Analysis of a stabilized finite element method for fluid flows through a porous interface

Alfonso Caiazzo[a,b], Miguel A. Fernández[b], Vincent Martin[c,b]

[a] *Weierstrass Institute, Mohrenstr. 39, 10117 Berlin, Germany*
[b] *INRIA Paris-Rocquencourt, BP 105, 78153 Le Chesnay Cedex, France*
[c] *UTC, LMAC, Royallieu, BP 20529, 60205 Compiègne, France*

Abstract

This work is devoted to the numerical simulation of an incompressible fluid through a porous interface, modeled as a macroscopic *resistive interface term* in the Stokes equations. We improve the results reported in [*M2AN* **42**(6):961–990, 2008], by showing that the standard Pressure Stabilized Petrov-Galerkin (PSPG) finite element method is stable, and optimally convergent, without the need for controlling the pressure jump across the interface.

Keywords: Stokes equation, porous interface, stabilized finite element method

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1. Introduction

We consider a regular domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, and a porous interface defined by a hyperplane domain $\Gamma \subset \mathbb{R}^{d-1}$, dividing Ω in two connected subdomains as $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$. We denote by $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ the outgoing normals from each subdomain Ω_i at the interface, with $\boldsymbol{n}_1 = -\boldsymbol{n}_2$, and we define $\boldsymbol{n} = \boldsymbol{n}_1$. The fluid velocity $\boldsymbol{u}$ and pressure p are governed by the following modified Stokes equations [1]:

$$\begin{aligned} \boldsymbol{\nabla} p - \mu \boldsymbol{\Delta} \boldsymbol{u} + r_\Gamma \delta_\Gamma \boldsymbol{u} = \boldsymbol{f} \quad &\text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 \quad &\text{in} \quad \Omega, \end{aligned} \tag{1}$$

with a homogeneous Dirichlet condition on $\partial\Omega$. In (1), the symbol μ stands for the fluid viscosity, $\boldsymbol{f}$ for a given volume force, δ_Γ for the Dirac measure on Γ, and $r_\Gamma > 0$ is a given interface resistance, related to the permeability and porosity of the interface. Without loss of generality, r_Γ is assumed to be a constant scalar. For the sake of conciseness we limit ourselves to this problem. Nevertheless, the analysis below could be generalized to other problems involving pressure discontinuities, such as two-phase flows.

Problem (1) can be reformulated equivalently as a two-domain Stokes problem, complemented with the interface conditions

$$[\![\boldsymbol{u}]\!] = \boldsymbol{0}, \quad [\![\mu \boldsymbol{\nabla} \boldsymbol{u} \cdot \boldsymbol{n} - p\boldsymbol{n}]\!] = -r_\Gamma \boldsymbol{u} \quad \text{on} \quad \Gamma, \tag{2}$$

where $[\![q]\!] \stackrel{\text{def}}{=} q_{1|\Gamma} - q_{2|\Gamma}$ denotes the jump across Γ and $q_i \stackrel{\text{def}}{=} q|_{\Omega_i}$ $(i = 1, 2)$.

In [1], problem (1) was discretized with an extension of the PSPG stabilized method (see [2]): an additional consistent term (based on (2)) was introduced to control the interface pressure jump. Numerical evidence showed, however, that this term did not improve noticeably the behavior of the numerical solution with respect to a standard PSPG stabilized formulation [1]. The aim of this note is to show that, indeed, stability and optimal accuracy can be derived without the need for this extra interface stabilization term (which is convenient in practice).

2. Finite element formulation

Let $\{\mathcal{T}_h\}_{0<h\leq 1}$ be a regular family of quasi-uniform triangulations of Ω, conforming with the interface Γ. The corresponding triangulation of the interface is denoted by $\mathcal{G}_h$ and we set $h \stackrel{\text{def}}{=} \max_{T\in\mathcal{T}_h} h_T$, where h_T is the diameter of the element T. We introduce the spaces $\boldsymbol{V} \stackrel{\text{def}}{=} [H_0^1(\Omega)]^d$, $Q \stackrel{\text{def}}{=} L_0^2(\Omega)$, and the finite element spaces of degree $k \geq 1$, $\boldsymbol{V}_h^k$ and N_h^k, equal order approximations of $\boldsymbol{V}$ and Q:

$$\begin{aligned} \boldsymbol{V}_h^k &\stackrel{\text{def}}{=} \left\{\boldsymbol{v}_h \in (\mathcal{C}^0(\overline{\Omega}))^d \mid \ \boldsymbol{v}_{h|T} \in (\mathbb{P}_k)^d \ \ \forall T \in \mathcal{T}_h\right\} \cap \boldsymbol{V}, \\ N_h^k &\stackrel{\text{def}}{=} \left\{q_{h|\Omega_i} \in \mathcal{C}^0(\overline{\Omega}_i),\ i = 1, 2 \mid \ q_{h|T} \in \mathbb{P}_k \ \ \forall T \in \mathcal{T}_h\right\} \cap Q. \end{aligned} \tag{3}$$

Note that the space N_h^k of discrete pressures allows discontinuity at the interface Γ. As underlined in [1], this is of utmost importance to get a correct approximation of the solution without excessive mesh refinement. Additionally, we introduce the spaces $\mathbf{V}_0 \stackrel{\text{def}}{=} \{\boldsymbol{v} \in \mathbf{V} \mid \boldsymbol{v}|_\Gamma = 0\}$ and $\mathbf{V}_{0,h}^k \stackrel{\text{def}}{=} \mathbf{V}_0 \cap \mathbf{V}_h^k$.

Let us consider the two following bilinear forms

$$\begin{aligned} \mathcal{A}_\delta^{r_\Gamma}(\boldsymbol{x}_h, \boldsymbol{y}_h) &\stackrel{\text{def}}{=} (\mu \boldsymbol{\nabla} \boldsymbol{u}_h, \boldsymbol{\nabla} \boldsymbol{v}_h) - (p_h, \operatorname{div} \boldsymbol{v}_h) + (r_\Gamma \boldsymbol{u}_h, \boldsymbol{v}_h)_\Gamma + (\operatorname{div} \boldsymbol{u}_h, q_h) \\ &\qquad + \delta \sum_{T\in\mathcal{T}_h} \frac{h_T^2}{\mu} (-\mu \boldsymbol{\Delta} \boldsymbol{u}_h + \boldsymbol{\nabla} p_h, \boldsymbol{\nabla} q_h)_T\,, \\ \mathcal{B}_\delta^{r_\Gamma}(\boldsymbol{x}_h, \boldsymbol{y}_h) &\stackrel{\text{def}}{=} \mathcal{A}_\delta^{r_\Gamma}(\boldsymbol{x}_h, \boldsymbol{y}_h) - \delta \sum_{E\in\mathcal{G}_h} \frac{h_E}{\mu} \left([\![\mu \boldsymbol{\nabla} \boldsymbol{u}_h \cdot \boldsymbol{n} - p_h \boldsymbol{n}]\!] + r_\Gamma \boldsymbol{u}_h, [\![q_h \boldsymbol{n}]\!]\right)_E \end{aligned}$$

for all $\boldsymbol{x}_h = (\boldsymbol{u}_h, p_h)$ and $\boldsymbol{y}_h = (\boldsymbol{v}_h, q_h)$ in $\boldsymbol{V}_h^k \times N_h^k$ and $\delta > 0$ is a stabilization parameter. The discrete formulation proposed and analyzed in [1] is based on $\mathcal{B}_\delta^{r_\Gamma}$. In this note, we consider the numerical analysis of the standard PSPG formulation

$$\mathcal{A}_\delta^{r_\Gamma}(\boldsymbol{x}_h, \boldsymbol{y}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) + \delta \sum_{T\in\mathcal{T}_h} \frac{h_T^2}{\mu} (\boldsymbol{f}, \boldsymbol{\nabla} q_h)_T \quad \forall \boldsymbol{y}_h \in \boldsymbol{V}_h^k \times N_h^k. \tag{4}$$

3. Stability analysis

Let us consider the mesh-dependent energy norm

$$\|\!|(\boldsymbol{u}_h, p_h)|\!\|_h^2 \stackrel{\text{def}}{=} \mu\|\boldsymbol{\nabla}\boldsymbol{u}_h\|_{0,\Omega}^2 + r_\Gamma\|\boldsymbol{u}_h\|_{0,\Gamma}^2 + \delta\sum_{T\in\mathcal{T}_h}\frac{h_T^2}{\mu}\|\boldsymbol{\nabla}p_h\|_{0,T}^2 + \frac{1}{\mu}\|p_h\|_{0,\Omega}^2.$$

Note that, unlike in [1], this norm provides no control on the interface pressure jump. We address now the stability of (4) in the $\|\!|\cdot|\!\|_h$ norm.

By applying the inverse inequality (see [3])

$$\|\boldsymbol{\Delta}\boldsymbol{v}_h\|_{0,T} \leq c_\Delta h^{-1}\|\boldsymbol{\nabla}\boldsymbol{v}_h\|_{0,T}, \quad \boldsymbol{v}_h \in \boldsymbol{V}_h^k,$$

and the Schwarz and Young inequalities to the term $\sum_{T\in\mathcal{T}_h} h_T^2\,(\boldsymbol{\Delta}\boldsymbol{u}_h, \boldsymbol{\nabla}p_h)_T$, we get the following coercivity estimate.

Proposition 3.1. *Let δ be such that $0 < \delta c_\Delta^2 \leq 1$. Then*

$$\mathcal{A}_\delta^{r_\Gamma}(\boldsymbol{x}_h, \boldsymbol{x}_h) \geq \frac{\mu}{2}\|\boldsymbol{\nabla}\boldsymbol{u}_h\|_{0,\Omega}^2 + r_\Gamma\|\boldsymbol{u}_h\|_{0,\Gamma}^2 + \frac{\xi^2}{2} \geq \frac{1}{2}\left(\|\!|(\boldsymbol{u}_h, p_h)|\!\|_h^2 - \frac{1}{\mu}\|p_h\|_{0,\Omega}^2\right) \tag{5}$$

for all $\boldsymbol{x}_h = (\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h^k \times Q_h^k$, with $\xi^2 \stackrel{\text{def}}{=} \delta\sum_{T\in\mathcal{T}_h}\frac{h_T^2}{\mu}\|\boldsymbol{\nabla}p_h\|_{0,T}^2$.

The stability and the optimal convergence are stated in the following result.

Proposition 3.2. *Under the assumption of Proposition 3.1 there holds:*
(i) *there exists a constant $\beta = \beta(\delta, \frac{\mu}{r_\Gamma})$ independent of h, such that*

$$\inf_{\boldsymbol{x}_h\in\boldsymbol{V}_h^k\times Q_h^k}\ \sup_{\boldsymbol{y}_h\in\boldsymbol{V}_h^k\times Q_h^k}\frac{\mathcal{A}_\delta^{r_\Gamma}(\boldsymbol{x}_h, \boldsymbol{y}_h)}{\|\!|\boldsymbol{x}_h|\!\|_h\|\!|\boldsymbol{y}_h|\!\|_h} \geq \beta. \tag{6}$$

Moreover, if $\delta \ll 1$ we have $\beta \sim \delta$, and $\beta = \mathcal{O}(\mu/r_\Gamma)$ for $r_\Gamma/\mu \gg 1$;
(ii) *let $(\boldsymbol{u}_h, p_h)$ be the solution of (4) and assume that $(\boldsymbol{u}, p)$, the solution of (1), is such that $\boldsymbol{u}_i \in \left[H^{k+1}(\Omega_i)\right]^d$, $p_i \in H^k(\Omega_i)$, $i = 1, 2$. There holds*

$$\|\!|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)|\!\|_h \leq c(\beta^{-1})\,h^k\sum_{i=1,2}\Big[\left(1 + r_\Gamma^{\frac{1}{2}}h^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \delta^{-\frac{1}{2}}\right)\mu^{\frac{1}{2}}\|\boldsymbol{u}\|_{k+1,\Omega_i} + \left(1 + \delta^{-\frac{1}{2}}\right)\mu^{-\frac{1}{2}}\|p\|_{k,\Omega_i}\Big], \tag{7}$$

where c is a positive constant, independent of h, that behaves as $1/\beta$.

We remark that the stability and convergence results are essentially the same as the ones given in [1], but without the need for the extra stabilization term. Note that the scaling $\sqrt{r_\Gamma/\mu}$ is present in both cases. The *inf-sup* constant β and the estimate constant c have also the same asymptotic behavior as in [1].

Proof. For the sake of conciseness, we prove only point **(i)**. The proof of **(ii)** follows [1], owing to the stability of $\mathcal{A}_\delta^{r_\Gamma}$. Let $\boldsymbol{x}_h = (\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h^k \times N_h^k$. Given (5), the inf-sup stability of $\mathcal{A}_\delta^{r_\Gamma}$ requires additional stability estimates needed to control the pressure.

A pressure $p \in L_0^2(\Omega)$ has zero mean in Ω, but this is not true in general for its restriction to Ω_i, $i = 1, 2$. Following an argument of [4], we decompose $p_h \in N_h^k \subset L_0^2(\Omega)$ as $p_h = p_h^0 + \overline{p}_h$, with $p_{h,i}^0 \in L_0^2(\Omega_i)$ and $\overline{p}_{h,i} \stackrel{\text{def}}{=} \frac{(p_{h,i},1)_{\Omega_i}}{|\Omega_i|}$ (*i.e.*, p_h^0 has zero mean over each subdomain and $\overline{p}_h$ is constant in each subdomain). The following relations hold:

$$\|p_h\|_{0,\Omega}^2 = \left\|p_h^0\right\|_{0,\Omega}^2 + \|\overline{p}_h\|_{0,\Omega}^2, \quad \overline{p}_{h,1}|\Omega_1| + \overline{p}_{h,2}|\Omega_2| = 0, \quad \|\overline{p}_h\|_{0,\Omega}^2 = \overline{p}_{h,1}^2|\Omega_1| + \overline{p}_{h,2}^2|\Omega_2|. \tag{8}$$

We show how to control separately p_h^0 and $\overline{p}_h$. Since $p_{h,i}^0 \in L_0^2(\Omega_i)$, $i = 1, 2$, there exists a function $\boldsymbol{v}^0 \in \mathbf{V}_0$, such that $\boldsymbol{v}_i^0 \in [H_0^1(\Omega_i)]^d$, $-\operatorname{div} \boldsymbol{v}_i^0 = \frac{1}{\mu} p_{h,i}^0$ and $\|\boldsymbol{v}^0\|_{1,\Omega} \le c_\Omega \frac{1}{\mu} \|p_h^0\|_{0,\Omega}$. We take $\boldsymbol{v}_h^0$ as the Scott-Zhang interpolant of $\boldsymbol{v}^0$ into $\mathbf{V}_{0,h}^k$, defined separately on each subdomain. Using the properties of the Scott-Zhang operator [3], we also have $\|\boldsymbol{v}_h^0\|_{1,\Omega} \le c'_\Omega \frac{1}{\mu} \|p_h^0\|_{0,\Omega}$ and $\|\boldsymbol{v}^0 - \boldsymbol{v}_h^0\|_{0,\Omega} \le c_\pi h \|\boldsymbol{v}^0\|_{1,\Omega}$. Since $\boldsymbol{v}_i^0, \boldsymbol{v}_{i,h}^0 \in [H_0^1(\Omega_i)]^d$, $i = 1, 2$, and $\overline{p}_h$ is constant on each subdomain, we have $(\boldsymbol{v}^0 - \boldsymbol{v}_h^0)|_\Gamma = 0$, $(\overline{p}_h, \operatorname{div} \boldsymbol{v}_h^0) = 0$ and $(\overline{p}_h, \operatorname{div} \boldsymbol{v}^0) = 0$. Hence, using the fact that $p_h \in N_h^k$ is continuous in Ω_1 and Ω_2 we obtain, integrating by parts in each subdomain:

$$\begin{aligned} -(p_h, \operatorname{div} \boldsymbol{v}_h^0) &= -\left(p_h^0, \operatorname{div} \boldsymbol{v}^0\right) + \left(p_h^0, \operatorname{div}(\boldsymbol{v}^0 - \boldsymbol{v}_h^0)\right) \\ &\ge \frac{1}{\mu} \left\|p_h^0\right\|_{0,\Omega}^2 - \xi c_\pi c_\Omega \delta^{-\frac{1}{2}} \frac{1}{\mu^{\frac{1}{2}}} \left\|p_h^0\right\|_{0,\Omega}, \end{aligned}$$

with ξ defined in Proposition 3.1. Using Young's inequality, this yields

$$\begin{aligned} \mathcal{A}_\delta^{r_\Gamma}\left(\boldsymbol{x}_h, (\boldsymbol{v}_h^0, 0)\right) &\ge -c'_\Omega \mu^{\frac{1}{2}} \|\boldsymbol{\nabla} \boldsymbol{u}_h\|_{0,\Omega} \frac{1}{\mu^{\frac{1}{2}}} \left\|p_h^0\right\|_{0,\Omega} - \delta^{-\frac{1}{2}} c_\pi c_\Omega \xi \frac{1}{\mu^{\frac{1}{2}}} \left\|p_h^0\right\|_{0,\Omega} + \frac{1}{\mu} \left\|p_h^0\right\|_{0,\Omega}^2 \\ &\ge \frac{1}{2\mu} \left\|p_h^0\right\|_{0,\Omega}^2 - (c'_\Omega)^2 \mu \|\boldsymbol{\nabla} \boldsymbol{u}_h\|_{0,\Omega}^2 - \frac{c_\pi^2 c_\Omega^2}{\delta} \xi^2. \end{aligned} \tag{9}$$

To handle the constant part of the pressure, we need the following Lemma (whose proof can be found in [4] in a more complex case):

Lemma 3.1. *There exist two (non-constant) functions $\overline{\boldsymbol{v}}_h^\alpha \in \boldsymbol{V}_h^1$, $\alpha = 1, 2$, defined over the whole domain Ω such that $\int_\Gamma \overline{\boldsymbol{v}}_{h,1}^\alpha \cdot \boldsymbol{n}_1 = -\int_\Gamma \overline{\boldsymbol{v}}_{h,2}^\alpha \cdot \boldsymbol{n}_2 = \frac{\overline{p}_{h,\alpha}}{\mu} |\Omega_\alpha|$ and*

$$\mu^{\frac{1}{2}} \|\boldsymbol{\nabla} \overline{\boldsymbol{v}}_h^\alpha\|_{0,\Omega} \le \overline{c} \mu^{-\frac{1}{2}} \left\|\overline{p}_{h,\alpha}\right\|_{0,\Omega_\alpha}, \quad r_\Gamma^{\frac{1}{2}} \|\overline{\boldsymbol{v}}_h^\alpha\|_{0,\Gamma} \le \overline{C} r_\Gamma^{\frac{1}{2}} \mu^{-1} \left\|\overline{p}_{h,\alpha}\right\|_{0,\Omega_\alpha}.$$

Let $\overline{\boldsymbol{v}}_h \stackrel{\text{def}}{=} \overline{\boldsymbol{v}}_h^2 - \overline{\boldsymbol{v}}_h^1 \in \boldsymbol{V}_h^k$. Since $\boldsymbol{\nabla}\overline{p}_{h,i} = 0$ and using (8) and Lemma 3.1, we have

$$
\begin{aligned}
-(\overline{p}_h, \operatorname{div}\overline{\boldsymbol{v}}_h) &= \sum_{i=1,2} (\overline{p}_{h,i}, (\overline{\boldsymbol{v}}_h^1 - \overline{\boldsymbol{v}}_h^2)\cdot \boldsymbol{n}_i)_\Gamma \\
&= \mu^{-1}\left(\overline{p}_{h,1}^2|\Omega_1| - \overline{p}_{h,1}\overline{p}_{h,2}\left(|\Omega_2| + |\Omega_1|\right) + \overline{p}_{h,2}^2|\Omega_2|\right) \\
&= 2\mu^{-1}\left(\overline{p}_{h,1}^2|\Omega_1| + \overline{p}_{h,2}^2|\Omega_2|\right) = 2\mu^{-1}\|\overline{p}_h\|_{0,\Omega}^2,
\end{aligned}
$$

and, by applying Lemma 3.1 once more, and using the fact that $\left\|\overline{p}_{h,1}\right\|_{0,\Omega_1} + \left\|\overline{p}_{h,2}\right\|_{0,\Omega_2} \leq 2\|\overline{p}_h\|_{0,\Omega}$, there follows

$$
\begin{aligned}
-(p_h, \operatorname{div}\overline{\boldsymbol{v}}_h) &= -(\overline{p}_h, \operatorname{div}\overline{\boldsymbol{v}}_h) - (\overline{p}_h^0, \operatorname{div}\overline{\boldsymbol{v}}_h) \\
&\geq 2\mu^{-1}\|\overline{p}_h\|_{0,\Omega}^2 - \left\|p_h^0\right\|_{0,\Omega} d^{\frac{1}{2}} \left\|\boldsymbol{\nabla}(\overline{\boldsymbol{v}}_h^1 - \overline{\boldsymbol{v}}_h^2)\right\|_{0,\Omega} \\
&\geq \frac{2}{\mu}\|\overline{p}_h\|_{0,\Omega}^2 - d\overline{c}^2\frac{1}{\mu}\left\|p_h^0\right\|_{0,\Omega}^2 - \frac{1}{4\mu}\left(\left\|\overline{p}_{h,1}\right\|_{0,\Omega_1} + \left\|\overline{p}_{h,2}\right\|_{0,\Omega_2}\right)^2 \\
&\geq \frac{1}{\mu}\|\overline{p}_h\|_{0,\Omega}^2 - d\overline{c}^2\frac{1}{\mu}\left\|p_h^0\right\|_{0,\Omega}^2,
\end{aligned}
$$

where we recall that d denotes the spatial dimension. Hence,

$$
\begin{aligned}
\mathcal{A}_\delta^{r_\Gamma}\left(\boldsymbol{x}_h, (\overline{\boldsymbol{v}}_h, 0)\right) \geq & - 2\overline{c}\mu^{\frac{1}{2}}\|\boldsymbol{\nabla}\boldsymbol{u}_h\|_{0,\Omega}\mu^{-\frac{1}{2}}\|\overline{p}_h\|_{0,\Omega} - 2\overline{C}r_\Gamma\mu^{-\frac{1}{2}}\|\boldsymbol{u}_h\|_{0,\Gamma}\mu^{-\frac{1}{2}}\|\overline{p}_h\|_{0,\Omega} \\
& + \mu^{-1}\|\overline{p}_h\|_{0,\Omega}^2 - d\overline{c}^2\mu^{-1}\left\|p_h^0\right\|_{0,\Omega}^2 \\
\geq & \frac{1}{2\mu}\|\overline{p}_h\|_{0,\Omega}^2 - d\overline{c}^2\frac{1}{\mu}\left\|p_h^0\right\|_{0,\Omega}^2 - 4\overline{c}^2\mu\|\boldsymbol{\nabla}\boldsymbol{u}_h\|_{0,\Omega}^2 - 4\overline{C}^2\frac{r_\Gamma^2}{\mu}\|\boldsymbol{u}_h\|_{0,\Gamma}^2 .
\end{aligned}
\tag{10}
$$

Therefore, by taking $\boldsymbol{y}_h = (\lambda\boldsymbol{v}_h^0 + (1-\lambda)\overline{\boldsymbol{v}}_h, 0)$, with $\lambda \stackrel{\text{def}}{=} \frac{1+2d\overline{c}^2}{2(1+d\overline{c}^2)} \in (0,1)$, and using (9) and (10), we obtain

$$
\begin{aligned}
\mathcal{A}_\delta^{r_\Gamma}(\boldsymbol{x}_h, \boldsymbol{y}_h) \geq & \left(\frac{\lambda}{2} - (1-\lambda)d\overline{c}^2\right)\frac{1}{\mu}\left\|p_h^0\right\|_{0,\Omega}^2 + \frac{1-\lambda}{2}\frac{1}{\mu}\|\overline{p}_h\|_{0,\Omega}^2 \\
& - \left(\lambda(c'_\Omega)^2 + (1-\lambda)4\overline{c}^2\right)\mu\|\boldsymbol{\nabla}\boldsymbol{u}_h\|_{0,\Omega}^2 \\
& - \lambda\frac{c_\pi^2 c_\Omega^2}{\delta}\xi^2 - (1-\lambda)4\overline{C}^2\frac{r_\Gamma^2}{\mu}\|\boldsymbol{u}_h\|_{0,\Gamma}^2 \\
\geq & \frac{1}{4\tilde{c}\mu}\|p_h\|_{0,\Omega}^2 - c_{\max}^2\left(\|\!|\!|(\boldsymbol{u}_h, p_h)|\!|\!|_h^2 - \frac{1}{\mu}\|p_h\|_{0,\Omega}^2\right),
\end{aligned}
\tag{11}
$$

where we have introduced $\tilde{c} \stackrel{\text{def}}{=} 1+d\overline{c}^2$, and $c_{\max}^2 \stackrel{\text{def}}{=} \max\left\{\left(c_\Omega'^2+4\overline{c}^2\right), \frac{1}{\delta}c_\pi^2 c_\Omega^2, 4\overline{C}^2\frac{r_\Gamma}{\mu}\right\}$. Note that, unlike the other constants that are dimensionless, $\overline{C}^2$ has the dimension of the inverse of a distance. Equation (11) provides a control on the pressure.

To conclude the proof, we take a test function $\boldsymbol{z}_h \stackrel{\text{def}}{=} (1-\omega)\boldsymbol{x}_h + \omega \boldsymbol{y}_h$, with $\omega \stackrel{\text{def}}{=} \frac{2\tilde{c}}{1+2\tilde{c}(1+2c_{\max}^2)} \in (0,1)$, and apply (5) and (11), to obtain

$$
\begin{aligned}
\mathcal{A}_\delta^{r_\Gamma}(\boldsymbol{x}_h, \boldsymbol{z}_h) \geq & \frac{1}{2}(1-\omega)\left(|\!|\!| (\boldsymbol{u}_h, p_h) |\!|\!|_h^2 - \frac{1}{\mu}\|p_h\|_{0,\Omega}^2 \right) \\
& + \omega \left(\frac{1}{4\tilde{c}\mu}\|p_h\|_{0,\Omega}^2 - c_{\max}^2 \left(|\!|\!| (\boldsymbol{u}_h, p_h) |\!|\!|_h^2 - \frac{1}{\mu}\|p_h\|_{0,\Omega}^2 \right) \right) \\
\geq & \left(\frac{1-\omega}{2} - \omega c_{\max}^2 \right) \left(\mu \|\boldsymbol{\nabla} \boldsymbol{u}_h\|_{0,\Omega}^2 + r_\Gamma \|\boldsymbol{u}_h\|_{0,\Gamma}^2 + \xi^2 \right) + \frac{\omega}{4\tilde{c}\mu}\|p_h\|_{0,\Omega}^2 \\
\geq & \frac{1}{2\big(1+2\tilde{c}(1+2c_{\max}^2)\big)} |\!|\!| \boldsymbol{x}_h |\!|\!|_h^2 .
\end{aligned} \tag{12}
$$

Moreover, it can be shown that $\boldsymbol{z}_h$ can be controlled by $\boldsymbol{x}_h$ as

$$
\begin{aligned}
|\!|\!| \boldsymbol{z}_h |\!|\!|_h & \leq (1-\omega) |\!|\!| \boldsymbol{x}_h |\!|\!|_h + \omega \left(|\!|\!| (\boldsymbol{v}_h^0, 0) |\!|\!|_h + |\!|\!| (\overline{\boldsymbol{v}}_h^1, 0) |\!|\!|_h + |\!|\!| (\overline{\boldsymbol{v}}_h^2, 0) |\!|\!|_h \right) \\
& \leq (1-\omega) |\!|\!| \boldsymbol{x}_h |\!|\!|_h + \omega c'_\Omega \mu^{-\frac{1}{2}} \left\| p_h^0 \right\|_{0,\Omega} + \omega \sqrt{2} (\overline{c}^2 + \frac{r_\Gamma}{\mu}\overline{C}^2)^{\frac{1}{2}} \mu^{-\frac{1}{2}} \|\overline{p}_h\|_{0,\Omega} \\
& \leq (1-\omega) |\!|\!| \boldsymbol{x}_h |\!|\!|_h + \omega \sqrt{2}\, c_{\max} |\!|\!| \boldsymbol{x}_h |\!|\!|_h \;=\; \left(1 - \omega + \omega\sqrt{2}\, c_{\max} \right) |\!|\!| \boldsymbol{x}_h |\!|\!|_h \\
& \leq \frac{1 + 2\tilde{c} c_{\max}(2c_{\max} + \sqrt{2})}{1+2\tilde{c}(1+2c_{\max}^2)} |\!|\!| \boldsymbol{x}_h |\!|\!|_h .
\end{aligned} \tag{13}
$$

Combining (12) and (13) we obtain that the global inf-sup condition (6) follows with a constant $\beta \stackrel{\text{def}}{=} \left[2(1 + 2\tilde{c} c_{\max}(2c_{\max} + \sqrt{2}) \right]^{-1}$. The stated asymptotic behavior of β follows from the definition of $c_{\max}$. □

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