Asymptotics beats Monte Carlo: The case of correlated local vol baskets

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Outline

1. Introduction
2. Outline of our approach
3. Heat kernel expansions
4. Numerical examples
1 Introduction

2 Outline of our approach

3 Heat kernel expansions

4 Numerical examples
Methods of European option pricing

\[ u(t, S_t) = e^{-r(T-t)} E \left[ f(S_T) \mid S_t \right] \]

Example (Example treated in this work)

\[ f(S) = \left( \sum_{i=1}^{n} w_i S_i - K \right)^+ \], at least one weight positive

\( n \) large (e.g., \( n = 500 \) for SPX)

- PDE methods
- (Quasi) Monte Carlo method
- Fourier transform based methods
- Approximation formulas

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Page 4 (32)
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- PDE methods
  - **Pros**: fast, general
  - **Cons**: curse of dimensionality, path-dependence may or may not be easy to include
    - (Quasi) Monte Carlo method
    - Fourier transform based methods
    - Approximation formulas
Methods of European option pricing

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- \( n \) large (e.g., \( n = 500 \) for SPX)

- PDE methods
- (Quasi) Monte Carlo method
  
  **Pros:** very general, easy to adapt, no curse of dimensionality
  
  **Cons:** slow, quasi MC may be difficult in high dimensions

- Fourier transform based methods
- Approximation formulas
Methods of European option pricing

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- \( n \) large (e.g., \( n = 500 \) for SPX)

- PDE methods
- (Quasi) Monte Carlo method
- Fourier transform based methods

**Pros:** very fast to evaluate (“explicit formula”)

**Cons:** only available for affine models, difficult to generalize, curse of dimensionality

- Approximation formulas
Methods of European option pricing

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- PDE methods
- (Quasi) Monte Carlo method
- Fourier transform based methods
- Approximation formulas

**Pros:** very fast evaluation

**Cons:** derived on case by case basis, therefore very restrictive
Methods of European option pricing

\[ u(t, S_t) = e^{-r(T-t)}E[f(S_T)|S_t] \]

Example (Example treated in this work)

- \[ f(S) = \left(\sum_{i=1}^{n} w_i S_i - K\right)^+, \text{ at least one weight positive} \]
- \[ n \text{ large (e.g., } n = 500 \text{ for SPX)} \]

- PDE methods
- (Quasi) Monte Carlo method
- Fourier transform based methods
- Approximation formulas
- Work horse methods: PDE methods and (in particular) (Q)MC
- Particular models allowing approximation formulas (e.g., SABR formula) or FFT (Heston model) very popular
Approximation formulas based on expansions in option parameters

- Expansions in large/small strike or large/small maturity

Example (Large strike expansion, Lee formula)

- \( \partial_K \text{Call}(S_0, T, K) = -P(S_T \geq K) \)
- For \( K \gg 1 \), this is a rare event (large deviation)
- Lee formula: \( m := \log(S_0/K) \), \( \beta \) related to moment explosion

\[
\lim_{m \to \pm\infty} \frac{T}{m} \sigma_I^2(S_0, T, K) = \beta_\pm
\]

- Extensions e.g., by P. Friz et al.
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- Extensions e.g., by P. Friz et al.
Consider the (one-dimensional) model

\[ dS_t = \sigma(t, S_t)dW_t, \quad S_0 \in \mathbb{R} \]

Expansion: \( S^\epsilon_t = S_0 + \epsilon S_{1,t} + \frac{1}{2} \epsilon^2 S_{2,t} + o(\epsilon^2), \) with

\[ S_{1,t} = \int_0^t \sigma(s, S_0)dW_s, \]
\[ S_{2,t} = 2 \int_0^t \partial_x \sigma(s, S_0)S_{1,s}dW_s \]

Wiener chaos decomposition

\[ E[f(S_T)] \approx f(S_0) + \epsilon f'(S_0)E[S_{1,T}] + \frac{1}{2} \epsilon^2 \left( f''(S_0)E[S_{2,T}] + f'''(S_0)E[S_{1,T}^2] \right) + \cdots \]

For non-smooth payoffs, extensions possible by Malliavin weights.
Small noise expansion (stochastic approach)

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For non-smooth payoffs, extensions possible by Malliavin weights.
Price $u^\epsilon(t, S_0)$ solves $L^\epsilon u = 0$ with $L^\epsilon = \partial_t + \frac{1}{2} \epsilon^{2} \sigma^{2} \partial_{x}^{2} =: L_0 + \epsilon^{2} L_2$

Ansatz $u^\epsilon = u_0 + \epsilon u_1 + \frac{1}{2} \epsilon^{2} u_2 + \cdots$ gives (regular perturbation)

$$L_0 u_0 + \epsilon L_0 u_1 + \epsilon^{2} \left( \frac{1}{2} L_0 u_2 + L_2 u_0 \right) + o(\epsilon^{2}) = 0$$

Formally, we get

$$u_0(T, S_0) = f(S_0), \quad L_0 u_0 = 0, \quad L_0 u_1 = 0, \quad \frac{1}{2} L_0 u_2 + L_2 u_0 = 0, \ldots$$

$$u_0(t, S_0) = f(S_0), \quad u_1 = 0, \quad u_2(t, S_0) = 2 \int_{t}^{T} L_2 u_0(s, S_0) ds$$

For non-smooth payoffs, a singular perturbation in the fast variable $y := (x - K)/\epsilon$ can be used
Small noise expansion (PDE approach)

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Local volatility model for forward prices

\[ dF_i(t) = \sigma_i(F_i(t))dW_i(t), \quad i = 1, \ldots, n, \]

\[ \langle dW_i(t), dW_j(t) \rangle = \rho_{ij}dt \]

Generalized spread option with payoff \((\sum_{i=1}^{n} w_i F_i - K)^+\), at least one \(w_i\) positive

Goal: fast and accurate approximation formulas, even for high \(n\)

\(n = 100\) or \(n = 500\) not uncommon (index options)

Example

- Black-Scholes model: \(\sigma_i(F_i) = \sigma_i F_i\)
- CEV model: \(\sigma_i(F_i) = \sigma_i F_i^{\beta_i}\)
Setting

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Basket Carr-Jarrow formula

Consider the basket (index) $\sum_{i=1}^{n} w_i F_i$:

$$d \sum_{i=1}^{n} w_i F_i(t) = \sum_{i=1}^{n} w_i \sigma_i(F_i(t)) dW_i(t)$$

Ito’s formula formally implies that

Let $p(F_0, F, t) := P(F(t) \in dF | F(0) = F_0)$ and $H_{n-1}$ be the Hausdorff measure on $\mathcal{E}(K)$, then we have the Carr-Jarrow formula

$$C(F_0, K, T) = \left( \sum_{i=1}^{n} w_i F_i(0) - K \right)^+ +$$

$$+ \frac{1}{2} \int_{0}^{T} \frac{1}{|w|} \int_{\mathcal{E}(K)} \sum_{i,j=1}^{n} w_i w_j \sigma_i(F_i) \sigma_j(F_j) \rho_{ij} p(F_0, F, u) H_{n-1}(dF) du.$$
Basket Carr-Jarrow formula

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\left( \sum_{i=1}^{n} w_i F_i(t) - K \right)^+ = \left( \sum_{i=1}^{n} w_i F_i(0) - K \right)^+ + \\
+ \sum_{i=1}^{n} w_i \int_{0}^{T} \mathbf{1}_{\sum w_i F_i(u) > K} dF_i(u) + \frac{1}{2} \int_{0}^{T} \delta_{\sum w_i F_i(u) = K} \sigma^2_{\mathcal{N}, \mathcal{B}(\mathbf{F}(u))} du
\]

- Let \( p(F_0, F, t) := P(F(t) \in dF | F(0) = F_0) \) and \( H_{n-1} \) be the Hausdorff measure on \( \mathcal{E}(K) \), then we have the Carr-Jarrow formula

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\]
Consider the basket (index) \( \sum_{i=1}^{n} w_i F_i \):

Ito's formula formally implies (with \( \mathcal{E}(K) = \{ F \mid \sum w_i F_i = K \} \)) that

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\]

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\]
Heat kernel expansion (to be discussed in detail later):

\[ \sigma_{N,B}^2 \mathcal{B}(\mathbf{F}) p(\mathbf{F}_0, \mathbf{F}, t) \approx \frac{1}{(2\pi t)^{n/2}} \exp \left( -\frac{d(\mathbf{F}_0, \mathbf{F})^2}{2t} - C(\mathbf{F}_0, \mathbf{F}) \right) \]

By change of variables \( F_n = \frac{1}{w_n} \left( K - \sum_{i=1}^{n-1} w_i F_i \right) \) on \( \mathcal{E}_K \):

\[ H_{n-1}(d\mathbf{F}) = \frac{|w|}{|w_n|} dF_1 \cdots dF_{n-1} \]

Laplace approximation: with \( \mathbf{F}^* = \text{argmin}_{\mathbf{F} \in \mathcal{E}_K} d(\mathbf{F}_0, \mathbf{F}) \) and \( \mathcal{G}_K = \{ (F_1, \ldots, F_{n-1}) | \sum_{i=1}^{n-1} w_i F_i < K \} \)

\[ \int_{\mathcal{G}_K} e^{-\frac{d(\mathbf{F}_0, \mathbf{F})^2}{2t}} - C(\mathbf{F}_0, \mathbf{F}) dF_1 \cdots dF_{n-1} \approx e^{-\frac{d(\mathbf{F}_0, \mathbf{F}^*)^2}{2t}} - C(\mathbf{F}_0, \mathbf{F}^*) \int_{\mathbb{R}^{n-1}} e^{-\frac{z^T Q z}{2t}} d\mathbf{z} \]

\[ = t^{\frac{n-1}{2}} e^{-\frac{d(\mathbf{F}_0, \mathbf{F}^*)^2}{2t}} - C(\mathbf{F}_0, \mathbf{F}^*) \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{\det Q}} \]

We rely on the principle of not feeling the boundary.
Heat kernel expansion (to be discussed in detail later):

\[
\sigma_{N,B}^2 \approx \frac{1}{(2\pi t)^{n/2}} \exp \left(-\frac{d(F_0, F)^2}{2t} - C(F_0, F)\right)
\]

By change of variables \( F_n = \frac{1}{w_n} \left( K - \sum_{i=1}^{n-1} w_i F_i \right) \) on \( E_K \):

\[
H_{n-1}(dF) = \frac{|w|}{|w_n|} dF_1 \cdots dF_{n-1}
\]

Laplace approximation: with \( F^* = \arg\min_{F \in E_K} d(F_0, F) \) and \( G_K = \{(F_1, \ldots, F_{n-1})\mid \sum_{i=1}^{n-1} w_i F_i < K\} \)

\[
\int_{G_K} e^{-\frac{d(F_0,F)^2}{2t} - C(F_0,F)} dF_1 \cdots dF_{n-1} \approx e^{-\frac{d(F_0,F^*)^2}{2t} - C(F_0,F^*)} \int_{\mathbb{R}^{n-1}} e^{-\frac{z^T Q z}{2t}} d\mathbf{z}
\]

\[
= t^{\frac{n-1}{2}} e^{-\frac{d(F_0,F^*)^2}{2t} - C(F_0,F^*)} \frac{(2\pi)^{n-1}}{\sqrt{\det Q}}
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Approximations

- Heat kernel expansion (to be discussed in detail later):
  \[ \sigma_{N,B}^2(F)p(F_0,F,t) \approx \frac{1}{(2\pi t)^{n/2}} \exp \left( -\frac{d(F_0,F)^2}{2t} - C(F_0,F) \right) \]

- By change of variables \( F_n = \frac{1}{w_n} \left( K - \sum_{i=1}^{n-1} w_i F_i \right) \) on \( \mathcal{E}_K \):
  \[ H_{n-1}(dF) = \frac{|w|}{|w_n|} dF_1 \cdots dF_{n-1} \]

- Laplace approximation: with \( F^* = \arg\min_{F \in \mathcal{E}_K} d(F_0,F) \) and
  \[ \mathcal{G}_K = \{ (F_1, \ldots, F_{n-1}) | \sum_{i=1}^{n-1} w_i F_i < K \} \]
  \[ \int_{\mathcal{G}_K} e^{-\frac{d(F_0,F)^2}{2t} - C(F_0,F)} dF_1 \cdots dF_{n-1} \approx e^{-\frac{d(F_0,F^*)^2}{2t} - C(F_0,F^*)} \int_{\mathbb{R}^{n-1}} e^{-\frac{z^TQz}{2t}} dz \]
  \[ = t^{\frac{n-1}{2}} e^{-\frac{d(F_0,F^*)^2}{2t} - C(F_0,F^*)} \frac{(2\pi)^{n-1}}{\sqrt{\det Q}} \]

We rely on the principle of not feeling the boundary.
Heat kernel expansion (to be discussed in detail later):

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\sigma^2_{N,B}(\mathbf{F}) p(\mathbf{F}_0, \mathbf{F}, t) \approx \frac{1}{(2\pi t)^{n/2}} \exp \left( -\frac{d(\mathbf{F}_0, \mathbf{F})^2}{2t} - C(\mathbf{F}_0, \mathbf{F}) \right)
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By change of variables \( F_n = \frac{1}{w_n} \left( K - \sum_{i=1}^{n-1} w_i F_i \right) \) on \( \mathcal{E}_K \):

\[
H_{n-1}(d\mathbf{F}) = \frac{|w|}{|w_n|} dF_1 \cdots dF_{n-1}
\]

Laplace approximation: with \( \mathbf{F}^* = \text{argmin}_{\mathbf{F} \in \mathcal{E}_K} d(\mathbf{F}_0, \mathbf{F}) \) and 
\[
\mathcal{G}_K = \{ (F_1, \ldots, F_{n-1}) | \sum_{i=1}^{n-1} w_i F_i < K \}
\]

\[
\int_{\mathcal{G}_K} e^{-\frac{d(\mathbf{F}_0, \mathbf{F})^2}{2t} - C(\mathbf{F}_0, \mathbf{F})} dF_1 \cdots dF_{n-1} \approx e^{-\frac{d(\mathbf{F}_0, \mathbf{F}^*)^2}{2t} - C(\mathbf{F}_0, \mathbf{F}^*)} \int_{\mathbb{R}^{n-1}} e^{-\frac{z^T Q z}{2t}} dz
\]

\[
= t^{\frac{n-1}{2}} e^{-\frac{d(\mathbf{F}_0, \mathbf{F}^*)^2}{2t} - C(\mathbf{F}_0, \mathbf{F}^*)} \frac{(2\pi)^{n-1/2}}{\sqrt{\det Q}}
\]

We rely on the principle of not feeling the boundary.
Matching to implied volatilities

Theorem

\[ C_B(F_0, K, T) = \left( \sum_{i=1}^{n} w_i F_i(0) - K \right)^+ + \]
\[ + \frac{1}{2 \sqrt{2\pi} |w_n| d(F_0, F^*)^2 \sqrt{\det Q}} e^{-C(F_0,F^*)} - \frac{d(F_0,F^*)}{2T} T^{3/2} + o(T^{3/2}), \text{ as } T \to 0. \]

- Bachelier implied vol (with \( \bar{F}_0 = \sum_{i=1}^{n} w_i F_{0,i} \)):
  \[ \sigma_B \sim \sigma_{B,0} + T \sigma_{B,1} \text{ with } \sigma_{B,0} = \frac{|\bar{F}_0 - K|}{d(F_0, F^*) |\bar{F}_0|}, \sigma_{B,1} = \cdots \]

- Black-Scholes implied voila:
  \[ \sigma_{BS} \sim \sigma_{BS,0} + T \sigma_{BS,1} \text{ with } \sigma_{BS,0} = \frac{|\log (\bar{F}_0/K)|}{d(F_0, F^*)}, \sigma_{BS,1} = \cdots \]
Matching to implied volatilities

**Theorem**

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Greeks

- Goal: sensitivity w. r. t. model parameter $\kappa$ of the option price

$$C_B(F_0, K, T) \approx C_{BS}(\bar{F}_0, K, \sigma_{BS}, T)$$

- Sensitivity: $\partial_\kappa C_{BS}(\bar{F}_0, K, \sigma_{BS}, T) + \nu_{BS}(\bar{F}_0, K, \sigma_{BS}, T) \partial_\kappa \sigma_{BS}$

- Recall that $\sigma_{BS,0}, \sigma_{BS,1}$ explicit up to $F^*$
  - By the minimizing property: $\partial_{F_i} \partial^2 (F_0, F_K(G))\big|_{G=G^*} = 0$
  - Differentiating with respect to $\kappa$ gives

$$\partial_\kappa \partial_{F_i} \partial^2 (F_0, F_K(G))\big|_{G^*} + \sum_{l=1}^{n-1} \partial_{F_i} \partial_{F_j} \partial^2 (F_0, F_K(G))\big|_{G^*} \partial_\kappa F_l^* = 0$$

Up to the above system of linear equations for $\partial_\kappa F^*$, there are explicit expression for the sensitivities of the approximate option prices.
Goal: sensitivity w. r. t. model parameter $\kappa$ of the option price

$$C_B(F_0, K, T) \approx C_{BS}(\overline{F}_0, K, \sigma_{BS}, T)$$

Sensitivity: $\frac{\partial}{\partial \kappa} C_{BS}(\overline{F}_0, K, \sigma_{BS}, T) + \nu_{BS}(\overline{F}_0, K, \sigma_{BS}, T) \frac{\partial}{\partial \kappa} \sigma_{BS}$

Recall that $\sigma_{BS,0}, \sigma_{BS,1}$ explicit up to $\overline{F}^*$

By the minimizing property: $\frac{\partial F_i}{\partial \kappa} d^2 (F_0, F_K(G)) \bigg|_{G=G^*} = 0$

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Up to the above system of linear equations for $\frac{\partial \kappa \overline{F}^*}{\partial \kappa}$, there are explicit expression for the sensitivities of the approximate option prices.
Heat kernels and geometry

\[ d\mathbf{X}_t = b(\mathbf{X}_t)dt + \sigma(\mathbf{X}_t)dW_t, \]

\[ L = \frac{1}{2} a^{i,j} \frac{\partial^2}{\partial x^i \partial x^j} + b^i \frac{\partial}{\partial x^i}, \quad a = \sigma^T \sigma \]

- Heat kernel: fundamental solution \( p(x, y, t) \) of \( \frac{\partial}{\partial t} u = Lu \)
- Transition density of \( \mathbf{X}_t \)

"Can you hear the shape of the drum?" (Kac '66)

Take \( L = \Delta \) on a domain \( D \) and relate:

- Geometrical properties of the domain \( D \)
- Partition function \( Z = \sum_{k \in \mathbb{N}} e^{\gamma_k t} \)
- Heat kernel
- E.g. \( -\gamma_k \sim C(n)(k/\text{vol } D)^{2/n} \) (Weyl, '46)
- E.g. (for \( n = 2 \)): \( Z = \frac{\text{area}}{4\pi t} - \frac{\text{circ.}}{\sqrt{4\pi t}} + O(1) \) (McKean & Singer, '67)
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The Riemannian metric associated to a diffusion

\[ dX_t = b(X_t)dt + \sigma(X_t)dW_t, \]

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- On \( \mathbb{R}^n \) (or a submanifold), introduce \( g^{ij} := a^{ij} \), Riemannian metric tensor \((g_{ij}(x))_{i,j=1}^n := \left((g^{ij}(x))_{i,j=1}^n\right)^{-1}\)
- Geodesic distance:
  \[ d(x, y) := \inf_{z(0)=x, z(1)=y} \int_0^1 \sqrt{\sum g_{ij}(z(t))\dot{z}^i(t)\dot{z}^j(t)} \, dt \]
  inf attained by a smooth curve, the geodesic
- Laplace-Beltrami operator: \( \Delta_g = \left(\det(g_{ij})\right)^{-\frac{1}{2}} \frac{\partial}{\partial x^i} \left(\det(g_{ij})\right)^{\frac{1}{2}} g^{ij} \frac{\partial}{\partial x^j} \)
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Heat kernel expansion

\[ p_N(x_0, x, T) = \sqrt{\det(g(x)_{ij})} U_N(x_0, x, T) e^{-\frac{d^2(x_0, x)}{2T}} \]

\[ U_N(x_0, x, T) = \sum_{k=0}^{N} u_k(x_0, x) T^k, \text{ the heat kernel coefficients} \]

\[ u_0(x_0, x) = \sqrt{\Delta(x_0, x)} e^{\int_z \langle h(z(t)) \, , \dot{z}(t) \rangle_g dt} \]

\[ \Delta is the Van Vleck-DeWitt determinant: \]
\[ \Delta(x_0, x) = \frac{1}{\sqrt{\det(g(x_0)_{ij}) \det(g(x)_{ij})}} \det \left( -\frac{1}{2} \frac{\partial^2 d^2}{\partial x_0 \partial x} \right). \]

\[ e^{\int_z \langle h(z(t)) \, , \dot{z}(t) \rangle_g dt} is the exponential of the work done by the vector field \( h \) along the geodesic \( z \) joining \( x_0 \) to \( x \) with \]
\[ h^i = b^i - \frac{1}{2 \sqrt{\det(g_{ij})}} \frac{\partial}{\partial x^j} \left[ \sqrt{\det(g_{ij})} g^{ij} \right] \]
Heat kernel expansion

\[ p_N(x_0, x, T) = \sqrt{\text{det}(g(x)_{ij})} U_N(x_0, x, T) \frac{e^{-\frac{d^2(x_0, x)}{2T}}}{(2\pi T)^{n/2}} \]

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Assumption

The cut-locus of any point is empty, i.e., any two points are connected by a unique minimizing geodesic.

Theorem (Varadhan ’67)

\( b = 0, \sigma \) uniformly Hölder continuous, system uniformly elliptic, then

\[
\lim_{T \to 0} T \log p(x, y, T) = -\frac{1}{2} d(x, y)^2.
\]

Theorem (Yosida ’53)

On a compact Riemannian manifold, assume smooth vector fields and an ellipticity property. Then

\[
p(x, y, T) - p_N(x, y, T) = O(T^N) \text{ as } T \to 0.
\]

Theorem (Azencott ’84)

For a locally elliptic system in an open set \( U \subset \mathbb{R}^n, x, y \in U \) such that \( d(x, y) < d(x, \partial U) + d(y, \partial U) \), we have

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Heat kernel expansion – 2

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The local vol case

- **Domain** $\mathbb{R}^n_+$, $dF_i(t) = \sigma_i(F_i(t))dW_i(t), \quad i = 1, \ldots, n$

- $L = \frac{1}{2} \rho_{ij} \sigma_i(x^i) \sigma_j(x^j) \frac{\partial^2}{\partial x^i \partial x^j}$

- Let $A \in \mathbb{R}^{n \times n}$ be such that $A \rho A^T = I_n$. Change variables $F \rightarrow y \rightarrow x$ according to

  $$y_i = \int_0^{F_i} \frac{du}{\sigma_i(u)}, \quad i = 1, \ldots, n, \quad x = Ay, \quad L \rightarrow \frac{1}{2} \frac{\partial^2}{\partial x_i^2} - \frac{1}{2} A_{ik} \sigma'_k(F_k) \frac{\partial}{\partial x_i}$$

- Isomorphic (up to boundary) to Euclidean geometry:

  $$d(F_0, F) = |x_0 - x|$$

- Geodesics known in closed form

- CEV case: $\sigma_i(F_i) = \sigma_i F_i^{\beta_i}$, zeroth and first order heat kernel coefficients given explicitly
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Outline

1 Introduction

2 Outline of our approach

3 Heat kernel expansions

4 Numerical examples
Optimization problem for $F^*$ is non-linear with a linear constraint

With $q_i := \int_{F_{0,i}}^{F_i} \frac{du}{\sigma_i(u)}$, it is a quadratic optimization problem with non-linear constraint.

Fast convergence of Newton iteration

Given $F^*$, $C(F_0, F^*)$ is a line integral along the geodesic; this integral can be calculated in closed form in the CEV model.

Formulas can be evaluated in less than 2 seconds for $n = 100$.

Our work relies on the principle of not feeling the boundary.
Implementation

- Optimization problem for $\mathbf{F}^*$ is non-linear with a linear constraint
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Our work relies on the principle of not feeling the boundary.
The initial guess in the Newton iteration

- Change of variable: \( q_i = \frac{F_i^{1-\beta_i} - F_{0,i}^{1-\beta_i}}{1-\beta_i}, \quad F_i = \left( F_{0,i}^{1-\beta_i} + (1 - \beta_i)q_i \right)^{1/(1-\beta_i)} \)

- \( \Lambda^{-1} = (\sigma_i \sigma_j \rho_{ij})_{i,j=1}^n \)

- Optimization problem: \( \min \mathbf{q}^T \Lambda \mathbf{q} : \sum_{i=1}^n w_i F_i(q_i) = K \)

- Linearized constraint: \( \sum_{i=1}^n w_i \left( F_{0,i} + F_{0,i}^{\beta_i}q_i \right) = K \)

- Minimizer \( q_0^* = \frac{K - \overline{F}_0}{F_{0}^T \Lambda^{-1} \overline{F}_0} \Lambda^{-1} \overline{F}_0 \) with Lagrange multiplier \( \lambda = 2 \frac{K - \overline{F}_0}{F_{0}^T \Lambda^{-1} \overline{F}_0} \), where \( \overline{F}_{0,i} = w_i F_{0,i} \)

- \( q_0^* \) not good enough (unless coupled with “1/2-slope rule”)

- Use as initial guess in Newton iteration
The initial guess in the Newton iteration

- Change of variable: \( q_i = \frac{F_{1-i} - F_{0,i}}{1-i} \), \( F_i = \left( F_{0,i} + (1 - i)q_i \right)^{1/(1-i)} \)

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Numerical examples

- CEV model framework
- For CEV, the formulas are fully explicit apart from the minimizing configuration $F^*$
- We observe very fast convergence of the iteration, but the initial guess is crucial.
- Reference values obtained using:
  - Ninomiya Victoir discretization
  - Quasi Monte Carlo based on Sobol numbers, Monte Carlo for very high dimensions ($n \approx 100$)
  - Variance (dimension) reduction using Mean value Monte Carlo based on one-dimensional Black-Scholes prices
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CEV index implied vol – three-dimensional visualization

Approximations for local vol baskets · November 28, 2013 · Page 24 (32)
CEV index implied vol – three-dimensional visualization

Approximations for local vol baskets · November 28, 2013 · Page 24 (32)
Spread option in dimension 10

- Recall: \(dF_i(t) = \sigma_i F_i(t)^{\beta_i} dW_i(t)\)
- \(\beta = (0.7, 0.2, 0.8, 0.3, 0.5, 0.5, 0.6, 0.6, 0.3, 0.3)\)
- \(\sigma = (0.8, 0.6, 0.9, 0.6, 0.8, 0.4, 0.9, 0.9, 0.3, 0.8)\)
- \(F_0 = (10, 13, 11, 18, 9, 10, 17, 16, 13, 17)\)
- \(w = (-1, -1, 1, 1, 1, -1, -1, 1, 1, 1)\)
Spread option in dimension 10

<table>
<thead>
<tr>
<th>$T$</th>
<th>$K = 32.9$</th>
<th>$K = 33.8$</th>
<th>$K = 34.1$</th>
<th>$K = 34.4$</th>
<th>$K = 35.3$</th>
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<td>6.2385</td>
<td>6.0924</td>
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<td>5.5322</td>
</tr>
</tbody>
</table>

**Table**: Quasi Monte Carlo prices.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$K = 32.9$</th>
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<tbody>
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<td>6.0648</td>
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<td>5.5046</td>
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</tbody>
</table>

**Table**: Zero order asymptotic prices.
Spread option in dimension 10

<table>
<thead>
<tr>
<th>$T$</th>
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<td>4.8976</td>
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<td>6.7015</td>
<td>6.2487</td>
<td>6.1027</td>
<td>5.9590</td>
<td>5.5423</td>
</tr>
</tbody>
</table>

**Table**: First order asymptotic prices.
Normalized errors

- Approximation error supposed to depend on “dimension-free”
time to maturity $\sigma^2 T$
- Use $\bar{\sigma} := \sigma_{N, B}(F_0)/\left(\sum_{i=1}^n w_i F_{0,i}\right)$ as proxy in local vol framework
- Normalized error: $\frac{\text{Rel. error}}{\sigma^2 T}$

<table>
<thead>
<tr>
<th>$T$</th>
<th>Dim. 5</th>
<th>Dim. 10</th>
<th>Dim. 15</th>
<th>Dim. 100</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.3085</td>
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<tr>
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<td>0.1481</td>
<td>-0.0261</td>
<td>0.3162</td>
<td>-0.0105</td>
</tr>
<tr>
<td>2</td>
<td>0.1429</td>
<td>-0.0218</td>
<td>0.3222</td>
<td>-0.0075</td>
</tr>
<tr>
<td>5</td>
<td>0.1376</td>
<td>-0.0129</td>
<td>0.3252</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.1328</td>
<td>-0.0035</td>
<td>0.3198</td>
<td></td>
</tr>
<tr>
<td>$\bar{\sigma}$</td>
<td>0.1704</td>
<td>0.3187</td>
<td>0.1073</td>
<td>0.2964</td>
</tr>
</tbody>
</table>

Table: Normalized relative error of the zero-order asymptotic prices.
Approximation error supposed to depend on “dimension-free”
time to maturity $\sigma^2 T$

- Use $\overline{\sigma} := \sigma_{N,B}(F_0)/\left(\sum_{i=1}^{n} w_i F_{0,i}\right)$ as proxy in local vol framework

- Normalized error: $\frac{\text{Rel. error}}{\overline{\sigma}^2 T}$

<table>
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<tr>
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<th>Dim. 10</th>
<th>Dim. 15</th>
<th>Dim. 100</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>$-4.02 \times 10^{-4}$</td>
<td>$1.76 \times 10^{-4}$</td>
<td>$8.76 \times 10^{-3}$</td>
<td>$5.06 \times 10^{-5}$</td>
</tr>
<tr>
<td>1</td>
<td>$-9.47 \times 10^{-4}$</td>
<td>$3.58 \times 10^{-3}$</td>
<td>$1.53 \times 10^{-3}$</td>
<td>$2.08 \times 10^{-3}$</td>
</tr>
<tr>
<td>2</td>
<td>$-1.63 \times 10^{-3}$</td>
<td>$8.09 \times 10^{-3}$</td>
<td>$-3.92 \times 10^{-3}$</td>
<td>$3.89 \times 10^{-3}$</td>
</tr>
<tr>
<td>5</td>
<td>$-3.41 \times 10^{-3}$</td>
<td>$1.71 \times 10^{-2}$</td>
<td>$-1.33 \times 10^{-2}$</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>$-7.15 \times 10^{-3}$</td>
<td>$2.67 \times 10^{-2}$</td>
<td>$-2.82 \times 10^{-2}$</td>
<td></td>
</tr>
</tbody>
</table>

$\overline{\sigma}$: $0.1704$ $0.3187$ $0.1073$ $0.2964$

**Table**: Normalized error of the first order asymptotic prices.
First order prices

Approximations for local vol baskets · November 28, 2013 · Page 27 (32)
Relative errors

Approximations for local vol baskets · November 28, 2013 · Page 28 (32)
Objective: Compute the sensitivity (delta) w.r.t. $F_{0,3}$. 

Note that the option payoff is 

$$P(F) = (F_1 + F_2 - F_3 - K)^+$$
Relative error of delta

T = 0.5

T = 5

Approximations for local vol baskets · November 28, 2013 · Page 31 (32)
M. Avellaneda, D. Boyer-Olson, J. Busca, P. Friz: *Application of large

R. Azencott: *Densité des diffusions en temps petit: développements

C. Bayer, P. Laurence: *Asymptotics beats Monte Carlo: The case of

J. Gatheral, E. P. Hsu, P. Laurence, C. Ouyang, T.-H. Wang: *Asymptotics

P. Henry-Labordère: *Analysis, geometry, and modeling in finance*, CRC

R. S. Varadhan: *Diffusion processes in a small time interval*, Comm. Pure

K. Yosida: *On the fundamental solution of the parabolic equation in a