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Lecture 10

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The Gershgorin Circle Theorem (Semyon Gershgorin,1931)

(everywhere, we assume $n \geq 2$)

Theorem (Varga, Th. 1.11) Let A be an $n \times n$ (real or complex) matrix. Let Λ_i be the sum of the absolute values of the i-th rowoff-diagonal entries:

$$\Lambda_i = \sum_{\substack{j=1\dots n \\ j \neq i}} |a_{ij}|$$

If λ is an eigenvalue of A, then there exists r, $1 \leq r \leq n$ such that λ lies on the disk defined by the circle of radius Λ_r around a_{rr}:

$$|\lambda - a_{rr}| \leq \Lambda_r.$$

Gershgorin Circle Corollaries

Corollary: Any eigenvalue of A lies in the union of the disks defined by the Gershgorin circles

$$\lambda \in \bigcup_{i=1\dots n} \{\mu \in \mathbb{V} : |\mu - a_{ii}| \leq \Lambda_i\}$$

Corollary: The Gershgorin circle theorem allows to estimate the spectral radius $\rho(A)$:

$$\rho(A) \leq \max_{i=1\dots n} \sum_{j=1}^{n} |a_{ij}| = ||A||_\infty,$$

$$\rho(A) \leq \max_{j=1\dots n} \sum_{i=1}^{n} |a_{ij}| = ||A||_1.$$

Proof

$$|\mu - a_{ii}| \leq \Lambda_i \quad \Rightarrow \quad |\mu| \leq \Lambda_i + |a_{ii}| = \sum_{j=1}^{n} |a_{ij}|$$

Furthermore, $\sigma(A) = \sigma(A^T)$. □

Gershgorin circles: heat example I

$$A = \begin{pmatrix} \frac{2}{h} & -\frac{1}{h} & & & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & & \\ & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} \\ & & & & -\frac{1}{h} & \frac{2}{h} \end{pmatrix}$$

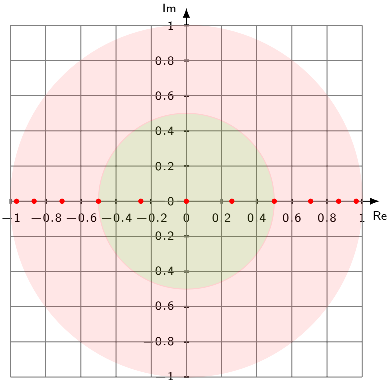
$$B = (I - D^{-1}A) = \begin{pmatrix} 0 & \frac{1}{2} & & & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & & & \\ & \frac{1}{2} & 0 & \frac{1}{2} & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & \frac{1}{2} & 0 & \frac{1}{2} & & \\ & & & \frac{1}{2} & 0 & \frac{1}{2} \\ & & & & \frac{1}{2} & 0 \end{pmatrix}$$

We have $b_{ii} = 0$, $\Lambda_i = \begin{cases} \frac{1}{2}, & i = 1, n \\ 1 & i = 2 \dots n-1 \end{cases} \Rightarrow$ estimate $|\lambda_i| \leq 1$

Gershgorin circles: heat example II

Let n=11, h=0.1:

$$\lambda_i = \cos\left(\frac{ih\pi}{1+2h}\right) \quad (i = 1 \ldots n)$$



⇒ the Gershgorin circle theorem is too pessimistic, we need a better theory . . .

Permutation matrices

- Permutation matrices are matrices which have exactly one non-zero entry in each row and each column which has value 1.
- There is a one-to-one correspondence permutations π of the the numbers $1 \dots n$ and $n \times n$ permutation matrices $P = (p_{ij})$ such that

$$p_{ij} = \begin{cases} 1, & \pi(i) = j \\ 0, & \text{else} \end{cases}$$

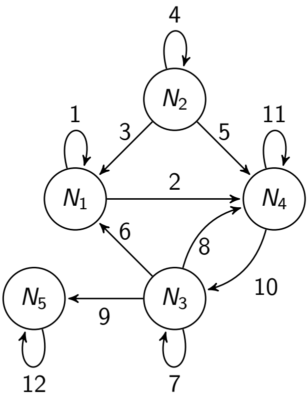
- Permutation matrices are orthogonal, and we have $P^{-1} = P^T$
- $A \to PA$ permutes the rows of A
- $A \to AP^T$ permutes the columns of A

Weighted directed graph representation of matrices

Define a directed graph from the nonzero entries of a matrix $A = (a_{ik})$:

- Nodes: $\mathcal{N} = \{N_i\}_{i=1\dots n}$
- Directed edges: $\mathcal{E} = \{\overrightarrow{N_k N_l} | a_{kl} \neq 0\}$
- Matrix entries $\equiv$ weights of directed edges

$$A = \begin{pmatrix} 1. & 0. & 0. & 2. & 0. \\ 3. & 4. & 0. & 5. & 0. \\ 6. & 0. & 7. & 8. & 9. \\ 0. & 0. & 10. & 11. & 0. \\ 0. & 0. & 0. & 0. & 12. \end{pmatrix}$$



- 1:1 equivalence between matrices and weighted directed graphs
- Convenient e.g. for sparse matrices

Reducible and irreducible matrices

Definition A is *reducible* if there exists a permutation matrix P such that

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

A is *irreducible* if it is not reducible.

Theorem (Varga, Th. 1.17): A is irreducible $\Leftrightarrow$ the matrix graph is connected, i.e. for each *ordered* pair (N_i, N_j) there is a path consisting of directed edges, connecting them.

Equivalently, for each i, j there is a sequence of consecutive nonzero matrix entries $a_{ik_1}, a_{k_1k_2}, a_{k_2k_3} \dots, a_{k_{r-1}k_r} a_{k_rj}$.

$\square$

Taussky theorem (Olga Taussky, 1948)

Theorem (Varga, Th. 1.18) Let A be irreducible. Assume that the eigenvalue λ is a boundary point of the union of all the disks

$$\lambda \in \partial \bigcup_{i=1\dots n} \{\mu \in \mathbb{C} : |\mu - a_{ii}| \le \Lambda_i\}$$

Then, all n Gershgorin circles pass through λ, i.e. for $i = 1 \dots n$,

$$|\lambda - a_{ii}| = \Lambda_i$$

Consequences for heat example from Taussky theorem

- $B = I - D^{-1}A$
- We had $b_{ii} = 0$, $\Lambda_i = \begin{cases} \frac{1}{2}, & i = 1, n \\ 1 & i = 2 \dots n-1 \end{cases} \Rightarrow$ estimate $|\lambda_i| \leq 1$
- Assume $|\lambda_i| = 1$. Then λ_i lies on the boundary of the union of the Gershgorin circles. But then it must lie on the boundary of both circles with radius $\frac{1}{2}$ and 1 around 0.
- Contradiction $\Rightarrow |\lambda_i| < 1$, $\rho(B) < 1$!

Diagonally dominant matrices

Definition Let $A = (a_{ij})$ be an $n \times n$ matrix.

- A is *diagonally dominant* if

  (i) for $i = 1 \dots n$, $|a_{ii}| \geq \sum_{\substack{j=1\dots n \\ j \neq i}} |a_{ij}|$

- A is *strictly diagonally dominant* (sdd) if

  (i) for $i = 1 \dots n$, $|a_{ii}| > \sum_{\substack{j=1\dots n \\ j \neq i}} |a_{ij}|$

- A is *irreducibly diagonally dominant* (idd) if

  (i) A is irreducible

  (ii) A is diagonally dominant –
  for $i = 1 \dots n$, $|a_{ii}| \geq \sum_{\substack{j=1\dots n \\ j \neq i}} |a_{ij}|$

  (iii) for at least one r, $1 \leq r \leq n$, $|a_{rr}| > \sum_{\substack{j=1\dots n \\ j \neq r}} |a_{rj}|$

A very practical nonsingularity criterion

Theorem (Varga, Th. 1.21): Let A be strictly diagonally dominant or irreducibly diagonally dominant. Then A is nonsingular.

If in addition, $a_{ii} > 0$ is real for $i = 1 \dots n$, then all real parts of the eigenvalues of A are positive:

$$\mathrm{Re}\lambda_i > 0, \quad i = 1 \dots n$$

Corollary

Theorem: If A is complex hermitian or real symmetric, sdd or idd, with positive diagonal entries, it is positive definite.

Proof: All eigenvalues of A are real, and due to the nonsingularity criterion, they must be positive, so A is positive definite.

□

Perron-Frobenius Theorem (1912/1907)

Definition: A real n-vector $\mathbf{x}$ is

- positive ($\mathbf{x} > 0$) if all entries of $\mathbf{x}$ are positive
- nonnegative ($\mathbf{x} \geq 0$) if all entries of $\mathbf{x}$ are nonnegative

Definition: A real $n \times n$ matrix A is

- positive ($A > 0$) if all entries of A are positive
- nonnegative ($A \geq 0$) if all entries of A are nonnegative

Theorem(Varga, Th. 2.7) Let $A \geq 0$ be an irreducible $n \times n$ matrix. Then

(i) A has a positive real eigenvalue equal to its spectral radius $\rho(A)$.
(ii) To $\rho(A)$ there corresponds a positive eigenvector $\mathbf{x} > 0$.
(iii) $\rho(A)$ increases when any entry of A increases.
(iv) $\rho(A)$ is a simple eigenvalue of A.

Proof: See Varga. □

Perron-Frobenius for general nonnegative matrices

Each $n \times n$ matrix can be brought to the normal form

$$PAP^T = \begin{pmatrix} R_{11} & R_{12} & \dots & R_{1m} \\ 0 & R_{22} & \dots & R_{2m} \\ \vdots & & \ddots & \\ 0 & 0 & \dots & R_{mm} \end{pmatrix}$$

where for $j = 1 \dots m$, either R_{jj} irreducible or $R_{jj} = (0)$.

Theorem(Varga, Th. 2.20) Let $A \geq 0$ be an $n \times n$ matrix. Then

(i) A has a nonnegative eigenvalue equal to its spectral radius $\rho(A)$. This eigenvalue is positive unless A is reducible and its normal form is strictly upper triangular

(ii) To $\rho(A)$ there corresponds a nonzero eigenvector $\mathbf{x} \geq 0$.

(iii) $\rho(A)$ does not decrease when any entry of A increases.

Proof: See Varga; $\sigma(A) = \bigcup_{j=1}^{m} \sigma(R_{jj})$, apply irreducible Perron-Frobenius to R_{jj}. □

Theorem on Jacobi matrix

Theorem: Let A be sdd or idd, and D its diagonal. Then

$$\rho(|I - D^{-1}A|) < 1$$

Proof: Let $B = (b_{ij}) = I - D^{-1}A$. Then

$$b_{ij} = \begin{cases} 0, & i = j \\ -\frac{a_{ij}}{a_{ii}}, & i \neq j \end{cases}$$

If A is sdd, then for $i = 1 \dots n$,

$$\sum_{j=1\dots n} |b_{ij}| = \sum_{\substack{j=1\dots n\\ j\neq i}} |\frac{a_{ij}}{a_{ii}}| = \frac{\Lambda_i}{|a_{ii}|} < 1$$

Therefore, $\rho(|B|) < 1$.

Jacobi method convergence

Corollary: Let A be sdd or idd, and D its diagonal. Assume that $a_{ii} > 0$ and $a_{ij} \leq 0$ for $i \neq j$. Then $\rho(I - D^{-1}A) < 1$, i.e. the Jacobi method converges.

Proof In this case, $|B| = B$ $\square$.

- Here, we made assumptions on the sign pattern and the diagonal dominance of the matrix. No additional information on the nonzero pattern or the symmetry has been used.
- Does this generalize to other iterative methods ?

Regular splittings

- $A = M - N$ is a regular splitting if
  - M is nonsingular
  - M^{-1}, N are nonnegative, i.e. have nonnegative entries
- Regard the iteration $u_{k+1} = M^{-1}Nu_k + M^{-1}b$.
- $B = I - M^{-1}A = M^{-1}N$ is a nonnegative matrix.

Convergence theorem for regular splitting

Theorem: Assume A is nonsingular, $A^{-1} \geq 0$, and $A = M - N$ is a regular splitting. Then $\rho(M^{-1}N) < 1$.

Proof: Let $B = M^{-1}N$. Then $A = M(I - B)$, therefore $I - B$ is nonsingular.

In addition

$$A^{-1}N = (M(I - M^{-1}N))^{-1}N = (I - M^{-1}N)^{-1}M^{-1}N = (I - B)^{-1}B$$

By Perron-Frobenius (for general matrices), $\rho(B)$ is an eigenvalue with a nonnegative eigenvector $\mathbf{x}$. Thus,

$$0 \leq A^{-1}N\mathbf{x} = \frac{\rho(B)}{1 - \rho(B)}\mathbf{x}$$

Therefore $0 \leq \rho(B) \leq 1$.
Assume that $\rho(B) = 1$. Then there exists $\mathbf{x} \neq \mathbf{0}$ such that $B\mathbf{x} = \mathbf{x}$.
Consequently, $(I - B)\mathbf{x} = \mathbf{0}$, contradicting the nonsingularity of $I - B$.
Therefore, $\rho(B) < 1$. $\square$

Convergence rate comparison

Corollary: $\rho(M^{-1}N) = \frac{\tau}{1+\tau}$ where $\tau = \rho(A^{-1}N)$.

Proof: Rearrange $\tau = \frac{\rho(B)}{1-\rho(B)}$ $\square$

Corollary: Let $A^{-1} \geq 0$, $A = M_1 - N_1$ and $A = M_2 - N_2$ be regular splittings.

If $N_2 \geq N_1$, then $1 > \rho(M_2^{-1}N_2) \geq \rho(M_1^{-1}N_1)$.

Proof: $\tau_2 = \rho(A^{-1}N_2) \geq \rho(A^{-1}N_1) = \tau_1$

But $\frac{\tau}{1+\tau}$ is strictly increasing. $\square$

Convergence rate comparison II

- Let $A^{-1} \geq 0$, $A = D - E - F$, $D > 0$ diagonal, $E, F \geq 0$ upper resp. lower triangular parts.
- Jacobi: $M_J = D$, $N_J = E + F$. $M_J^{-1} > 0 \Rightarrow$ regular splitting
- Gauss-Seidel: $M_{GS} = D - E$, $N_{GS} = F \geq 0$. Show $M_{GS}^{-1} \geq 0$:

$$M_{GS} = \begin{pmatrix} d_{11} & -e_{12} & -e_{13} & \dots & -e_{1n} \\ & d_{22} & -e_{23} & \dots & -e_{2n} \\ & & \ddots & \ddots & \vdots \\ & & & d_{n-1,n-1} & -e_{n-1,n} \\ & & & \dots & d_{nn} \end{pmatrix}$$

Elimination steps for $M_{GS} v = r$:

$$v_n = \frac{r_n}{d_{nn}}, \qquad v_{n-1} = \frac{r_n + e_{n-1,n} v_n}{d_{n-1,n-1}} \dots$$

All coefficients are nonnegative $\Rightarrow M_{GS} - N_{GS}$: regular splitting

- $N_{GS} \leq N_J \Rightarrow \rho(M_{GS}^{-1} N_{GS}) \leq \rho(M_J^{-1} N_J)$

M-Matrix definition

Definition Let A be an $n \times n$ real matrix. A is called M-Matrix if

(i) $a_{ij} \leq 0$ for $i \neq j$

(ii) A is nonsingular

(iii) $A^{-1} \geq 0$

Corollary: If A is an M-Matrix, then $A^{-1} > 0 \Leftrightarrow A$ is irreducible.

Proof: See Varga. □

M-Matrix main diagonal

Theorem: If A is an M-matrix, than is diagonal $D_A > 0$ is positive.

Proof: Let $C = A^{-1} \geq 0$. The $AC = I$ and $(AC)_{ii} = 1$.

$$\sum_{k=1}^{n} a_{ik} c_{ki} = 1$$

$$a_{ii} c_{ii} = 1 - \sum_{k=1, k\neq i}^{n} a_{ik} c_{ki} \geq 1$$

The last inequality is due to $c_{ki} \geq 0$ and $a_{ik} < 0$ for $k \neq i$. As $a_{ii} c_{ii} \geq 1$, neither factor can be 0. So $c_{ii} > 0$ and $a_{ii} > 0$.

Theorem: (Saad, Th. 1.31) Assume

(i) $a_{ij} \leq 0$ for $i \neq j$

(ii) $a_{ii} > 0$

Then A is an M-Matrix if and only if $\rho(I - D^{-1}A) < 1$.

Proof: See Saad. □

Main practical M-Matrix criterion

Corollary: Let A be sdd or idd. Assume that $a_{ii} > 0$ and $a_{ij} \leq 0$ for $i \neq j$. Then A is an M-Matrix.

Proof: We know that A is nonsingular, but we have to show $A^{-1} \geq 0$.

- Let $B = I - D^{-1}A$. Then $\rho(B) < 1$, therefore $I - B$ is nonsingular.
- We have for $k > 0$:

$$
\begin{aligned}
I - B^{k+1} &= (I - B)(I + B + B^2 + \cdots + B^k) \\
(I - B)^{-1}(I - B^{k+1}) &= (I + B + B^2 + \cdots + B^k)
\end{aligned}
$$

The left hand side for $k \to \infty$ converges to $(I - B)^{-1}$, therefore

$$
(I - B)^{-1} = \sum_{k=0}^{\infty} B^k
$$

As $B \geq 0$, we have $(I - B)^{-1} = A^{-1}D \geq 0$. As $D > 0$ we must have $A^{-1} \geq 0$. □

M-Matrix comparison criterion

Theorem(Saad, Th. 1.33): Let A, B $n \times n$ matrices such that

(i) $A \leq B$

(ii) $b_{ij} \leq 0$ for $i \neq j$.

Then, if A is an M-Matrix, so is B.

Proof: From M-property of A and $A \leq B$ we have $0 < D_A \leq D_B$. We have $D_B - B \geq 0$ and

$$\begin{aligned} D_A - A &\geq D_B - B \\ I - D_A^{-1}A &\geq D_A^{-1}(D_B - B) \\ &\geq D_B^{-1}(D_B - B) \\ &\geq I - D_B^{-1}B =: G \geq 0 \end{aligned}$$

Perron-Frobenius $\Rightarrow \rho(G) = \rho(I - D_B^{-1}B) \leq \rho(I - D_A^{-1}A) < 1$
$\Rightarrow I - G$ is nonsingular. From the proof of the M-matrix criterion, $D_B^{-1}B = (I - G)^{-1} = \sum_{k=0}^{\infty} G^k \geq 0$. As $D_B > 0$, we get $B \geq 0$.

$\square$

Corollary $A \leq M_{GS} \Rightarrow M_{GS}$ is an M-Matrix.

Incomplete LU factorizations (ILU)

Idea (Varga, Buleev, 1960):

- fix a predefined zero pattern
- apply the standard LU factorization method, but calculate only those elements, which do not correspond to the given zero pattern
- Result: incomplete LU factors L, U, remainder R:

$$A = LU - R$$

- Problem: with complete LU factorization procedure, for any nonsingular matrix, the method is stable, i.e. zero pivots never occur. Is this true for the incomplete LU Factorization as well ?

M-Property propagation in Gaussian Elimination

Theorem:(Ky Fan; Saad Th 1.10) Let A be an M-matrix. Then the matrix A_1 obtained from the first step of Gaussian elimination is an M-matrix.

Proof: One has $a^1_{ij} = a_{ij} - \frac{a_{i1}a_{1j}}{a_{11}}$,
$a_{ij}, a_{i1}, a_{1j} \leq 0$, $a_{11} > 0$
$\Rightarrow a^1_{ij} \leq 0$ for $i \neq j$

$$A = L_1 A_1 \text{ with } L_1 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \frac{-a_{12}}{a_{11}} & 1 & \dots & 0 \\ \vdots & & \ddots & 0 \\ \frac{-a_{1n}}{a_{11}} & 0 & \dots & 1 \end{pmatrix} \text{ nonsingular, nonnegative}$$

$\Rightarrow A_1$ nonsingular

Let $e_1 \dots e_n$ be the unit vectors. Then $A_1^{-1} e_1 = \frac{1}{a_1 1} e_1 \geq 0$. For $j > 1$,
$A_1^{-1} e_j = A^{-1} L^{-1} e_j = A^{-1} e_j \geq 0$.
$\Rightarrow A_1^{-1} \geq 0$

□

Stability of ILU

Theorem (Saad, Th. 10.2): If A is an M-Matrix, then the algorithm to compute the incomplete LU factorization with a given nonzero pattern

$$A = LU - R$$

is stable. Moreover, $A = LU - R$ is a regular splitting.

Stability of ILU decomposition II

Proof

Let $\tilde{A}_1 = A_1 + R_1 = L_1 A + R_1$ where R_1 is a nonnegative matrix which occurs from dropping some off diagonal entries from A_1. Thus, $\tilde{A}_1 \geq A_1$ and $\tilde{A}_1$ is an M-matrix. We can repeat this recursively

$$\begin{aligned}\tilde{A}_k = A_k + R_k &= L_k A_{k-1} + R_k \\ &= L_k L_{k-1} A_{k-2} + L_k R_{k-1} + R_k \\ &= L_k L_{k-1} \cdot \ldots \cdot L_1 A + L_k L_{k-1} \cdot \ldots \cdot L_2 R_1 + \cdots + R_k\end{aligned}$$

Let $L = (L_{n-1} \cdot \ldots \cdot L_1)^{-1}$, $U = \tilde{A}_{n-1}$. Then $U = L^{-1}A + S$ with

$$S = L_{n-1} L_{n-2} \cdot \ldots \cdot L_2 R_1 + \cdots + R_{n-1} = L_{n-1} L_{n-2} \cdot \ldots \cdot L_2 (R_1 + R_2 + \ldots R_{n-1})$$

Let $R = R_1 + R_2 + \ldots R_{n-1}$, then $A = LU - R$ where $U^{-1}L^{-1}$, R are nonnegative.

□

ILU(0)

- Special case of ILU: ignore any fill-in.
- Representation:

$$M = (\tilde{D} - E)\tilde{D}^{-1}(\tilde{D} - F)$$

- $\tilde{D}$ is a diagonal matrix (wich can be stored in one vector) which is calculated by the incomplete factorization algorithm.

- Setup:

```
for i=1:n
    d[i]=a[i,i]
end

for i=1:n
    d[i]=1.0/d[i]
    for j=i+1:n
        d[j]=d[j]-a[i,j]*d[i]*a[j,i]
    end
end
```

ILU(0)

Solve $Mu = v$

```
for i=1:n
    x=0.0
    for j=1:i-1
        x=x+a[i,j]*u[j]
        u[i]=d[i]*(v[i]-x)
    end
end
for i=n:-1:1
    x=0.0
    for j=i+1:n
        x=x+a[i,j]*u[j]
     u[i]=u[i]-d[i]*x
    end
end
```

ILU(0)

- Generally better convergence properties than Jacobi, Gauss-Seidel
- One can develop block variants
- Alternatives:
  - ILUM: ("modified"): add ignored off-diagonal entries to $\tilde{D}$
  - ILUT: zero pattern calculated dynamically based on drop tolerance
- Dependence on ordering
- Can be parallelized using graph coloring
- Not much theory: experiment for particular systems
- I recommend it as the default initial guess for a sensible preconditioner
- Incomplete Cholesky: symmetric variant of ILU

Intermediate Summary

- Given some matrix, we now have some nice recipies to establish nonsingularity and iterative method convergence:
- **Check if the matrix is irreducible.**
  This is mostly the case for elliptic and parabolic PDEs.
- **Check if the matrix is strictly or irreducibly diagonally dominant**.
  If yes, it is in addition nonsingular.
- **Check if main diagonal entries are positive and off-diagonal entries are nonpositive.**
  If yes, in addition, the matrix is an M-Matrix, its inverse is nonnegative, and elementary iterative methods converge.
- These critera do not depend on the symmetry of the matrix!

Example: 1D finite difference matrix:

$$Au = \begin{pmatrix} \alpha + \frac{1}{h} & -\frac{1}{h} & & & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & & \\ & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & & & -\frac{1}{h} & \frac{2}{h} & & -\frac{1}{h} \\ & & & & -\frac{1}{h} & & \frac{1}{h} + \alpha \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-2} \\ u_{N-1} \\ u_N \end{pmatrix} = f = \begin{pmatrix} \alpha v_1 \\ hf_2 \\ hf_3 \\ \vdots \\ hf_{N-2} \\ hf_{N-1} \\ \alpha v_n \end{pmatrix}$$

- idd
- positive main diagonal entries, nonpositive off-diagonal entries

$\Rightarrow$ A is nonsingular, has the M-property, and we can e.g. apply the Jacobi and Gauss-Seidel iterative method to solve it (ok, in 1D we already know this is a bad idea ...).

$\Rightarrow$ for $f \geq 0$ and $v \geq 0$ it follows that $u \geq 0$.
$\equiv$ heating and positive environment temperatures cannot lead to negative temperatures in the interior.