vec{\nabla} \\ &= \Delta : u \mapsto \Delta u = \partial_{11} u + \cdots + \partial_{dd} u \end{aligned}$$

Lipschitz domains

Definition: A connected open subset $\Omega \subset \mathbb{R}^d$ is called *domain*. If Ω is a bounded set, the domain is called *bounded*.

Definition:

- Let $D \subset \mathbb{R}^n$. A function $f : D \to \mathbb{R}^m$ is called *Lipschitz continuous* if there exists $c > 0$ such that $\|f(x) - f(y)\| \leq c\|x - y\|$ for any $x, y \in D$
- A hypersurface in $\mathbb{R}^n$ is a *graph* if for some k it can be represented as

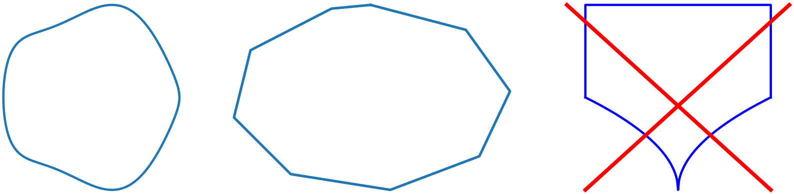
  $$x_k = f(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$$

  defined on some domain $D \subset \mathbb{R}^{n-1}$
- A domain $\Omega \subset \mathbb{R}^n$ is a *Lipschitz domain* if for all $x \in \partial\Omega$, there exists a neigborhood of x on $\partial\Omega$ which can be represented as the graph of a Lipschitz continuous function.

Lipschitz domains II

Standard PDE calculus happens in Lipschitz domains

- Boundaries of Lipschitz domains are continuous
- Polygonal domains are Lipschitz
- Boundaries of Lipschitz domains have no cusps (e.g. the graph of $y = \sqrt{|x|}$ has a cusp at $x = 0$)



Divergence theorem (Gauss' theorem)

Theorem: Let Ω be a bounded Lipschitz domain and $\vec{v} : \Omega \to \mathbb{R}^d$ be a continuously differentiable vector function. Let $\vec{n}$ be the outward normal to Ω. Then,

$$\int_\Omega \nabla \cdot \vec{v} \, d\vec{x} = \int_{\partial\Omega} \vec{v} \cdot \vec{n} \, ds$$

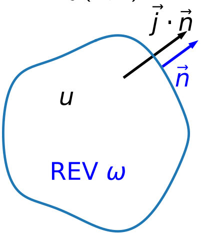
□

This is a generalization of the Newton-Leibniz rule of calculus:
Let $d = 1$, $\Omega = (a, b)$. Then $n_a = (-1)$, $n_b = (1)$, $\nabla \cdot v = v'$.

$$\int_a^b v'(x) \, dx = v(a)n_a + v(b)n_b = v(b) - v(a)$$

Species flux through boundary of an REV

- $u(\vec{x}, t) : \Omega \times [0, T] \to \mathbb{R}$: time dependent local amount of species
- $f(\vec{x}, t) : \Omega \times [0, T] \to \mathbb{R}$: species sources/sinks
- $\vec{j}(\vec{x}, t)$: vector field of the species flux



- $\omega \subset \Omega$: *representative elementary volume (REV)*
- $(t_0, t_1) \subset (0, T)$: subset of the time interval

- $J(t) = \int_{\partial\omega} \vec{j}(\vec{x}, t) \cdot \vec{n}\, ds$: flux of species trough $\partial\omega$ at moment t
- $U(t) = \int_{\omega} u(\vec{x}, t)\, d\vec{x}$: amount of species in ω at moment t
- $F(t) = \int_{\omega} f(\vec{x}, t)\, d\vec{x}$: rate of creation/destruction at moment t

Continuity equation

- Change of amount of species in ω during (t_0, t_1) proportional to the sum of the amount transported through the boundary and the amount created/destroyed

$$U(t_1) - U(t_0) = \int_{t_0}^{t_1} J(t)\, dt + \int_{t_0}^{t_1} F(t)\, dt$$

$$\int_\omega (u(\vec{x}, t_1) - u(\vec{x}, t_0))\, d\vec{x} = \int_{t_0}^{t_1} \int_{\partial\omega} \vec{j} \cdot \vec{n}\, ds\, dt = \int_{t_0}^{t_1} \int_\omega f(\vec{x}, t)\, ds$$

- Using Gauss' theorem, rewrite this as

$$0 = \int_{t_0}^{t_1} \int_\omega \partial_t u(\vec{x}, t)\, d\vec{x}\, dt - \int_{t_0}^{t_1} \int_\omega \nabla \cdot \vec{j}\, d\vec{x}\, dt - \int_{t_0}^{t_1} \int_\omega f(\vec{x}, t)\, ds$$

- True for all $\omega \subset \Omega$, $(t_0, t_1) \subset (0, T) \Rightarrow$
  Continuity equation in differential form

$$\partial_t u(\vec{x}, t) - \nabla \cdot \vec{j}(\vec{x}, t) = f(\vec{x}, t)$$

Flux expressions

- In many cases: species flux $\vec{j}$ is proportional to $-\vec{\nabla} u$
- Assumption: $\vec{j} = -\delta \vec{\nabla} u$, where $\delta > 0$ can be constant, space dependent or even depend on u. For simplicity, we assume δ to be constant, unless stated otherwise.
- Heat conduction:
  $u = T$: temperature
  $\delta = \lambda$: heat conduction coefficient
  f: heat source
  $\vec{j} = -\lambda \vec{\nabla} T$: "Fourier law"
- Diffusion of molecules in a given medium (for low concentrations)
  $u = c$: concentration
  $\delta = D$: diffusion coefficient
  f: species source (e.g due to reactions)
  $\vec{j} = -D \vec{\nabla} c$: "Fick's law"

More flux expressions

- Flow in a saturated porous medium:
  $u = p$: pressure
  $\delta = k$: permeability
  $\vec{j} = -k\vec{\nabla}p$: "Darcy's law"
- Electrical conduction:
  $u = \varphi$: electric potential
  $\delta = \sigma$: electric conductivity
  $\vec{j} = -\sigma\vec{\nabla}\varphi \equiv$ current density: "Ohms's law"
- Electrostatics in a constant magnetic field:
  $u = \varphi$: electric potential
  $\delta = \varepsilon$: dielectric permittivity
  $\vec{E} = \vec{\nabla}\phi$: electric field
  $\vec{j} = \vec{D} = \varepsilon\vec{E} = \varepsilon\vec{\nabla}\varphi$: electric displacement field: "Gauss's Law"
  $f = \rho$: charge density

Second order partial differential equaions (PDEs)

Combine continuity equation with flux expression:

- Transient problem:

Parabolic PDE:

$$\partial_t u(\vec{x}, t) - \nabla \cdot (\delta \vec{\nabla} u(\vec{x}, t)) = f(\vec{x}, t)$$

- Stationary case: $\partial_t u = 0 \Rightarrow$

Elliptic PDE

$$-\nabla \cdot (\delta \vec{\nabla} u(\vec{x})) = f(\vec{x})$$

- For solvability we need additional conditions:
  - Initial condition in the time dependent case: $u(\vec{x}, 0) = u_0(\vec{x})$
  - Boundary conditions: behavior of solution on $\partial\Omega$

Second order PDEs: boundary conditions

- Assume $\partial\Omega = \cup_{i=1}^{N_\Gamma}\Gamma_i$ is the union of a finite number of non-intersecting subsets Γ_i which are locally Lipschitz.
- On each Γ_i, specify one of
  - Fixed solution at boundary $\Rightarrow$ *Dirichlet* ("first kind") BC: let $g_i : \Gamma_i \to \mathbb{R}$ (*homogeneous* for $g_i = 0$)

$$u(\vec{x}, t) = g_i(\vec{x}, t) \quad \text{for } \vec{x} \in \Gamma_i$$

  - Fixed boundary flux $\Rightarrow$ *Neumann* ("second kind") BC: Let $g_i : \Gamma_i \to \mathbb{R}$ (*homogeneus* for $g_i = 0$)

$$\delta\vec{\nabla} u(\vec{x}, t) \cdot \vec{n} = g_i(\vec{x}, t) \quad \text{for } \vec{x} \in \Gamma_i$$

  - Boundary flux proportional to solution $\Rightarrow$ *Robin* ("third kind") BC: let $\alpha_i > 0, g_i : \Gamma_i \to \mathbb{R}$

$$\delta\vec{\nabla} u(\vec{x}, t) \cdot \vec{n} + \alpha_i(\vec{x}, t)u(\vec{x}, t) = g_i(\vec{x}, t) \quad \text{for } \vec{x} \in \Gamma_i$$

PDEs: generalizations

- δ may depend on $\vec{x}$, u, $|\vec{\nabla} u|$ ... $\Rightarrow$ equations become nonlinear
- Coefficients can depend on other processes
  - temperature can influence conductvity
  - source terms can describe chemical reactions between different species
  - chemical reactions can generate/consume heat
  - Electric current generates heat ("Joule heating")
  - ...

  $\Rightarrow$ coupled PDEs
- Convective terms: $\vec{j} = -\delta \vec{\nabla} u + u\vec{v}$ where $\vec{v}$ is a convective velocity
- PDEs for vector unknowns
  - Momentum balance $\Rightarrow$ Navier-Stokes equations for fluid dynamics
  - Elasticity
  - Maxwell's electromagnetic field equations

Problems with “strong formulation”

Writing the PDE with divergence and gradient assumes smoothness of coefficients and at least second derivatives for the solution.

- δ may not be continuous (e.g. for heat conduction in a piece consisting of different materials) – what is then $\vec{\nabla} \cdot (\delta \vec{\nabla} u)$?
- Approximation of solution u e.g. by piecewise linear functions what does $\vec{\nabla} u$ mean ?
- Solution spaces of twice, and even once continuously differentiable functions is not well suited:
  - Favorable approximation functions (e.g. piecewise linear ones) are not contained
  - Though they can be equipped with norms ($\Rightarrow$ Banach spaces) they have no scalar product $\Rightarrow$ no Hilbert spaces
  - Not complete: Cauchy sequences of functions may not converge to elements in these spaces

Cauchy sequences of functions

- Let Ω be a Lipschitz domain, let V be a metric space of functions $f : \Omega \to \mathbb{R}$
- Regard sequences of functions $f_n = \{f_n\}_{n=1}^{\infty} \subset V$
- A *Cauchy sequence* is a sequence f_n of functions where the norm of the difference between two elements can be made arbitrarily small by increasing the element indices:

$$\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N} : \forall m, n > n_0, ||f_n - f_m|| < \varepsilon$$

- All convergent sequences of functions are Cauchy sequences
- A metric space V is *complete* if all Cauchy sequences f_n of its elements have a limit $f = \lim_{n\to\infty} f_n \in V$ within this space

Completion of a metric space

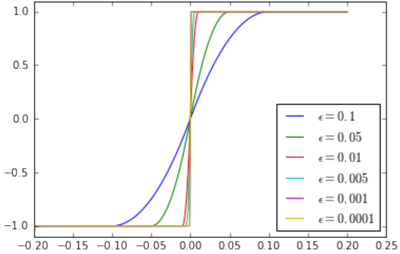
- Let V be a metric space. Its completion is the space $\bar{V}$ consisting of all elements of V and all possible limits of Cauchy sequences of elements of V.
- This procedure allows to carry over definitions which are applicable only to elements of V to more general ones
- This process depends on the norm which is part of the definition of the metric space

Example: construction of real numbers $\bar{V} = \mathbb{R}$ from rational numbers $V = \mathbb{Q}$ via Cauchy sequences: every real number is an equivalence class of Cauchy sequences with the same limit.

Completion in function spaces

- Example: step function

$$\theta_\epsilon(\vec{x}) = \begin{cases} 1, & \vec{x} \geq \epsilon \\ -(\frac{\vec{x}-\epsilon}{\epsilon})^2 + 1, & 0 \leq \vec{x} < \epsilon \\ (\frac{\vec{x}+\epsilon}{\epsilon})^2 - 1, & -\epsilon \leq \vec{x} < 0 \\ -1, & \vec{x} < -\epsilon \end{cases} \xrightarrow{\epsilon \to 0} \theta(\vec{x}) = \begin{cases} 1, & \vec{x} \geq 0 \\ -1, & \text{else} \end{cases}$$



- The discontinuous function $\theta(x)$ is the limit of a sequence of continuously differentiable functions θ_ϵ.

Riemann integral → Lebesgue integral

- Let Ω be a Lipschitz domain, let $C_c(\Omega)$ be the set of continuous functions $f : \Omega \to \mathbb{R}$ with compact support. ($\Rightarrow$ they vanish on $\partial\Omega$)
- For these functions, the Riemann integral $\int_\Omega f(\vec{x})\, d\vec{x}$ is well defined, and $\|f\|_{L^1} := \int_\Omega |f(\vec{x})| d\vec{x}$ provides a norm, and induces a metric.
- Let $L^1(\Omega)$ be the completion of $C_c(\Omega)$ with respect to the metric defined by the norm $\|\cdot\|_{L^1}$. That means that $L^1(\Omega)$ consists of all elements of $C_c(\Omega)$, and of all limits of Cauchy sequences of elements of $C_c(\Omega)$. Such functions are called *measurable*.
- For any measurable $f = \lim_{n\to\infty} f_n \in L^1(\Omega)$ with $f_n \in C_c(\Omega)$, define the *Lebesque integral*

$$\int_\Omega f(\vec{x})\, d\vec{x} := \lim_{n\to\infty} \int_\Omega f_n(\vec{x})\, d\vec{x}$$

as the limit of a sequence of Riemann integrals of continuous functions

Lebesgue integrable (measurable) functions

- Examples:
  - Step functions
  - Bounded functions which are continuous except in a finite number of points
- As the product of the completion process, measurable functions are equivalence classes, and saying f, g belong to the same equivalence class amounts to saying that $\|f - g\| = 0$. In this we say that f, g are equal *almost everywhere*.
- In particular, L^1 functions whose values differ in a finite number of points are equal almost everywhere.

Space of square integrable functions

- Let $L^2(\Omega)$ be the space of measureable functions such that

$$\int_\Omega |f(\vec{x})|^2 \, d\vec{x} < \infty$$

equipped with the norm

$$\|f\|_{L^2} = \left(\int_\Omega |f(\vec{x})|^2 \, d\vec{x} \right)^{\frac{1}{2}}$$

- The space $L^2(\Omega)$ is a *Hilbert space*, i.e. a Banach space equipped with a scalar product $(\cdot, \cdot)$ whose norm is induced by that scalar product, i.e. $||u|| = \sqrt{(u, u)}$. The scalar product in L^2 is

$$(f, g)_{L^2} = \int_\Omega f(\vec{x}) g(\vec{x}) \, d\vec{x}.$$

- Similar definitions for L^p, $0 < p \leq \infty$

Derivatives of measurable functions

- Derivatives of functions $f \in L^2(\Omega)$ are not a priori defined
- Due to the fact that they even may be not continuous, the usual way to define derivatives as the limit of difference quotients fails.
- $\Rightarrow$ introduce derivatives in an indirect, *weak* way

Green's theorem for smooth functions

Theorem Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and $u, v \in C^1(\overline{\Omega})$ (continuously differentiable). Let $\vec{n} = (n_1 \dots n_d)$ being the outward normal $\partial\Omega$. Then Ω,

$$\int_\Omega u \partial_i v \, d\vec{x} = \int_{\partial\Omega} u v n_i \, ds - \int_\Omega v \partial_i u \, d\vec{x}$$

□

This is a generalization of the integration by parts rule of calculus:
Let $d = 1$, $\Omega = (a, b)$. Then $n_a = (-1)$, $n_b = 1$, $\partial_i(\cdot) = (\cdot)'$.

$$\begin{aligned} \int_a^b u v'(x) \, dx &= n_a u(a) v(a) + n_b u(b) v(b) - \int_a^b u' v \, dx \\ &= u v \Big|_a^b - \int_a^b u' v \, dx \end{aligned}$$

Corollaries of Green's theorem

- Let $\vec{u} = (u_1 \dots u_d) : \Omega \to \mathbb{R}^d$ and $v : \Omega \to \mathbb{R}$. Then

$$\int_\Omega \left(\sum_{i=1}^{d} u_i \partial_i v \right) \, d\vec{x} = \int_{\partial\Omega} v \sum_{i=1}^{d} (u_i n_i) \, ds - \int_\Omega v \sum_{i=1}^{d} (\partial_i u_i) \, d\vec{x}$$

$$\int_\Omega \vec{u} \cdot \vec{\nabla} v \, d\vec{x} = \int_{\partial\Omega} v \vec{u} \cdot \vec{n} \, ds - \int_\Omega v \nabla \cdot \vec{u} \, d\vec{x}$$

- If $v = 0$ on $\partial\Omega$:

$$\int_\Omega u \partial_i v \, d\vec{x} = - \int_\Omega v \partial_i u \, d\vec{x}$$

$$\int_\Omega \vec{u} \cdot \vec{\nabla} v \, d\vec{x} = - \int_\Omega v \vec{\nabla} \cdot \vec{u} \, d\vec{x}$$

Weak derivative

- Let $L^1_{loc}(\Omega)$ be the set of functions which are Lebesgue integrable on every compact subset $K \subset \Omega$. Let $C_0^\infty(\Omega)$ be the set of functions infinitely differentiable with zero values on the boundary.
- For $u \in L^1_{loc}(\Omega)$ we *define* $\partial_i u$ by

$$\int_\Omega v \partial_i u \, d\vec{x} = - \int_\Omega u \partial_i v \, d\vec{x} \quad \forall v \in C_0^\infty(\Omega)$$

  and $\partial^\alpha u$ by

$$\int_\Omega v \partial^\alpha u \, d\vec{x} = (-1)^{|\alpha|} \int_\Omega u \partial_i v \, d\vec{x} \quad \forall v \in C_0^\infty(\Omega)$$

  if these integrals exist.
- For smooth functions, weak derivatives coincide with with the usual derivative

Sobolev spaces of square integrable functios

- For $k \geq 0$ the *Sobolev space* $H^k(\Omega)$ is the space functions where all up to the k-th derivatives are in L^2:

$$H^k(\Omega) = \{u \in L^2(\Omega) : \partial^\alpha u \in L^2(\Omega)\ \forall |\alpha| \leq k\}$$

with then norm

$$||u||_{H^k(\Omega)} = \left(\sum_{|\alpha| \leq k} ||\partial^\alpha u||^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$$

- Alternatively, H^k is the completion of C^∞ in the norm $||u||_{H^k(\Omega)}$
- $H^k_0(\Omega)$ is the completion of C^∞_0 in the norm $||u||_{H^k(\Omega)}$
- These Sobolev spaces are Banach spaces.
- Similar definitions exist for $p \neq 2$

Important function spaces

- $H^k(\Omega)$ is a Hilbert space with the scalar product

$$(u, v)_{H^k(\Omega)} = \sum_{|\alpha| \le k} \int_\Omega \partial^\alpha u \partial^\alpha v \; d\vec{x}$$

- $H_0^k(\Omega)$ is a Hilbert space with the scalar product

$$(u, v)_{H_0^k(\Omega)} = \sum_{|\alpha| = k} \int_\Omega \partial^\alpha u \partial^\alpha v \; d\vec{x}$$

Hilbert space structure

- For this course the most important:
  - $L^2(\Omega)$, scalar product $(u,v)_{L^2(\Omega)} = (u,v)_{0,\Omega} = \int_\Omega uv\ d\vec{x}$
  - $H^1(\Omega)$, scalar product $(u,v)_{H^1(\Omega)} = (u,v)_{1,\Omega} = \int_\Omega (uv + \vec{\nabla} u \cdot \vec{\nabla} v)\ d\vec{x}$
  - $H^1_0(\Omega)$, scalar product $(u,v)_{H^1_0(\Omega)} = \int_\Omega (\vec{\nabla} u \cdot \vec{\nabla} v)\ d\vec{x}$
- All of them are metric spaces with a scalar product and we have in each of them

$$
\begin{aligned}
|(u,v)|^2 &\leq (u,u)(v,v) \quad \text{Cauchy-Schwarz}\\
||u+v|| &\leq ||u|| + ||v|| \quad \text{Triangle inequality}
\end{aligned}
$$

A trace theorem

The notion of function values on the boundary initially is only well defined for continouos functions. So we need an extension of this notion to functions from Sobolev spaces.

Theorem: Let Ω be a bounded Lipschitz domain. Then there exists a bounded linear mapping

$$\mathrm{tr}: H^1(\Omega) \to L^2(\partial\Omega)$$

such that

(i) $\exists c > 0$ such that $\|\mathrm{tr}\, u\|_{0,\partial\Omega} \leq c\|u\|_{1,\Omega}$

(ii) $\forall u \in C^1(\bar{\Omega})$, $\mathrm{tr}\, u = u|_{\partial\Omega}$

□

Corollary: If $u \in H_0^1(\Omega)$ then $\mathrm{tr}\, u = 0$.

Derivation of weak formulation

- Sobolev space theory provides a convenient framework to formulate existence, uniqueness and approximations of solutions of PDEs.
- Let us first consider the stationary heat conduction equation with homogeneous Dirichlet boundary conditions and constant heat conduction coefficient $\lambda > 0$:

$$\begin{aligned} -\nabla \cdot \lambda \vec{\nabla} u(\vec{x}) &= f(\vec{x}) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

Multiply and integrate with an arbitrary *test function* $v \in C_0^\infty(\Omega)$ and apply Green's theorem using $v = 0$ on $\partial\Omega$

$$\begin{aligned} & -\int_\Omega (\nabla \cdot \lambda \vec{\nabla} u) v \, d\vec{x} = \int_\Omega f v \, d\vec{x} \\ \Rightarrow \quad & \int_\Omega \lambda \vec{\nabla} u \cdot \vec{\nabla} v \, d\vec{x} = \int_\Omega f v \, d\vec{x} \end{aligned}$$

Weak formulation of homogeneous Dirichlet problem

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \lambda \vec{\nabla} u \cdot \vec{\nabla} v \, d\vec{x} = \int_\Omega fv \, d\vec{x} \; \forall v \in H_0^1(\Omega)$$

- Then,

$$a(u, v) := \int_\Omega \lambda \vec{\nabla} u \cdot \vec{\nabla} v \, d\vec{x}$$

  is a self-adjoint bilinear form defined on the Hilbert space $H_0^1(\Omega)$.

- It is bounded due to Cauchy-Schwarz:

$$|a(u, v)| = \lambda \cdot \left| \int_\Omega \vec{\nabla} u \cdot \vec{\nabla} v \, d\vec{x} \right| \leq \lambda ||u||_{H_0^1(\Omega)} \cdot ||v||_{H_0^1(\Omega)}$$

- $f(v) = \int_\Omega fv \, d\vec{x}$ is a linear functional on $H_0^1(\Omega)$. For Hilbert spaces V the dual space V' (the space of linear functionals) can be identified with the space itself.

The Lax-Milgram lemma

Theorem: Let V be a Hilbert space. Let $a : V \times V \to \mathbb{R}$ be a self-adjoint bilinear form, and f a linear functional on V. Assume a is coercive, i.e.

$$\exists \alpha > 0 : \forall u \in V, a(u,u) \geq \alpha ||u||_V^2.$$

Then the problem: find $u \in V$ such that

$$a(u,v) = f(v) \ \forall v \in V$$

admits one and only one solution with an a priori estimate

$$||u||_V \leq \frac{1}{\alpha} ||f||_{V'}$$

□

Coercivity of weak formulation

Theorem: Assume $\lambda > 0$. Then the weak formulation of the heat conduction problem: search $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \lambda \vec{\nabla} u \cdot \vec{\nabla} v \, d\vec{x} = \int_\Omega fv \, d\vec{x} \; \forall v \in H_0^1(\Omega)$$

has an unique solution.

Proof: $a(u, v)$ is cocercive:

$$a(u, u) = \int_\Omega \lambda \vec{\nabla} u \cdot \vec{\nabla} u \, d\vec{x} = \lambda ||u||^2_{H_0^1(\Omega)}$$

□

Weak formulation of inhomogeneous Dirichlet problem

$$-\nabla \cdot \lambda \vec{\nabla} u = f \text{ in } \Omega$$
$$u = g \text{ on } \partial\Omega$$

If g is smooth enough, there exists a *lifting* $u_g \in H^1(\Omega)$ such that $u_g|_{\partial\Omega} = g$. Then, we can re-formulate:

$$-\nabla \cdot \lambda \vec{\nabla}(u - u_g) = f + \nabla \cdot \lambda \vec{\nabla} u_g \text{ in } \Omega$$
$$u - u_g = 0 \text{ on } \partial\Omega$$

Find $u \in H^1(\Omega)$ such that

$$u = u_g + \phi$$
$$\int_\Omega \lambda \vec{\nabla}\phi \cdot \vec{\nabla} v \, d\vec{x} = \int_\Omega f v \, d\vec{x} + \int_\Omega \lambda \vec{\nabla} u_g \cdot \vec{\nabla} v \quad \forall v \in H_0^1(\Omega)$$

Here, necessarily, $\phi \in H_0^1(\Omega)$ and we can apply the theory for the homogeneous Dirichlet problem.

Weak formulation of Robin problem

$$-\nabla \cdot \lambda \vec{\nabla} u = f \text{ in } \Omega$$
$$\lambda \vec{\nabla} u \cdot \vec{n} + \alpha u = g \text{ on } \partial\Omega$$

- Multiply and integrate with an arbitrary *test function* from $C_c^\infty(\Omega)$:

$$-\int_\Omega (\nabla \cdot \lambda \vec{\nabla} u) v \, d\vec{x} = \int_\Omega f v \, d\vec{x}$$
$$\int_\Omega \lambda \vec{\nabla} u \cdot \vec{\nabla} v \, d\vec{x} + \int_{\partial\Omega} \lambda \vec{\nabla} u \cdot \vec{n} v ds = \int_\Omega f v \, d\vec{x}$$
$$\int_\Omega \lambda \vec{\nabla} u \cdot \vec{\nabla} v \, d\vec{x} + \int_{\partial\Omega} \alpha u v \; ds = \int_\Omega f v \, d\vec{x} + \int_{\partial\Omega} g v \; ds$$

Weak formulation of Robin problem II

Let

$$a^R(u,v) := \int_\Omega \lambda \vec{\nabla} u \cdot \vec{\nabla} v \, d\vec{x} + \int_{\partial\Omega} \alpha uv \; ds$$

$$f^R(v) := \int_\Omega fv \, d\vec{x} + \int_{\partial\Omega} gv \, ds$$

Find $u \in H^1(\Omega)$ such that

$$a^R(u,v) = f^R(v) \; \forall v \in H^1(\Omega)$$

If $\lambda > 0$ and $\alpha > 0$ then $a^R(u,v)$ is cocercive, and by Lax-Milgram we establish the existence of a weak solution

Neumann boundary conditions

- Homogeneous Neumann:

$$\lambda \vec{\nabla} u \cdot \vec{n} = 0 \text{ on } \partial\Omega$$

- Inhomogeneous Neumann:

$$\lambda \vec{\nabla} u \cdot \vec{n} = g \text{ on } \partial\Omega$$

Find $u \in H^1(\Omega)$ such that

$$\int_\Omega \vec{\nabla} u \cdot \vec{\nabla} v \, d\vec{x} = \int_{\partial\Omega} gv \, ds \; \forall v \in H^1(\Omega)$$

Not coercive due to the fact that we can add an arbitrary constant to u and $a(u, u)$ stays the same!

Further discussion on boundary conditions

- Mixed boundary conditions:
  One can have differerent boundary conditions on different parts of the boundary. In particular, if Dirichlet or Robin boundary conditions are applied on at least a part of the boundary of measure larger than zero, the binlinear form becomes coercive.
- Natural boundary conditions: Robin, Neumann
  These are imposed in a "natural" way in the weak formulation
- Essential boundary conditions: Dirichlet
  explicitely imposed on the function space
- Coefficients $\lambda, \alpha \dots$ can be functions from Sobolev spaces as long as they do not change integrability of terms in the bilinear forms

Inhomogeneous Dirichlet problem: solution extrema

$$-\nabla \cdot \lambda \vec{\nabla} u = f \text{ in } \Omega$$
$$u = g \text{ on } \partial\Omega$$

- What can we say about minimum and maximum of the solution ?
- u has local local extremum in $\vec{x}_0 \in \Omega$ if
  - $\vec{x}_0$ is a critical point: $\vec{\nabla}_u|_{\vec{x}_0} = 0$
  - The matrix of second derivatives in $\vec{x}_0$ is definite
  - This is linked to the sign of the right hand side: if $f = 0$ the main diagonal entries have different signs (as their sum is zero), so perhaps we would get a saddle point

Solution extrema: weak formulation

- Search $u \in H^1(\Omega)$ such that

$$u = u_g + \phi$$
$$\int_\Omega \lambda \vec{\nabla}\phi \cdot \vec{\nabla} v \, d\vec{x} = \int_\Omega fv \, d\vec{x} - \int_\Omega \lambda \vec{\nabla} u_g \cdot \vec{\nabla} v \quad \forall v \in H_0^1(\Omega)$$

  Here, necessarily, $\phi \in H_0^1(\Omega)$ and we can apply the theory for the homogeneous Dirichlet problem.
- if u is a solution, we also have

$$\int_\Omega \lambda \vec{\nabla} u \cdot \vec{\nabla} v \, d\vec{x} = \int_\Omega fv \, d\vec{x} \quad \forall v \in H_0^1(\Omega)$$

  as we can add $\int_\Omega \lambda \vec{\nabla} \cdot u_g \vec{\nabla} v$ on left and right side

Inhomogeneous Dirichlet problem: minimum principle

- Let u be a solution
- Let $f \geq 0$ almost everywhere
- Let $g^\flat = \inf_{\partial\Omega} g$
- Let $w = (u - g^\flat)^- = \min\{u - g^\flat, 0\} \in H_0^1(\Omega)$
- Consequently, $w \leq 0$ almost everywhere
- As $\vec{\nabla} u = \vec{\nabla}(u - g^\flat)$ and $\vec{\nabla} w = 0$ where $w \neq u - g^\flat$, one has

$$
\begin{aligned}
0 \geq \int_\Omega f w \, d\vec{x} &= \int_\Omega \lambda \vec{\nabla} u \cdot \vec{\nabla} w \, d\vec{x} \\
&= \int_\Omega \lambda \vec{\nabla} w \cdot \vec{\nabla} w \, d\vec{x} \geq 0
\end{aligned}
$$

- Therefore: $(u - g^\flat)^- = 0$ and $u \geq g^\flat$ almost everywhere

Inhomogeneous Dirichlet problem: maximum principle

- Let u be a solution
- Let $f \leq 0$ almost everywhere
- Let $g^\sharp = \sup_{\partial\Omega} g$
- Let $w = (u - g^\sharp)^+ = \max\{u - g^\sharp, 0\} \in H_0^1(\Omega)$
- Consequently, $w \geq 0$ almost everywhere
- As $\vec{\nabla} u = \vec{\nabla}(u - g^\sharp)$ and $\vec{\nabla} w = 0$ where $w \neq u - g^\sharp$, one has

$$
\begin{aligned}
0 \geq \int_\Omega f w \, d\vec{x} &= \int_\Omega \lambda \vec{\nabla} u \cdot \vec{\nabla} w \, d\vec{x} \\
&= \int_\Omega \lambda \vec{\nabla} w \cdot \vec{\nabla} w \, d\vec{x} \geq 0
\end{aligned}
$$

- Therefore: $(u - g^\sharp)^- = 0$ and $u \leq g^\sharp$ almost everywhere

Inhomogeneous Dirichlet problem: minmax principle

Theorem: The weak solution of the inhomogeneous Dirichlet problem

$$-\nabla \cdot \lambda \vec{\nabla} u = f \text{ in } \Omega$$
$$u = g \text{ on } \partial\Omega$$

fulfills the global minimax principle: it attains its maximum at the boundary if $f \leq 0$ and attains its minimum at the boundary if $f \geq 0$.

Corollary: If $f = 0$ then u attains both its minimum and its maximum at the boundary.

Corollary: Local minimax principle:
This is true of any subdomain $\omega \subset \Omega$.

Corollary: Nonnegativity of the solution:
if $g \geq 0$ and $f \geq 0$ then $u \geq 0$

Interpretation of minimax principle

- Positive right hand side $\Rightarrow$ "production" of heat, matter ...
- No local minimum in the interior of domain if matter is produced.
- Also, positivity/nonnegativity of solutions if boundary conditions are positive/nonnegative
- Negative right hand side $\Rightarrow$ "consumption" of heat, matter ...
- No local maximum in the interior of domain if matter is consumed.
- Basic physical principle !