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Lecture 17

Jürgen Fuhrmann

juergen.fuhrmann@wias-berlin.de

Reference finite element

- Let $\{\widehat{P}, \widehat{K}, \widehat{\Sigma}\}$ be a fixed finite element
- Let T_K be some affine transformation and $K = T_K(\widehat{K})$
- There is a linear bijective mapping ψ_K between functions on K and functions on $\widehat{K}$:

$$\begin{aligned} \psi_K : V(K) &\to V(\widehat{K}) \\ f &\mapsto f \circ T_K \end{aligned}$$

- Let
  - $K = T_K(\widehat{K})$
  - $P_K = \{\psi_K^{-1}(\widehat{p}); \widehat{p} \in \widehat{P}\}$,
  - $\Sigma_K = \{\sigma_{K,i}, i = 1 \dots s : \sigma_{K,i}(p) = \widehat{\sigma}_i(\psi_K(p))\}$

  Then $\{K, P_K, \Sigma_K\}$ is a finite element.
- This construction allows to develop theory for a reference element and to lift it later to an arbitrary element.

Commutativity of interpolation and reference mapping

- $\mathcal{I}_{\hat{K}} \circ \psi_K = \psi_K \circ \mathcal{I}_K$,
  i.e. the following diagram is commutative:

$$
\begin{array}{ccc}
V(K) & \xrightarrow{\psi_K} & V(\widehat{K}) \\
\Big\downarrow \mathcal{I}_K & & \Big\downarrow \mathcal{I}_{\hat{K}} \\
P_K & \xrightarrow{\psi_K} & P_{\widehat{K}}
\end{array}
$$

- $\equiv$ Interpolation and reference mapping are interchangeable

Affine transformation estimates I

- $\widehat{K}$: reference element
- Let $K \in \mathcal{T}_h$. Affine mapping described by matrix J_K and shift vertor b_K:

$$
\begin{aligned}
T_K : \widehat{K} &\to K \\
\widehat{x} &\mapsto J_K \widehat{x} + b_K
\end{aligned}
$$

  with $J_K \in \mathbb{R}^{d,d}$, $b_K \in \mathbb{R}^d$, J_K nonsingular
- Diameter of K: $h_K = \max_{x_1, x_2 \in K} ||x_1 - x_2||$
  $\equiv$ longest edge if K is triangular
- ρ_K diameter of largest ball that can be inscribed into K
- $\sigma_K = \frac{h_K}{\rho_K}$: local shape regularity measure
  $\sigma_K = 2\sqrt{3}$ for equilateral triangle
  $\sigma_K \to \infty$ if largest angle approaches π.

Affine transformation estimates II

Lemma T:

- $|\det J_K| = \frac{meas(K)}{meas(\widehat{K})}$
- $||J_K|| \leq \frac{h_K}{\rho_{\hat{K}}}$, $||J_K^{-1}|| \leq \frac{h_{\hat{K}}}{\rho_K}$
- $\Rightarrow ||J_K|| \cdot ||J_K^{-1}|| \leq c_{\hat{K}} \sigma_K$

Proof:

- $|\det J_K| = \frac{meas(K)}{meas(\widehat{K})}$: basic property of affine mappings
- Further:

$$||J_K|| = \sup_{\hat{x} \neq 0} \frac{||J_K \hat{x}||}{||\hat{x}||} = \frac{1}{\rho_{\hat{K}}} \sup_{||\hat{x}|| = \rho_{\hat{K}}} ||J_K \hat{x}||$$

  Set $\hat{x} = \hat{x}_1 - \hat{x}_2$ with $\hat{x}_1, \hat{x}_2 \in \widehat{K}$. Then $J_K \hat{x} = T_K \hat{x}_1 - T_K \hat{x}_2$ and one can estimate $||J_K \hat{x}|| \leq h_K$.

- For $||J_K^{-1}||$ regard the inverse mapping □

Estimate of derivatives under affine transformation

- For $w \in H^s(K)$ recall the H^s seminorm $|w|^2_{s,K} = \sum_{|\beta|=s} ||\partial^\beta w||^2_{L^2(K)}$

Lemma D: Let $w \in H^s(K)$ and $\widehat{w} = w \circ T_K$. There exists a constant c such that

$$|\hat{w}|_{s,\hat{K}} \leq c||J_K||^s |\det J_K|^{-\frac{1}{2}} |w|_{s,K}$$
$$|w|_{s,K} \leq c||J_K^{-1}||^s |\det J_K|^{\frac{1}{2}} |\hat{w}|_{s,\hat{K}}$$

Proof: Let $|\alpha| = s$. By affinity and chain rule one obtains

$$||\partial^\alpha \hat{w}||_{L^2(\hat{K})} \leq c||J_K||^s \sum_{|\beta|=s} ||\partial^\beta w \circ T_K||_{L^2(K)}$$

Changing variables in the right hand side yields

$$||\partial^\alpha \hat{w}||_{L^2(\hat{K})} \leq c||J_K||^s |\det J_K|^{-\frac{1}{2}} |w|_{s,K}$$

Summation over α yields the first inequality. Regarding the inverse mapping yields the second estimate. □

Local interpolation error estimate I

Theorem: Let $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ be a finite element with associated normed vector space $V(\widehat{K})$. Assume there exists k such that

$$\mathbb{P}_K \subset \widehat{P} \subset H^{k+1}(\widehat{K}) \subset V(\widehat{K})$$

and

$$H^{l+1}(\widehat{K}) \subset V(\widehat{K}) \quad \text{for} \quad 0 \leq l \leq k$$

Then there exists $c > 0$ such that for all $m = 0 \dots l+1$, $K \in \mathcal{T}_h$, $v \in H^{l+1}(K)$:

$$|v - \mathcal{I}_K^k v|_{m,K} \leq c h_K^{l+1-m} \sigma_K^m |v|_{l+1,K}.$$

I.e. the the local interpolation error can be estimated through h_K, σ_K and the norm of a higher derivative.

Local interpolation error estimate II

Draft of Proof

- Estimate on reference element $\hat{K}$ using deeper results from functional analysis:

$$|\hat{w} - \mathcal{I}^k_{\hat{K}}\hat{w}|_{m,\hat{K}} \leq c|\hat{w}|_{l+1,\hat{K}} \qquad (*)$$

(From Poincare like inequality, e.g. for $v \in H^1_0(\Omega)$, $c||v||_{L^2} \leq ||\nabla v||_{L^2}$: under certain circumstances, we can can estimate the norms of lower derivatives by those of the higher ones)

- Derive estimate on K from estimate on $\hat{K}$: Let $v \in H^{l+1}(K)$ and set $\hat{v} = v \circ T_K$. We know that $(\mathcal{I}^k_K v) \circ T_K = \mathcal{I}^k_{\hat{K}}\hat{v}$.

$$\begin{aligned}
|v - \mathcal{I}^k_K v|_{m,K} &\leq c||J_K^{-1}||^m |\det J_K|^{\frac{1}{2}} |\hat{v} - \mathcal{I}^k_{\hat{K}}\hat{v}|_{m,\hat{K}} && \text{(Lemma E)}\\
&\leq c||J_K^{-1}||^m |\det J_K|^{\frac{1}{2}} |\hat{v}|_{l+1,\hat{K}} && (*)\\
&\leq c||J_K^{-1}||^m ||J_K||^{l+1} |v|_{l+1,K} && \text{(Lemma E)}\\
&= c(||J_K|| \cdot ||J_K^{-1}||)^m ||J_K||^{l+1-m} |v|_{l+1,K} &&\\
&\leq c h_K^{l+1-m} \sigma_K^m |v|_{l+1,K} && \text{(Lemma T)}
\end{aligned}$$

Local interpolation: special cases for Lagrange finite elements

- General condition

$$\mathbb{P}_K \subset \widehat{P} \subset H^{k+1}(\widehat{K}) \subset V(\widehat{K})$$
$$H^{l+1}(\widehat{K}) \subset V(\widehat{K}) \quad \text{for} \quad 0 \leq l \leq k$$

- $k = 1$:

$$\mathbb{P}_K \subset \widehat{P} \subset H^{2}(\widehat{K}) \subset V(\widehat{K})$$
$$H^{1}(\widehat{K}) \subset V(\widehat{K})$$

- $k = 1, l = 1, m = 0$: $|v - \mathcal{I}_K^k v|_{0,K} \leq c h_K^2 |v|_{2,K}$
- $k = 1, l = 1, m = 1$: $|v - \mathcal{I}_K^k v|_{1,K} \leq c h_K \sigma_K |v|_{2,K}$

Shape regularity

- Now we discuss a family of meshes $\mathcal{T}_h$ for $h \to 0$. We want to estimate global interpolation errors and see how they possibly diminish
- For given $\mathcal{T}_h$, assume that $h = \max_{K\in\mathcal{T}_h} h_j$
- A family of meshes is called *shape regular* if

$$\forall h, \forall K \in \mathcal{T}_h, \sigma_K = \frac{h_K}{\rho_K} \leq \sigma_0$$

- In 1D, $\sigma_K = 1$
- In 2D, $\sigma_K \leq \frac{2}{\sin \theta_K}$ where θ_K is the smallest angle

Global interpolation error estimate

Theorem Let Ω be polyhedral, and let $\mathcal{T}_h$ be a shape regular family of affine meshes. Then there exists c such that for all h, $v \in H^{l+1}(\Omega)$,

$$||v - \mathcal{I}_h^k v||_{L^2(\Omega)} + \sum_{m=1}^{l+1} h^m \left(\sum_{K \in \mathcal{T}_h} |v - \mathcal{I}_h^k v|_{m,K}^2 \right)^{\frac{1}{2}} \leq ch^{l+1} |v|_{l+1,\Omega}$$

and

$$\lim_{h \to 0} \left(\inf_{v_h \in V_h^k} ||v - v_h||_{L^2(\Omega)} \right) = 0$$

Global interpolation error estimate for Lagrangian finite elements, $k = 1$

- Assume $v \in H^2(\Omega)$, e.g. if problem coefficients are smooth and the domain is convex

$$||v - \mathcal{I}_h^k v||_{0,\Omega} + h|v - \mathcal{I}_h^k v|_{1,\Omega} \leq ch^2 |v|_{2,\Omega}$$

$$|v - \mathcal{I}_h^k v|_{1,\Omega} \leq ch|v|_{2,\Omega}$$

$$\lim_{h\to 0} \left(\inf_{v_h \in V_h^k} |v - v_h|_{1,\Omega} \right) = 0$$

- If $v \in H^2(\Omega)$ cannot be guaranteed, estimates become worse. Example: L-shaped domain.
- These results immediately can be applied in Cea's lemma.

Error estimates for homogeneous Dirichlet problem

- Search $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \lambda \nabla u \nabla v \, d\mathbf{x} = \int_\Omega fv \, d\mathbf{x} \; \forall v \in H_0^1(\Omega)$$

Then, $\lim_{h\to 0} ||u - u_h||_{1,\Omega} = 0$. If $u \in H^2(\Omega)$ (e.g. on convex domains) then

$$||u - u_h||_{1,\Omega} \leq ch|u|_{2,\Omega}$$
$$||u - u_h||_{0,\Omega} \leq ch^2|u|_{2,\Omega}$$

Under certain conditions (convex domain, smooth coefficients) one also has

$$||u - u_h||_{0,\Omega} \leq ch|u|_{1,\Omega}$$

("Aubin-Nitsche-Lemma")

H^2-Regularity

- $u \in H^2(\Omega)$ may be *not* fulfilled e.g.
  - if Ω has re-entrant corners
  - if on a smooth part of the domain, the boundary condition type changes
  - if problem coefficients (λ) are discontinuos
- Situations differ as well between two and three space dimensions
- Delicate theory, ongoing research in functional analysis
- Consequence for simuations
  - Deterioration of convergence rate
  - Remedy: local refinement of the discretization mesh
    - using a priori information
    - using a posteriori error estimators + automatic refinement of discretizatiom mesh

Higher regularity

- If $u \in H^s(\Omega)$ for $s > 2$, convergence order estimates become even better for P^k finite elements of order $k > 1$.
- Depending on the regularity of the solution the combination of grid adaptation and higher oder ansatz functions may be successful

Quadrature rules

- *Quadrature rule*:

$$\int_K g(x)\ d\mathbf{x} \approx |K| \sum_{l=1}^{l_q} \omega_l g(\xi_l)$$

- ξ_l: *nodes, Gauss points*
- ω_l: *weights*
- The largest number k such that the quadrature is exact for polynomials of order k is called *order* k_q of the quadrature rule, i.e.

$$\forall k \leq k_q, \forall p \in \mathbb{P}^k \int_K p(x)\ d\mathbf{x} = |K| \sum_{l=1}^{l_q} \omega_l p(\xi_l)$$

- Error estimate:

$$\forall \phi \in \mathcal{C}^{k_q+1}(K), \left| \frac{1}{|K|} \int_K \phi(x)\ d\mathbf{x} - \sum_{l=1}^{l_q} \omega_l g(\xi_l) \right|$$
$$\leq c h_K^{k_q+1} \sup_{x \in K, |\alpha| = k_q+1} |\partial^\alpha \phi(x)|$$

Some common quadrature rules

Nodes are characterized by the barycentric coordinates

d	k_q	l_q	Nodes	Weights
1	1	1	$(\frac{1}{2}, \frac{1}{2})$	1
	1	2	$(1, 0), (0, 1)$	$\frac{1}{2}, \frac{1}{2}$
	3	2	$(\frac{1}{2} + \frac{\sqrt{3}}{6}, \frac{1}{2} - \frac{\sqrt{3}}{6}), (\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} + \frac{\sqrt{3}}{6})$	$\frac{1}{2}, \frac{1}{2}$
	5	3	$(\frac{1}{2},), (\frac{1}{2} + \sqrt{\frac{3}{20}}, \frac{1}{2} - \sqrt{\frac{3}{20}}), (\frac{1}{2} - \sqrt{\frac{3}{20}}, \frac{1}{2} + \sqrt{\frac{3}{20}})$	$\frac{8}{18}, \frac{5}{18}, \frac{5}{18}$
2	1	1	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	1
	1	3	$(1, 0, 0), (0, 1, 0), (0, 0, 1)$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$
	2	3	$(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$
	3	4	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{5}, \frac{1}{5}, \frac{3}{5}), (\frac{1}{5}, \frac{3}{5}, \frac{1}{5}), (\frac{3}{5}, \frac{1}{5}, \frac{1}{5}),$	$-\frac{9}{16}, \frac{25}{48}, \frac{25}{48}, \frac{25}{48}$
3	1	1	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	1
	1	4	$(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$
	2	4	$(\frac{5-\sqrt{5}}{20}, \frac{5-\sqrt{5}}{20}, \frac{5-\sqrt{5}}{20}, \frac{5+3\sqrt{5}}{20}) \dots$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$

Weak formulation of homogeneous Dirichlet problem

- Search $u \in V = H_0^1(\Omega)$ such that

$$\int_\Omega \kappa \nabla u \nabla v \, d\mathbf{x} = \int_\Omega fv \, d\mathbf{x} \; \forall v \in H_0^1(\Omega)$$

- Then,

$$a(u, v) := \int_\Omega \kappa \nabla u \nabla v \, d\mathbf{x}$$

is a self-adjoint bilinear form defined on the Hilbert space $H_0^1(\Omega)$.

Galerkin ansatz

- Let $V_h \subset V$ be a finite dimensional subspace of V
- "Discrete" problem $\equiv$ Galerkin approximation:
  Search $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h) \; \forall v_h \in V_h$$

- E.g. V_h is the space of P1 Lagrange finite element approximations

Stiffness matrix for Laplace operator for P1 FEM

- Element-wise calculation:

$$a_{ij} = a(\phi_i, \phi_j) = \int_\Omega \nabla\phi_i \nabla\phi_j \; d\mathbf{x} = \int_\Omega \sum_{K\in\mathcal{T}_h} \nabla\phi_i|_K \nabla\phi_j|_K \; d\mathbf{x}$$

- Standard assembly loop:

for $i, j = 1 \dots N$ **do**
  set $a_{ij} = 0$
end
for $K \in \mathcal{T}_h$ **do**
  for *m,n=0...d* **do**

$$s_{mn} = \int_K \nabla\lambda_m \nabla\lambda_n \; d\mathbf{x}$$

$$a_{j_{dof}(K,m),j_{dof}(K,n)} = a_{j_{dof}(K,m),j_{dof}(K,n)} + s_{mn}$$

  end
end

- Local stiffness matrix:

$$S_K = (s_{K;m,n}) = \int_K \nabla\lambda_m \nabla\lambda_n \; d\mathbf{x}$$

Local stiffness matrix calculation for P1 FEM

- $a_0 \dots a_d$: vertices of the simplex K, $a \in K$.
- Barycentric coordinates: $\lambda_j(a) = \frac{|K_j(a)|}{|K|}$
- For indexing modulo d+1 we can write

$$|K| = \frac{1}{d!}\det\left(a_{j+1} - a_j, \dots a_{j+d} - a_j\right)$$
$$|K_j(a)| = \frac{1}{d!}\det\left(a_{j+1} - a, \dots a_{j+d} - a\right)$$

- From this information, we can calculate explicitely$\nabla\lambda_j(x)$ (which are constant vectors due to linearity) and the corresponding entries of the local stiffness

$$s_{ij} = \int_K \nabla\lambda_i \nabla\lambda_j \; d\mathbf{x}$$

Local stiffness matrix calculation for P1 FEM in 2D

- $a_0 = (x_0, y_0) \dots a_d = (x_2, y_2)$: vertices of the simplex K, $a = (x, y) \in K$.
- Barycentric coordinates: $\lambda_j(x, y) = \frac{|K_j(x,y)|}{|K|}$
- For indexing modulo d+1 we can write

$$|K| = \frac{1}{2}\det\begin{pmatrix} x_{j+1} - x_j & x_{j+2} - x_j \\ y_{j+1} - y_j & y_{j+2} - y_j \end{pmatrix}$$
$$|K_j(x,y)| = \frac{1}{2}\det\begin{pmatrix} x_{j+1} - x & x_{j+2} - x \\ y_{j+1} - y & y_{j+2} - y \end{pmatrix}$$

- Therefore, we have

$$|K_j(x,y)| = \frac{1}{2}\left((x_{j+1} - x)(y_{j+2} - y) - (x_{j+2} - x)(y_{j+1} - y)\right)$$
$$\partial_x |K_j(x,y)| = \frac{1}{2}\left((y_{j+1} - y) - (y_{j+2} - y)\right) = \frac{1}{2}(y_{j+1} - y_{j+2})$$
$$\partial_y |K_j(x,y)| = \frac{1}{2}\left((x_{j+2} - x) - (x_{j+1} - x)\right) = \frac{1}{2}(x_{j+2} - x_{j+1})$$

Local stiffness matrix calculation for P1 FEM in 2D II

- $$s_{ij} = \int_K \nabla\lambda_i \nabla\lambda_j \; d\mathbf{x} = \frac{|K|}{4|K|^2} \left(y_{i+1} - y_{i+2}, x_{i+2} - x_{i+1}\right) \begin{pmatrix} y_{j+1} - y_{j+2} \\ x_{j+2} - x_{j+1} \end{pmatrix}$$

- So, let $V = \begin{pmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{pmatrix}$

- Then

$$\begin{aligned} x_1 - x_2 &= V_{00} - V_{01} \\ y_1 - y_2 &= V_{10} - V_{11} \end{aligned}$$

and

$$\begin{aligned} 2|K| \; \nabla\lambda_0 &= \begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \end{pmatrix} = \begin{pmatrix} V_{10} - V_{11} \\ V_{01} - V_{00} \end{pmatrix} \\ 2|K| \; \nabla\lambda_1 &= \begin{pmatrix} y_2 - y_0 \\ x_0 - x_2 \end{pmatrix} = \begin{pmatrix} V_{11} \\ -V_{01} \end{pmatrix} \\ 2|K| \; \nabla\lambda_2 &= \begin{pmatrix} y_0 - y_1 \\ x_1 - x_0 \end{pmatrix} = \begin{pmatrix} -V_{10} \\ V_{00} \end{pmatrix} \end{aligned}$$

Degree of freedom map representation for P1 finite elements

- List of global nodes $a_0 \dots a_N$: two dimensional array of coordinate values with N rows and d columns
- Local-global degree of freedom map: two-dimensional array C of index values with N_{el} rows and $d+1$ columns such that $C(i,m) = j_{dof}(K_i, m)$.
- The mesh generator triangle generates this information directly

Practical realization of boundary conditions

- Robin boundary value problem

$$
\begin{aligned}
-\nabla \cdot \kappa \nabla u &= f \quad \text{in } \Omega \\
\kappa \nabla u + \alpha(u - g) &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

- Weak formulation: search $u \in H^1(\Omega)$ such that

$$
\int_\Omega \kappa \nabla u \nabla v \, d\mathbf{x} + \int_{\partial\Omega} \alpha u v \, ds = \int_\Omega f v \, d\mathbf{x} + \int_{\partial\Omega} \alpha g v \, ds \; \forall v \in H^1(\Omega)
$$

- In 2D, for P^1 FEM, boundary integrals can be calculated by trapezoidal rule without sacrificing approximation order
- Use Dirichlet penalty method to handle Dirichlet boundary conditions

More complicated integrals

- Assume non-constant right hand side f, space dependent heat conduction coefficient κ.
- Right hand side integrals

$$f_i = \int_K f(x)\lambda_i(x)\ d\mathbf{x}$$

- P^1 stiffness matrix elements

$$a_{ij} = \int_K \kappa(x)\ \nabla\lambda_i\ \nabla\lambda_j\ d\mathbf{x}$$

- P^k stiffness matrix elements created from higher order ansatz functions

Matching of approximation order and quadrature order

- "Variational crime": instead of

$$a(u_h, v_h) = f(v_h)\ \forall v_h \in V_h$$

  we solve

$$a_h(u_h, v_h) = f_h(v_h)\ \forall v_h \in V_h$$

  where a_h, f_h are derived from their exact counterparts by quadrature
- For P^1 finite elements, zero order quadrature for volume integrals and first order quadrature for surface intergals is sufficient to keep the convergence order estimates stated before
- The rule of thumb for the volume quadrature is that the highest order terms must be evaluated exactly if the coefficients of the PDE are constant.

Practical realization of integrals

- Integral over barycentric coordinate function

$$\int_K \lambda_i(x)\ d\mathbf{x} = \frac{1}{3}|K|$$

- Right hand side integrals. Assume $f(x)$ is given as a piecewise linear function with given values in the nodes of the triangulation

$$f_i = \int_K f(x)\lambda_i(x)\ d\mathbf{x} \approx \frac{1}{3}|K|f(a_i)$$

- Integral over space dependent heat conduction coefficient: Assume $\kappa(x)$ is given as a piecewise linear function with given values in the nodes of the triangulation

$$a_{ij} = \int_K \kappa(x)\ \nabla\lambda_i\ \nabla\lambda_j\ d\mathbf{x} = \frac{1}{3}(\kappa(a_0)+\kappa(a_1)+\kappa(a_2))\int_K \nabla\lambda_i\ \nabla\lambda_j\ d\mathbf{x}$$

Convergence tests

P1 FEM, homogeneous Dirichlet

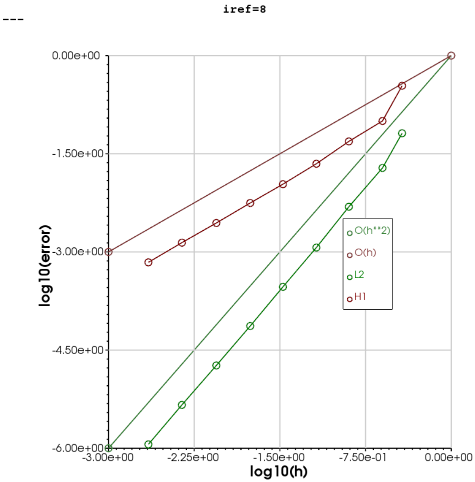
- Problem:

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

- Exact solution + rhs:

$$\begin{aligned} u(x, y) &= \sin(\pi x) \sin(\pi y) \\ f(x, y) &= 2\pi \sin(\pi x) \sin(\pi y) \end{aligned}$$

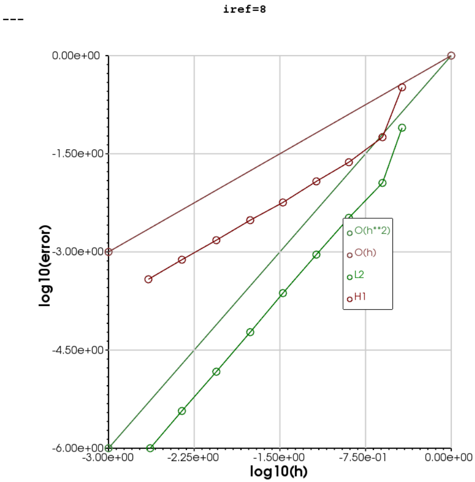
P1 FEM: error plot



- As expected:

$$||u - u_h||_{L^2} \leq Ch^2$$
$$||u - u_h||_{H^1} \leq Ch$$

FVM: error plot



- As with P1 FEM

$$||u - u_h||_{L^2} \leq Ch^2$$
$$||u - u_h||_{H^1} \leq Ch$$

Iterative solver complexity I

- Solve linear system iteratively until $||e_k|| = ||(I - M^{-1}A)^k e_0|| \leq \epsilon$

$$\rho^k e_0 \leq \epsilon$$
$$k \ln \rho < \ln \epsilon - \ln e_0$$
$$k \geq k_\rho = \left\lceil \frac{\ln e_0 - \ln \epsilon}{\ln \rho} \right\rceil$$

- $\Rightarrow$ we need at least k_ρ iteration steps to reach accuracy ϵ
- Optimal iterative solver complexity - assume:
  - $\rho < \rho_0 < 1$ independent of h resp. N
  - A sparse ($A \cdot u$ has complexity $O(N)$)
  - Solution of $Mv = r$ has complexity $O(N)$.

$\Rightarrow$ Number of iteration steps k_ρ independent of N
$\Rightarrow$ Overall complexity $O(N)$

Iterative solver complexity II

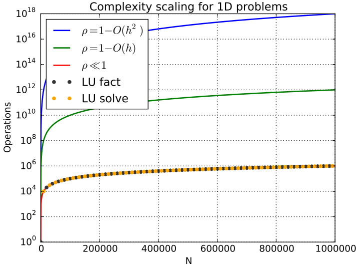
- Assume
  - $\rho = 1 - h^\delta \Rightarrow \ln\rho \approx -h^\delta \to k_\rho = O(h^{-\delta})$
  - d: space dimension $\Rightarrow h \approx N^{-\frac{1}{d}} \Rightarrow k_\rho = O(N^{\frac{\delta}{d}})$
  - $O(N)$ complexity of one iteration step (e.g. Jacobi, Gauss-Seidel)

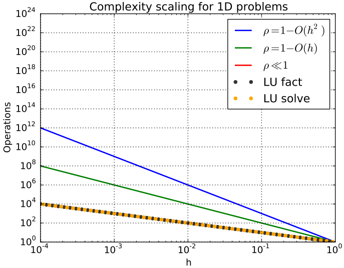
  $\Rightarrow$ Overall complexity $O(N^{1+\frac{\delta}{d}}) = O(N^{\frac{d+\delta}{d}})$
- Jacobi: $\delta = 2$
- Hypothetical "Improved iterative solver" with $\delta = 1$?
- Overview on complexity estimates

dim	$\rho = 1 - O(h^2)$	$\rho = 1 - O(h)$	LU fact.	LU solve
1	$O(N^3)$	$O(N^2)$	$O(N)$	$O(N)$
2	$O(N^2)$	$O(N^{\frac{3}{2}})$	$O(N^{\frac{3}{2}})$	$O(N\log N)$
3	$O(N^{\frac{5}{3}})$	$O(N^{\frac{4}{3}})$	$O(N^2)$	$O(N^{\frac{4}{3}})$

Solver complexity scaling for 1D problems

dim	$\rho = 1 - O(h^2)$	$\rho = 1 - O(h)$	LU fact.	LU solve
1	$O(N^3)$	$O(N^2)$	$O(N)$	$O(N)$

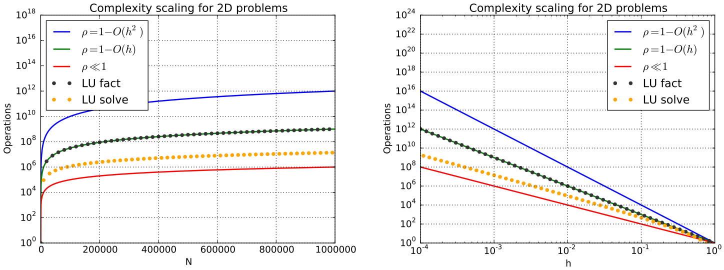




- Direct solvers significantly better than iterative ones

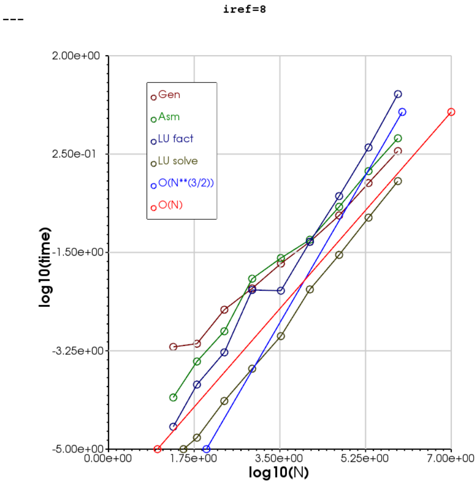
Solver complexity scaling for 2D problems

dim	$\rho = 1 - O(h^2)$	$\rho = 1 - O(h)$	LU fact.	LU solve
2	$O(N^2)$	$O(N^{\frac{3}{2}})$	$O(N^{\frac{3}{2}})$	$O(N \log N)$



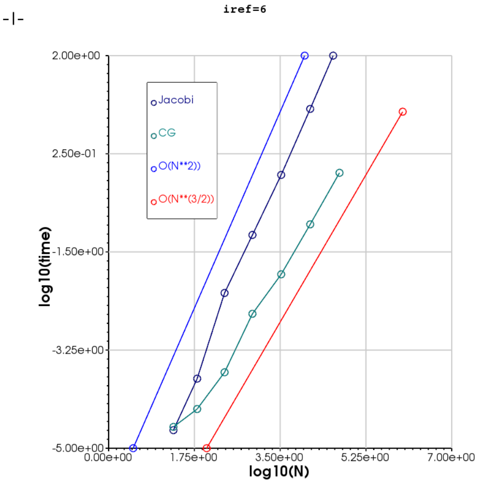
- Direct solvers better than simple iterative solvers (Jacobi etc.)
- On par with improved iterative solvers

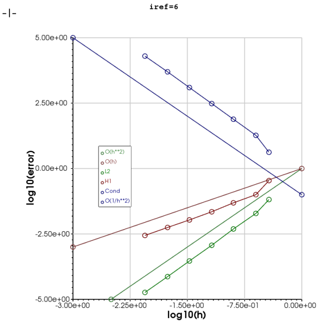
P1 FEM + LU Factorization: timing plot



asm:	Assembly
gen:	Mesh generation
luf:	LU factorization
lus:	LU solution
- For large problems, $t_{luf} > t_{asm} > t_{gen} > t_{lus}$
- $t_{luf} = O(N^{\frac{3}{2}})$

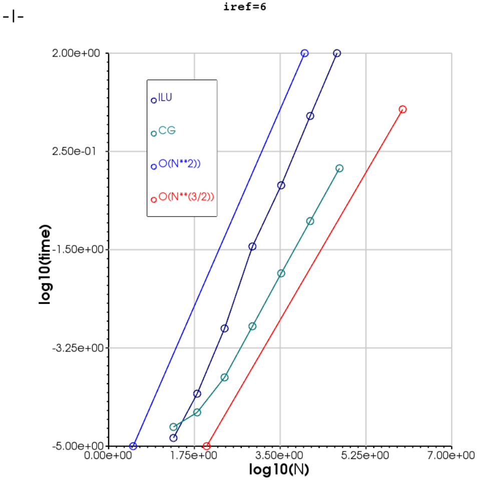
P1 FEM + Jacobi: timing plot

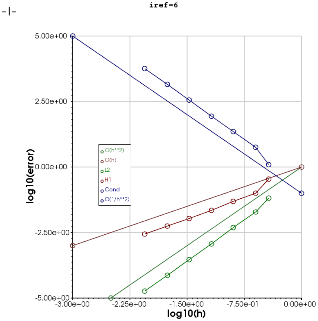




- $t_{cg} = O(N^{\frac{3}{2}})$
- $t_{jac} = O(N^2)$
- $\kappa = O(h^{-2})$

P1 FEM + ILU: timing plot





- $t_{cg} = O(N^{\frac{3}{2}})$
- $t_{ilu} = O(N^2)$
- $\kappa = O(h^{-2})$

Next lecture

Next lecture: Jan. 8, 2019

Happy holidays!