Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Pricing under rough volatility

Christian Bayer
Weierstrass Institute Berlin
Joint work with Peter Friz and Jim Gatheral.

Mohrenstrasse 39 · 10117 Berlin · Germany · Tel. +49 30 20372 0 · www.wias-berlin.de
November 3, 2015

- Given a traded asset S_t satisfying

$$dS_t = \sqrt{v_t} S_t dZ_t$$

- Interest rate $r = 0$; model formulated under Q
- In this talk, S corresponds to the S & P 500 index (SPX).
- *Realized variance* $w_{t,T} = \int_t^T v_s ds$
- Variance swaps are swaps on realized variances.
- Allow direct trades in volatility, not indirect via options
- For convenience, CBOE introduced an index (VIX) for the square root of (annualized) one month variance swaps.
- $\mathrm{VIX}_t \approx \sqrt{\frac{1}{\Delta} E_t w_{t,t+\Delta}}$, $\Delta = 1/12$

- Given a traded asset S_t satisfying

$$dS_t = \sqrt{v_t} S_t dZ_t$$

- Interest rate $r = 0$; model formulated under Q
- In this talk, S corresponds to the S & P 500 index (SPX).
- *Realized variance* $w_{t,T} = \int_t^T v_s ds$
- Variance swaps are swaps on realized variances.
- Allow direct trades in volatility, not indirect via options
- For convenience, CBOE introduced an index (VIX) for the square root of (annualized) one month variance swaps.
- $\mathrm{VIX}_t \approx \sqrt{\frac{1}{\Delta} E_t w_{t,t+\Delta}}$, $\Delta = 1/12$

- Given a traded asset S_t satisfying

$$dS_t = \sqrt{v_t} S_t dZ_t$$

- Interest rate $r = 0$; model formulated under Q
- In this talk, S corresponds to the S & P 500 index (SPX).
- *Realized variance* $w_{t,T} = \int_t^T v_s ds$
- Variance swaps are swaps on realized variances.
- Allow direct trades in volatility, not indirect via options
- For convenience, CBOE introduced an index (VIX) for the square root of (annualized) one month variance swaps.
- $\mathrm{VIX}_t \approx \sqrt{\frac{1}{\Delta} E_t w_{t,t+\Delta}}, \Delta = 1/12$

The log-strip

- Ito's formula gives for the payoff $\log(S_T)$

$$\log S_T - \log S_t = \int_t^T \frac{dS_u}{S_u} - \frac{1}{2}\int_t^T v_u du$$

- *Breeden-Litzenberger formula:* $p(S_T, T, S_t, t) = \left.\frac{\partial^2 C/P(S_t,K,t,T)}{\partial K^2}\right|_{K=S_t}$
- p . . . density, C, P call and put prices
- Integration by parts, put-call-parity give for smooth payoff g

$$E[g(S_T)|S_t] = g(S_t) + \int_0^{S_t} P(K)g''(K)dK + \int_{S_t}^{\infty} C(K)g''(K)dK$$

- For $g(S) = -2\log S$, we have $g''(K) = \frac{2}{K^2}$ and

$$E_t w_{t,T} = -2\left(\int_0^{S_t} \frac{P(K)}{K^2}dK + \int_{S_t}^{\infty} \frac{C(K)}{K^2}dK\right)$$

The log-strip

- Ito's formula gives for the payoff $\log(S_T)$

$$E_t w_{t,T} = E_t \int_t^T v_u du = 2\log S_t - 2E_t \log S_T$$

- *Breeden-Litzenberger formula:* $p(S_T, T, S_t, t) = \left.\frac{\partial^2 C/P(S_t,K,t,T)}{\partial K^2}\right|_{K=S_t}$
- p ... density, C, P call and put prices
- Integration by parts, put-call-parity give for smooth payoff g

$$E[g(S_T)|S_t] = g(S_t) + \int_0^{S_t} P(K)g''(K)dK + \int_{S_t}^{\infty} C(K)g''(K)dK$$

- For $g(S) = -2\log S$, we have $g''(K) = \frac{2}{K^2}$ and

$$E_t w_{t,T} = -2\left(\int_0^{S_t} \frac{P(K)}{K^2}dK + \int_{S_t}^{\infty} \frac{C(K)}{K^2}dK\right)$$

The log-strip

- Ito's formula gives for the payoff $\log(S_T)$

$$E_t w_{t,T} = E_t \int_t^T v_u du = 2\log S_t - 2E_t \log S_T$$

- *Breeden-Litzenberger formula:* $p(S_T, T, S_t, t) = \frac{\partial^2 C/P(S_t,K,t,T)}{\partial K^2}\Big|_{K=S_t}$
- p . . . density, C, P call and put prices
- Integration by parts, put-call-parity give for smooth payoff g

$$E[g(S_T)|S_t] = g(S_t) + \int_0^{S_t} P(K)g''(K)dK + \int_{S_t}^{\infty} C(K)g''(K)dK$$

- For $g(S) = -2\log S$, we have $g''(K) = \frac{2}{K^2}$ and

$$E_t w_{t,T} = -2\left(\int_0^{S_t} \frac{P(K)}{K^2}dK + \int_{S_t}^{\infty} \frac{C(K)}{K^2}dK\right)$$

$$dS_t = \sqrt{v_t} S_t dZ_t,$$
$$dv_t = \ldots$$

- Z, W Brownian motions with correlation ρ
- Goal: model consistent with the full SPX implied volatility surface
- $\mathrm{VIX}_t \approx \sqrt{v_t}$ (with $\Delta \approx 0$)
- VIX itself is not traded, but the following are:
  - VIX futures (rate given by $E_t \, \mathrm{VIX}_T$; traded on CBOE)
  - VIX options (i.e., options on VIX futures; traded on CBOE)
  - Variance swaps (swap rate $E_t w_{t,T}$; traded over the counter)
- Fundamental object: forward variance $\xi_t(u) = E_t v_u$, $t \le u$
- Variance swap $E_t w_{t,T} = E_t \int_t^T v_s ds = \int_t^T \xi_t(s) ds$

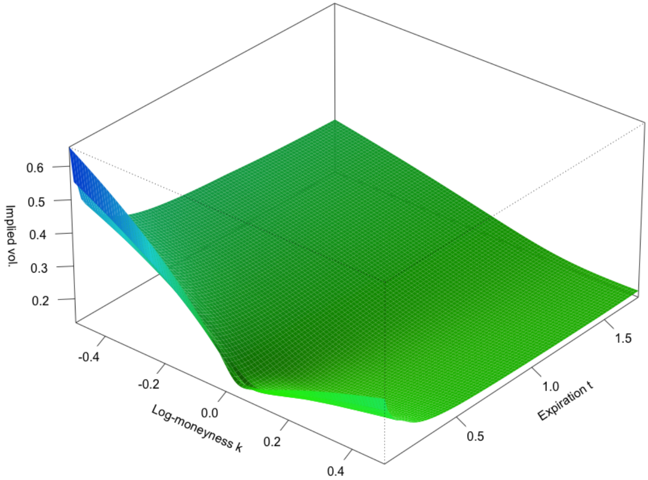
$$dS_t = \sqrt{v_t} S_t dZ_t,$$
$$dv_t = \dots$$

- Z, W Brownian motions with correlation ρ
- Goal: model consistent with the full SPX implied volatility surface
- $\mathrm{VIX}_t \approx \sqrt{v_t}$ (with $\Delta \approx 0$)
- VIX itself is not traded, but the following are:
  - VIX futures (rate given by $E_t \, \mathrm{VIX}_T$; traded on CBOE)
  - VIX options (i.e., options on VIX futures; traded on CBOE)
  - Variance swaps (swap rate $E_t w_{t,T}$; traded over the counter)
- Fundamental object: forward variance $\xi_t(u) = E_t v_u$, $t \le u$
- Variance swap $E_t w_{t,T} = E_t \int_t^T v_s ds = \int_t^T \xi_t(s) ds$

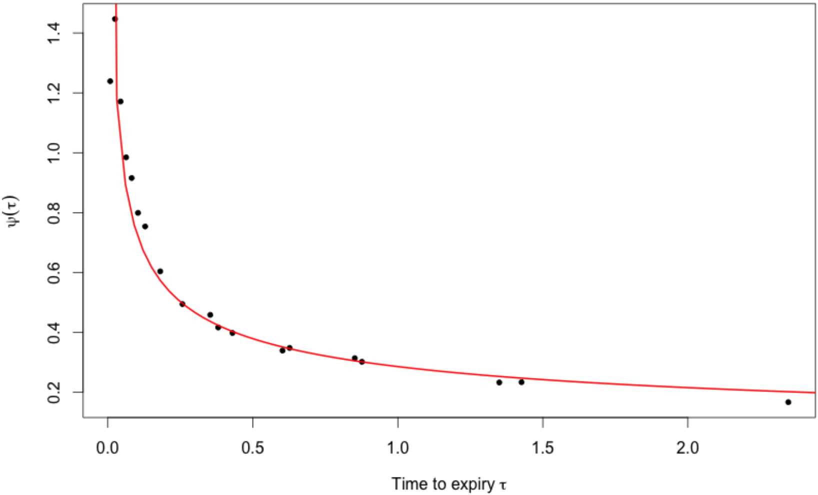
$$dS_t = \sqrt{v_t} S_t dZ_t,$$
$$dv_t = \ldots$$

- Z, W Brownian motions with correlation ρ
- Goal: model consistent with the full SPX implied volatility surface
- $\mathrm{VIX}_t \approx \sqrt{v_t}$ (with $\Delta \approx 0$)
- VIX itself is not traded, but the following are:
  - VIX futures (rate given by $E_t \, \mathrm{VIX}_T$; traded on CBOE)
  - VIX options (i.e., options on VIX futures; traded on CBOE)
  - Variance swaps (swap rate $E_t w_{t,T}$; traded over the counter)
- Fundamental object: forward variance $\xi_t(u) = E_t v_u$, $t \le u$
- Variance swap $E_t w_{t,T} = E_t \int_t^T v_s ds = \int_t^T \xi_t(s) ds$

Some SPX implied volatility surfaces



SPX ATM volatility skew



- Since the rough shape of volatility surfaces seems pretty stable, we look for time-homogeneous models.
- Term structure of ATM volatility skew ($k = \log(K/S_t)$)

$$\psi(\tau) = \left| \frac{\partial}{\partial k} \sigma_{BS}(k, \tau) \right|_{k=0} \sim 1/\tau^{\alpha}, \quad \alpha \in [0.3, 0.5]$$

- Conventional stochastic volatility models produce ATM skews which are constant for $\tau \ll 1$ and of order $1/\tau$ for $\tau \gg 1$. Hence, conventional stochastic volatility models cannot fit the full volatility surface.
- Do we need jumps?
- Stochastic variance has log-normal distribution (under P).

Outline

- Recall $\xi_t(u) = E_t v_u$

$$dS_t = \sqrt{\xi_t(t)} S_t dZ_t,$$
$$\xi_t(u) = \xi_0(u)\mathcal{E}\left(\sum_{i=1}^{n} \eta_i \int_0^t e^{-\kappa_i(u-s)} dW_s^i\right)$$

- $\mathcal{E}(X) = \exp(X - \frac{1}{2}E[|X|^2])$ for Gaussian r.v. X
- Market model
- In practice, $n = 2$ needed for good fit, contains seven parameters
- $\psi(\tau) \sim \sum_{i=1}^{n} \frac{\eta_i}{\kappa_i \tau}\left(1 - \frac{1 - e^{-\kappa_i \tau}}{\kappa_i \tau}\right)$
- Tempting to replace the exponential kernel by a power law kernel!

- Recall $\xi_t(u) = E_t v_u$

$$dS_t = \sqrt{\xi_t(t)} S_t dZ_t,$$
$$\xi_t(u) = \xi_0(u)\mathcal{E}\left(\sum_{i=1}^{n} \eta_i \int_0^t e^{-\kappa_i(u-s)} dW_s^i\right)$$

- $\mathcal{E}(X) = \exp(X - \frac{1}{2}E[|X|^2])$ for Gaussian r.v. X
- Market model
- In practice, $n = 2$ needed for good fit, contains seven parameters
- $\psi(\tau) \sim \sum_{i=1}^{n} \frac{\eta_i}{\kappa_i \tau}\left(1 - \frac{1 - e^{-\kappa_i \tau}}{\kappa_i \tau}\right)$
- Tempting to replace the exponential kernel by a power law kernel!

- Recall $\xi_t(u) = E_t v_u$

$$dS_t = \sqrt{\xi_t(t)} S_t dZ_t,$$
$$\xi_t(u) = \xi_0(u) \mathcal{E}\left(\sum_{i=1}^{n} \eta_i \int_0^t e^{-\kappa_i (u-s)} dW_s^i \right)$$

- $\mathcal{E}(X) = \exp(X - \frac{1}{2} E[|X|^2])$ for Gaussian r.v. X
- Market model
- In practice, $n = 2$ needed for good fit, contains seven parameters
- $\psi(\tau) \sim \sum_{i=1}^{n} \frac{\eta_i}{\kappa_i \tau} \left(1 - \frac{1 - e^{-\kappa_i \tau}}{\kappa_i \tau} \right)$
- Tempting to replace the exponential kernel by a power law kernel!

- Gatheral, Jaisson, and Rosenbaum (2014) study time series of realized variance and find amazing fits of a stochastic volatility model based on

$$\log v_u - \log v_t = 2\nu \left(W_u^H - W_t^H \right)$$

- Mandelbrot – Van Ness representation of fBm (with $\gamma = 1/2 - H$)
- v_u is not a Markov process (neither under P or Q).
- With $\widetilde{W}_t^P(u) = \sqrt{2H} \int_t^u \frac{dW_s^P}{(u-s)^\gamma}$, we get

- Gatheral, Jaisson, and Rosenbaum (2014) study time series of realized variance and find amazing fits of a stochastic volatility model based on

$$\log v_u - \log v_t = 2\nu \left(W_u^H - W_t^H \right)$$

- Mandelbrot – Van Ness representation of fBm (with $\gamma = 1/2 - H$)

$$W_t^H = C_H \left(\int_0^t \frac{dW_s^P}{(t-s)^\gamma} + \int_{-\infty}^0 \left[\frac{1}{(t-s)^\gamma} - \frac{1}{(-s)^\gamma} \right] dW_s^P \right)$$

- v_u is not a Markov process (neither under P or Q).
- With $\widetilde{W}_t^P(u) = \sqrt{2H} \int_t^u \frac{dW_s^P}{(u-s)^\gamma}$, we get

- Gatheral, Jaisson, and Rosenbaum (2014) study time series of realized variance and find amazing fits of a stochastic volatility model based on

$$\log v_u - \log v_t = 2\nu\left(W_u^H - W_t^H\right)$$

- Mandelbrot – Van Ness representation of fBm (with $\gamma = 1/2 - H$)

$$\log v_u - \log v_t = 2\nu C_H\left(\int_t^u \frac{dW_s^P}{(u-s)^\gamma} + \int_{-\infty}^t \left[\frac{1}{(u-s)^\gamma} - \frac{1}{(t-s)^\gamma}\right] dW_s^P\right)$$

- v_u is not a Markov process (neither under P or Q).
- With $\widetilde{W}_t^P(u) = \sqrt{2H}\int_t^u \frac{dW_s^P}{(u-s)^\gamma}$, we get

- Gatheral, Jaisson, and Rosenbaum (2014) study time series of realized variance and find amazing fits of a stochastic volatility model based on

$$\log v_u - \log v_t = 2\nu\left(W_u^H - W_t^H\right)$$

- Mandelbrot – Van Ness representation of fBm (with $\gamma = 1/2 - H$)

$$\log v_u - \log v_t = 2\nu C_H\left(\int_t^u \frac{dW_s^P}{(u-s)^\gamma} + \int_{-\infty}^t \left[\frac{1}{(u-s)^\gamma} - \frac{1}{(t-s)^\gamma}\right] dW_s^P\right)$$

- v_u is not a Markov process (neither under P or Q).
- With $\widetilde{W}_t^P(u) = \sqrt{2H}\int_t^u \frac{dW_s^P}{(u-s)^\gamma}$, we get

$$v_u = E^P[v_u|\mathcal{F}_t]\mathcal{E}\left(\eta\widetilde{W}_t^P(u)\right)$$

- Gatheral, Jaisson, and Rosenbaum (2014) study time series of realized variance and find amazing fits of a stochastic volatility model based on

$$\log v_u - \log v_t = 2\nu \left(W_u^H - W_t^H \right)$$

- Mandelbrot – Van Ness representation of fBm (with $\gamma = 1/2 - H$)

$$\log v_u - \log v_t = 2\nu C_H \left(\int_t^u \frac{dW_s^P}{(u-s)^\gamma} + \int_{-\infty}^t \left[\frac{1}{(u-s)^\gamma} - \frac{1}{(t-s)^\gamma} \right] dW_s^P \right)$$

- v_u is not a Markov process (neither under P or Q).
- With $\widetilde{W}_t^P(u) = \sqrt{2H} \int_t^u \frac{dW_s^P}{(u-s)^\gamma}$, we get

$$v_u = E^Q[v_u | \mathcal{F}_t] \mathcal{E} \left(\eta \widetilde{W}_t^Q(u) \right)$$

- Gatheral, Jaisson, and Rosenbaum (2014) study time series of realized variance and find amazing fits of a stochastic volatility model based on

$$\log v_u - \log v_t = 2\nu \left(W_u^H - W_t^H \right)$$

- Mandelbrot – Van Ness representation of fBm (with $\gamma = 1/2 - H$)

$$\log v_u - \log v_t = 2\nu C_H \left(\int_t^u \frac{dW_s^P}{(u-s)^\gamma} + \int_{-\infty}^t \left[\frac{1}{(u-s)^\gamma} - \frac{1}{(t-s)^\gamma} \right] dW_s^P \right)$$

- v_u is not a Markov process (neither under P or Q).
- With $\widetilde{W}_t^P(u) = \sqrt{2H} \int_t^u \frac{dW_s^P}{(u-s)^\gamma}$, we get

$$v_u = \xi_t(u) \mathcal{E}\left(\eta \widetilde{W}_t^Q(u) \right)$$

$$dS_t = \sqrt{v_t} S_t dZ_t$$
$$v_t = \xi_0(t) \mathcal{E}\left(\eta \widetilde{W}_t\right)$$

- $dW_t dZ_t = \rho dt, \widetilde{W}_t = \sqrt{2H} \int_0^t \frac{dW_s}{(t-s)^\gamma}, \gamma = 1/2 - H$
- $\widetilde{W}$ is a "Volterra" process (or "Riemann-Liouville fBm")
- Covariance:

$$E\left[\widetilde{W}_v \widetilde{W}_u\right] = \frac{2H}{1/2 + H} \frac{u^{1/2+H}}{v^{1/2-H}} {}_2F_1\left(1, 1/2 - H, 3/2 + H, u/v\right), \ u \leq v,$$
$$E\left[\widetilde{W}_v Z_u\right] = \rho \frac{\sqrt{2H}}{1/2 + H} \left(v^{1/2+H} - [v - \min(u, v)]^{1/2+H}\right)$$

- $\psi(\tau) \sim 1/\tau^\gamma$
- Typical parameter values: $H \approx 0.05$, $\eta \approx 2.5$

The Rough Bergomi model (under Q)

$$
\begin{aligned}
dS_t &= \sqrt{v_t} S_t dZ_t \\
v_t &= \xi_0(t) \mathcal{E}\left(\eta \widetilde{W}_t\right)
\end{aligned}
$$

- $dW_t dZ_t = \rho dt, \widetilde{W}_t = \sqrt{2H} \int_0^t \frac{dW_s}{(t-s)^\gamma}, \gamma = 1/2 - H$
- $\widetilde{W}$ is a “Volterra” process (or “Riemann-Liouville fBm”)
- Covariance:

$$
\begin{aligned}
E\left[\widetilde{W}_v \widetilde{W}_u\right] &= \frac{2H}{1/2+H} \frac{u^{1/2+H}}{v^{1/2-H}} {}_2F_1\left(1, 1/2-H, 3/2+H, u/v\right), \; u \leq v, \\
E\left[\widetilde{W}_v Z_u\right] &= \rho \frac{\sqrt{2H}}{1/2+H} \left(v^{1/2+H} - [v - \min(u,v)]^{1/2+H}\right)
\end{aligned}
$$

- $\psi(\tau) \sim 1/\tau^\gamma$
- Typical parameter values: $H \approx 0.05$, $\eta \approx 2.5$

$$dS_t = \sqrt{v_t} S_t dZ_t$$
$$v_t = \xi_0(t) \mathcal{E}\left(\eta \widetilde{W}_t\right)$$

- $dW_t dZ_t = \rho dt, \widetilde{W}_t = \sqrt{2H} \int_0^t \frac{dW_s}{(t-s)^\gamma}, \gamma = 1/2 - H$
- $\widetilde{W}$ is a “Volterra” process (or “Riemann-Liouville fBm”)
- Covariance:

$$E\left[\widetilde{W}_v \widetilde{W}_u\right] = \frac{2H}{1/2 + H} \frac{u^{1/2+H}}{v^{1/2-H}} {}_2F_1\left(1, 1/2 - H, 3/2 + H, u/v\right), \; u \leq v,$$
$$E\left[\widetilde{W}_v Z_u\right] = \rho \frac{\sqrt{2H}}{1/2 + H} \left(v^{1/2+H} - [v - \min(u, v)]^{1/2+H}\right)$$

- $\psi(\tau) \sim 1/\tau^\gamma$
- Typical parameter values: $H \approx 0.05$, $\eta \approx 2.5$

$$dS_t = \sqrt{v_t} S_t dZ_t$$
$$v_t = \xi_0(t) \mathcal{E}\left(\eta \widetilde{W}_t\right)$$

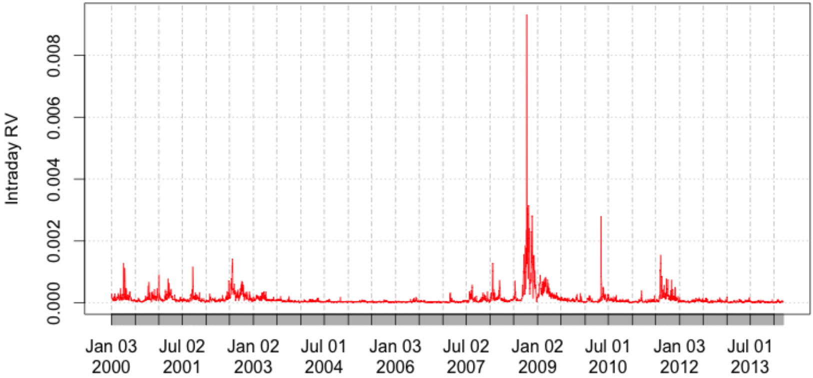
- $dW_t dZ_t = \rho dt, \widetilde{W}_t = \sqrt{2H} \int_0^t \frac{dW_s}{(t-s)^\gamma}, \gamma = 1/2 - H$
- $\widetilde{W}$ is a “Volterra” process (or “Riemann-Liouville fBm”)
- Covariance:

$$E\left[\widetilde{W}_v \widetilde{W}_u\right] = \frac{2H}{1/2+H} \frac{u^{1/2+H}}{v^{1/2-H}} {}_2F_1\left(1, 1/2-H, 3/2+H, u/v\right), \; u \leq v,$$
$$E\left[\widetilde{W}_v Z_u\right] = \rho \frac{\sqrt{2H}}{1/2+H} \left(v^{1/2+H} - [v - \min(u,v)]^{1/2+H}\right)$$

- $\psi(\tau) \sim 1/\tau^\gamma$
- Typical parameter values: $H \approx 0.05$, $\eta \approx 2.5$

Outline

KRV estimates of SPX realized variance from 2000 to 2014



- The Oxford Man Institute provides estimated realized variances v_t for numerous indices on a daily bases.
- Let $\sigma_t = \sqrt{v_t}$.
- For some lag $\Delta > 0$ fix a corresponding time-grid t_i (with $t_{i+1} - t_i = \Delta$) and define the moment of the log-differences by

$$m(q, \Delta) := \left\langle \left| \log \sigma_{t+\Delta} - \log \sigma_t \right|^q \right\rangle$$

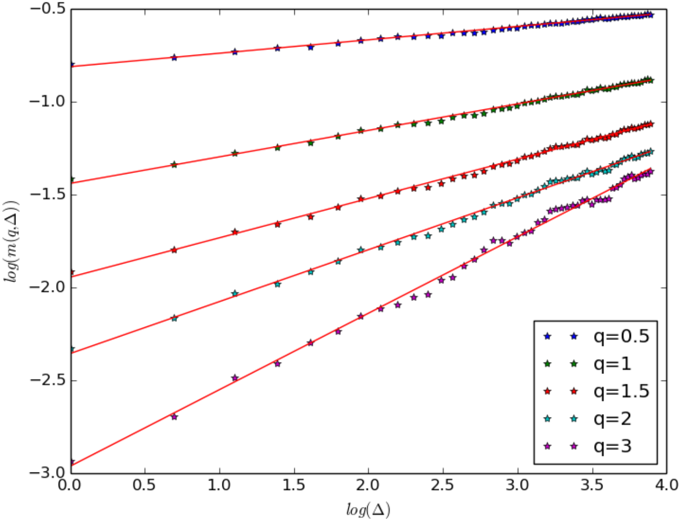
- $\langle \cdot \rangle$ denotes taking sample average.
- $m(q, \Delta)$ measures smoothness of realized volatility at various lags.

- The Oxford Man Institute provides estimated realized variances v_t for numerous indices on a daily bases.
- Let $\sigma_t = \sqrt{v_t}$.
- For some lag $\Delta > 0$ fix a corresponding time-grid t_i (with $t_{i+1} - t_i = \Delta$) and define the moment of the log-differences by

$$m(q, \Delta) := \left\langle \left| \log \sigma_{t+\Delta} - \log \sigma_t \right|^q \right\rangle$$

- $\langle \cdot \rangle$ denotes taking sample average.
- $m(q, \Delta)$ measures smoothness of realized volatility at various lags.

Scaling of m in Δ



- We see fractal behaviour: for each moment order q there is a coefficient ζ_q such that

$$m(q, \Delta) \sim \Delta^{\zeta_q}$$

- Different q show the same fractal behaviour in the sense that for some $H \approx 0.1$, $\zeta_q \approx qH$.
- Log-volatility is also approximately normal.
- These observations hold for all 21 indices in the Oxford Man database.

Log-volatility seems to be described by a fractional Brownian motion with Hurst index $H \approx 0.1$. This suggests models of the form

$$dS_t = S_t \exp\left(\eta W_t^H\right) dZ_t + \cdots$$

for $0 < H < 1/2$.

- We see fractal behaviour: for each moment order q there is a coefficient ζ_q such that

$$m(q, \Delta) \sim \Delta^{\zeta_q}$$

- Different q show the same fractal behaviour in the sense that for some $H \approx 0.1$, $\zeta_q \approx qH$.
- Log-volatility is also approximately normal.
- These observations hold for all 21 indices in the Oxford Man database.

Log-volatility seems to be described by a fractional Brownian motion with Hurst index $H \approx 0.1$. This suggests models of the form

$$dS_t = S_t \exp\left(\eta W_t^H\right) dZ_t + \cdots$$

for $0 < H < 1/2$.

- Several fractional stochastic volatility models have been proposed, inevitably with $H > 1/2$.
- Fractional Brownian motion with $H > 1/2$ has long memory, i.e., the auto-correlation function $\rho(\Delta)$ (at lag Δ) has power law decay as $\Delta \to \infty$.
- It was an accepted stylized fact that volatility has long memory.
- In our rough model:

$$\rho(\Delta) \sim \exp\left(-\frac{\eta^2}{2}\Delta^{2H}\right)$$

- Hence, no long term memory!
- Estimates and comparisons by Gatheral, Jaisson, Rosenbaum suggest that there really is no long term memory in volatility.
- Might be an effect of new, better (high-frequency) data.

- Several fractional stochastic volatility models have been proposed, inevitably with $H > 1/2$.
- Fractional Brownian motion with $H > 1/2$ has long memory, i.e., the auto-correlation function $\rho(\Delta)$ (at lag Δ) has power law decay as $\Delta \to \infty$.
- It was an accepted stylized fact that volatility has long memory.
- In our rough model:

$$\rho(\Delta) \sim \exp\left(-\frac{\eta^2}{2}\Delta^{2H}\right)$$

- Hence, no long term memory!
- Estimates and comparisons by Gatheral, Jaisson, Rosenbaum suggest that there really is no long term memory in volatility.
- Might be an effect of new, better (high-frequency) data.

- Several fractional stochastic volatility models have been proposed, inevitably with $H > 1/2$.
- Fractional Brownian motion with $H > 1/2$ has long memory, i.e., the auto-correlation function $\rho(\Delta)$ (at lag Δ) has power law decay as $\Delta \to \infty$.
- It was an accepted stylized fact that volatility has long memory.
- In our rough model:

$$\rho(\Delta) \sim \exp\left(-\frac{\eta^2}{2}\Delta^{2H}\right)$$

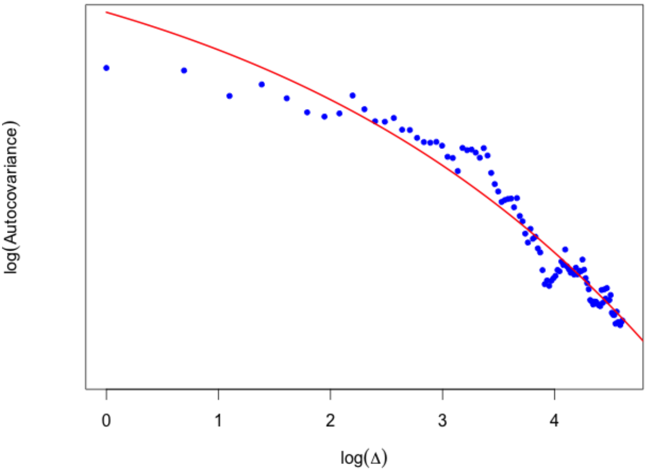
- Hence, no long term memory!
- Estimates and comparisons by Gatheral, Jaisson, Rosenbaum suggest that there really is no long term memory in volatility.
- Might be an effect of new, better (high-frequency) data.

- Several fractional stochastic volatility models have been proposed, inevitably with $H > 1/2$.
- Fractional Brownian motion with $H > 1/2$ has long memory, i.e., the auto-correlation function $\rho(\Delta)$ (at lag Δ) has power law decay as $\Delta \to \infty$.
- It was an accepted stylized fact that volatility has long memory.
- In our rough model:

$$\rho(\Delta) \sim \exp\left(-\frac{\eta^2}{2}\Delta^{2H}\right)$$

- Hence, no long term memory!
- Estimates and comparisons by Gatheral, Jaisson, Rosenbaum suggest that there really is no long term memory in volatility.
- Might be an effect of new, better (high-frequency) data.

Empirical auto-correlation against exponential decay



Fractional stochastic volatility model:

$$dS_t = \sigma_t S_t dZ_t,$$
$$d \log \sigma_t = -\alpha (\log \sigma_t - \theta) dt + \gamma d\hat{W}_t^H$$

with $\hat{W}_t^H = \int_0^t \frac{(t-s)^{H-1/2}}{\Gamma(H+1/2)} dW_s$, $\langle Z, W \rangle_t = \rho t$, $1/2 \leq H < 1$.

- Related to Hull-White stochastic volatility model
- FSV model equivalent to RFSV model of Gatheral, Jaisson, Rosenbaum (up to choice of H)
- rBergomi: replace fOU-process by fBm
- Get long memory for $1/2 < H < 1$.

Fractional stochastic volatility model:

$$dS_t = \sigma_t S_t dZ_t,$$
$$d\log\sigma_t = -\alpha(\log\sigma_t - \theta)\,dt + \gamma d\hat{W}_t^H$$

with $\hat{W}_t^H = \int_0^t \frac{(t-s)^{H-1/2}}{\Gamma(H+1/2)} dW_s$, $\langle Z, W\rangle_t = \rho t$, $1/2 \le H < 1$.

- Related to Hull-White stochastic volatility model
- FSV model equivalent to RFSV model of Gatheral, Jaisson, Rosenbaum (up to choice of H)
- rBergomi: replace fOU-process by fBm
- Get long memory for $1/2 < H < 1$.

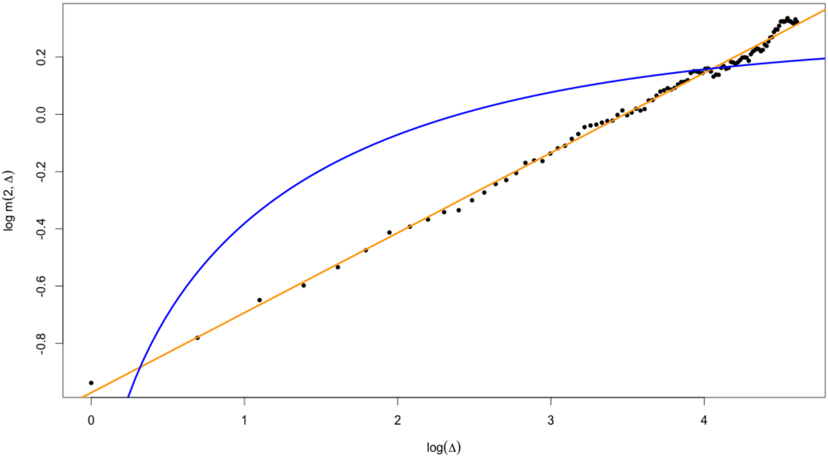
Fractional stochastic volatility model:

$$dS_t = \sigma_t S_t dZ_t,$$
$$d\log\sigma_t = -\alpha\left(\log\sigma_t - \theta\right)dt + \gamma d\hat{W}_t^H$$

with $\hat{W}_t^H = \int_0^t \frac{(t-s)^{H-1/2}}{\Gamma(H+1/2)} dW_s$, $\langle Z, W\rangle_t = \rho t$, $1/2 \le H < 1$.

- Related to Hull-White stochastic volatility model
- FSV model equivalent to RFSV model of Gatheral, Jaisson, Rosenbaum (up to choice of H)
- rBergomi: replace fOU-process by fBm
- Get long memory for $1/2 < H < 1$.

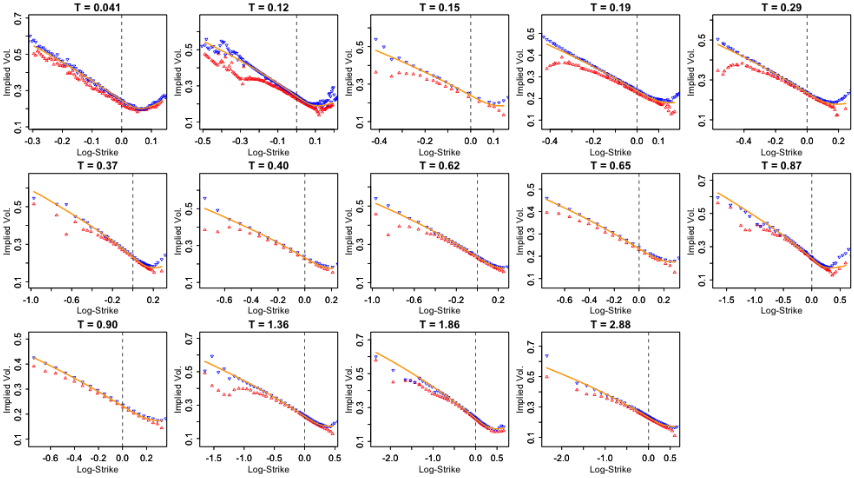
Moment comparison for realized variance



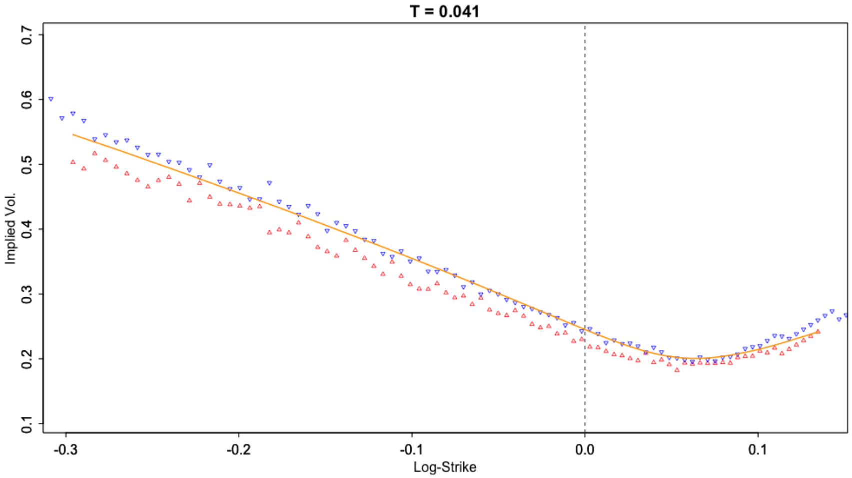
Blue: FSV model with $H = 0.53$, orange: rBergomi, $H = 0.15$

Outline

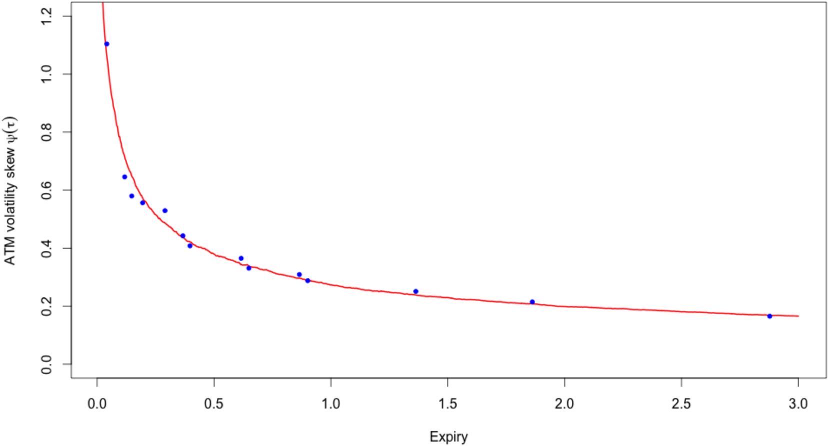
02/04/2010; SPX Vol surface for $H = 0.07$, $\eta = 1.9$, $\rho = -0.9$



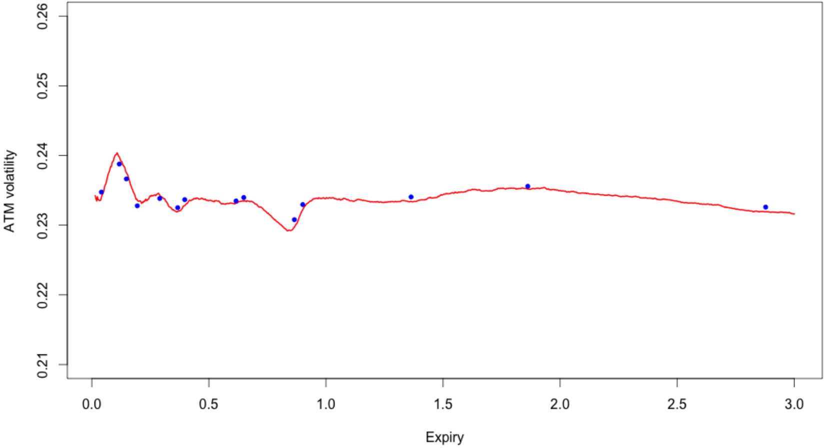
02/04/2010; SPX short maturity smile for $H = 0.07$, $\eta = 1.9$, $\rho = -0.9$



02/04/2010; SPX volatility skew for $H = 0.07$, $\eta = 1.9$, $\rho = -0.9$



02/04/2010; SPX ATM volatility for $H = 0.07$, $\eta = 1.9$, $\rho = -0.9$



- Variance v is not a martingale, hence non-trivial forecast.
- Formulate in RFSV model.

$$E^P\left[\log v_{t+\Delta}|\mathcal{F}_t\right] = \frac{\cos(H\pi)}{\pi}\Delta^{H+1/2}\int_{-\infty}^{t}\frac{\log v_s}{(t-s+\Delta)(t-s)^{H+1/2}}ds,$$

$$E^P\left[v_{t+\Delta}|\mathcal{F}_t\right] = \exp\left(E^P\left[\log v_{t+\Delta}|\mathcal{F}_t\right] + 2c\nu^2\Delta^{2H}\right).$$

- Use realized variance as proxy for v
- Problem: realized variance only available from opening to close, not from close to close
- Forecasts must be re-scaled by (time-varying) factor; hence should predict variance swap curve up to a factor

- Variance v is not a martingale, hence non-trivial forecast.
- Formulate in RFSV model.

$$E^P\left[\log v_{t+\Delta} | \mathcal{F}_t\right] = \frac{\cos(H\pi)}{\pi} \Delta^{H+1/2} \int_{-\infty}^{t} \frac{\log v_s}{(t-s+\Delta)(t-s)^{H+1/2}} ds,$$

$$E^P\left[v_{t+\Delta} | \mathcal{F}_t\right] = \exp\left(E^P\left[\log v_{t+\Delta} | \mathcal{F}_t\right] + 2c\nu^2 \Delta^{2H}\right).$$

- Use realized variance as proxy for v
- Problem: realized variance only available from opening to close, not from close to close
- Forecasts must be re-scaled by (time-varying) factor; hence should predict variance swap curve up to a factor

- Variance v is not a martingale, hence non-trivial forecast.
- Formulate in RFSV model.

$$E^P\left[\log v_{t+\Delta}|\mathcal{F}_t\right] = \frac{\cos(H\pi)}{\pi}\Delta^{H+1/2}\int_{-\infty}^{t}\frac{\log v_s}{(t-s+\Delta)(t-s)^{H+1/2}}ds,$$
$$E^P\left[v_{t+\Delta}|\mathcal{F}_t\right] = \exp\left(E^P\left[\log v_{t+\Delta}|\mathcal{F}_t\right] + 2c\nu^2\Delta^{2H}\right).$$

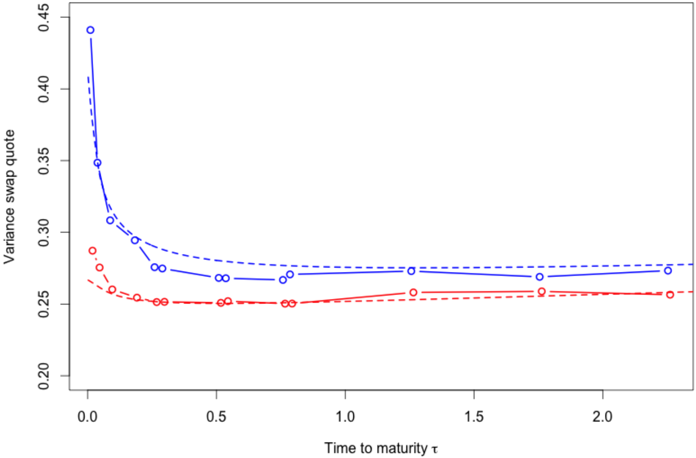
- Use realized variance as proxy for v
- Problem: realized variance only available from opening to close, not from close to close
- Forecasts must be re-scaled by (time-varying) factor; hence should predict variance swap curve up to a factor

- Variance v is not a martingale, hence non-trivial forecast.
- Formulate in RFSV model.

$$E^P\left[\log v_{t+\Delta}|\mathcal{F}_t\right] = \frac{\cos(H\pi)}{\pi}\Delta^{H+1/2}\int_{-\infty}^{t}\frac{\log v_s}{(t-s+\Delta)(t-s)^{H+1/2}}ds,$$
$$E^P\left[v_{t+\Delta}|\mathcal{F}_t\right] = \exp\left(E^P\left[\log v_{t+\Delta}|\mathcal{F}_t\right] + 2c\nu^2\Delta^{2H}\right).$$

- Use realized variance as proxy for v
- Problem: realized variance only available from opening to close, not from close to close
- Forecasts must be re-scaled by (time-varying) factor; hence should predict variance swap curve up to a factor

Forecasts for the Lehman weekend



Actual and predicted variance swap curves, 09/12/08 (red) and 09/15/08 (blue), after scaling.

Outline

- Bergomi and Guyon (2012) give a general expansion of implied volatility in terms of vol-of-vol and maturity for Bergomi-like stochastic volatility models.
- Expansion is based on auto-covariance $C = E\left[\langle \log S_\cdot, \xi_\cdot(u)\rangle_t\right]$
- We derived the formula for the rBergomi model. In the special case $\xi_0(\cdot) \equiv \overline{\sigma}$, we obtain

$$\psi(\tau) = \rho\eta F_H \frac{1}{\tau^\gamma} + \rho^2\eta^2\overline{\sigma}\tau^{2H} G_H + o\left(\eta^3\tau^{3H}\right)$$

- Very high accuracy for $\lambda = \eta\tau^H \ll 1$.
- Unsurprisingly, poor accuracy for $\lambda = \eta\tau^H$ not sufficiently small, as typically the case for real-life situations.

- Bergomi and Guyon (2012) give a general expansion of implied volatility in terms of vol-of-vol and maturity for Bergomi-like stochastic volatility models.
- Expansion is based on auto-covariance $C = E\left[\langle \log S_\cdot\,,\, \xi_\cdot(u)\rangle_t\right]$
- We derived the formula for the rBergomi model. In the special case $\xi_0(\cdot) \equiv \overline{\sigma}$, we obtain

$$\psi(\tau) = \rho\eta F_H \frac{1}{\tau^\gamma} + \rho^2\eta^2\overline{\sigma}\tau^{2H} G_H + o\left(\eta^3\tau^{3H}\right)$$

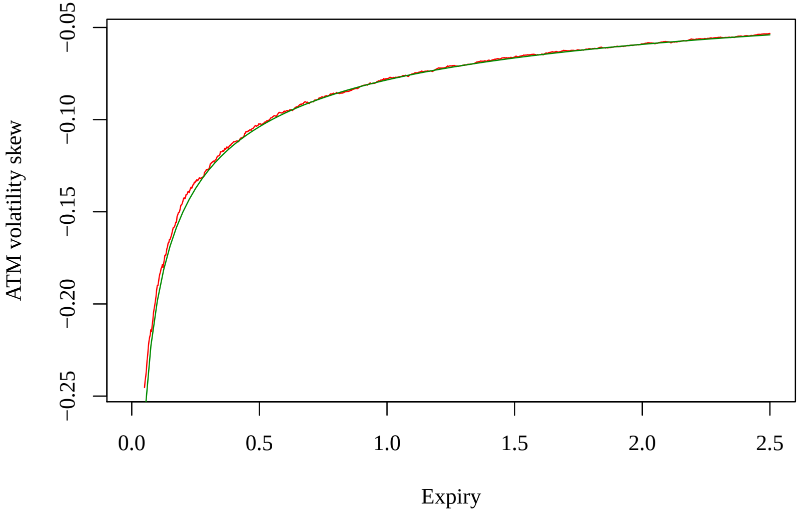
- Very high accuracy for $\lambda = \eta\tau^H \ll 1$.
- Unsurprisingly, poor accuracy for $\lambda = \eta\tau^H$ not sufficiently small, as typically the case for real-life situations.

- Bergomi and Guyon (2012) give a general expansion of implied volatility in terms of vol-of-vol and maturity for Bergomi-like stochastic volatility models.
- Expansion is based on auto-covariance $C = E\left[\langle \log S_{\cdot}\,, \xi_{\cdot}(u)\rangle_t\right]$
- We derived the formula for the rBergomi model. In the special case $\xi_0(\cdot) \equiv \overline{\sigma}$, we obtain

$$\psi(\tau) = \rho\eta F_H \frac{1}{\tau^\gamma} + \rho^2\eta^2\overline{\sigma}\tau^{2H} G_H + o\left(\eta^3\tau^{3H}\right)$$

- Very high accuracy for $\lambda = \eta\tau^H \ll 1$.
- Unsurprisingly, poor accuracy for $\lambda = \eta\tau^H$ not sufficiently small, as typically the case for real-life situations.

Bergomi-Guyon formula for $H = 0.1$, $\eta = 0.4$, $\rho = -0.85$, $\overline{\sigma} = 0.235$



- Maybe we can calibrate against VIX options, in particular VIX variance swaps / VVIX?
- Let $\sqrt{\zeta(T)}$ be the terminal value of VIX futures, i.e.,

$$\zeta(T) = \frac{1}{\Delta} \int_T^{T+\Delta} E_T v_u du$$

- Similar to the construction of VIX, we use the log-strip to construct VVIX (based on VIX options)

$$\mathrm{VVIX}^2_{t,T}(T-t) = -2E_t\left[\log\sqrt{\zeta(T)} - \log\sqrt{\zeta(t)}\right]$$

- Heuristic approximation gives

$$\mathrm{VVIX}^2_{t,T}\,\tau \approx \frac{1}{4}\eta^2\tau^{2H} f_H\left(\frac{\Delta}{\tau}\right),$$

$$f_H(\theta) = \frac{D_H^2}{\theta^2}\int_0^1 \left((1+\theta+x)^{1/2+H} - (1-x)^{1/2+H}\right) dx$$

- Maybe we can calibrate against VIX options, in particular VIX variance swaps / VVIX?
- Let $\sqrt{\zeta(T)}$ be the terminal value of VIX futures, i.e.,

$$\zeta(T) = \frac{1}{\Delta} \int_T^{T+\Delta} E_T v_u du$$

- Similar to the construction of VIX, we use the log-strip to construct VVIX (based on VIX options)

$$\mathrm{VVIX}_{t,T}^2 (T - t) = -2E_t \left[\log \sqrt{\zeta(T)} - \log \sqrt{\zeta(t)} \right]$$

- Heuristic approximation gives

$$\mathrm{VVIX}_{t,T}^2 \, \tau \approx \frac{1}{4} \eta^2 \tau^{2H} f_H \left(\frac{\Delta}{\tau} \right),$$

$$f_H(\theta) = \frac{D_H^2}{\theta^2} \int_0^1 \left((1 + \theta + x)^{1/2+H} - (1 - x)^{1/2+H} \right) dx$$

- Maybe we can calibrate against VIX options, in particular VIX variance swaps / VVIX?
- Let $\sqrt{\zeta(T)}$ be the terminal value of VIX futures, i.e.,

$$\zeta(T) = \frac{1}{\Delta} \int_T^{T+\Delta} E_T v_u du$$

- Similar to the construction of VIX, we use the log-strip to construct VVIX (based on VIX options)

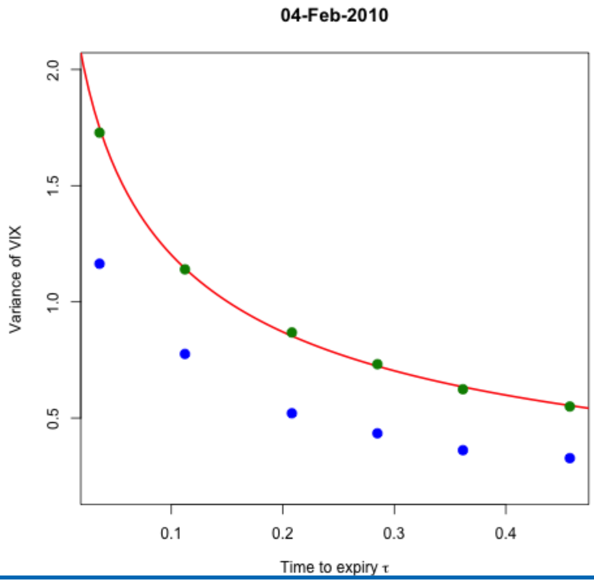
$$\mathrm{VVIX}^2_{t,T}(T-t) = -2E_t \left[\log \sqrt{\zeta(T)} - \log \sqrt{\zeta(t)} \right]$$

- Heuristic approximation gives

$$\mathrm{VVIX}^2_{t,T}\,\tau \approx \frac{1}{4} \eta^2 \tau^{2H} f_H\left(\frac{\Delta}{\tau}\right),$$

$$f_H(\theta) = \frac{D_H^2}{\theta^2} \int_0^1 \left((1+\theta+x)^{1/2+H} - (1-x)^{1/2+H} \right) dx$$

Comparison to market VVIX term structure



- *Rough* fractional stochastic volatility models (with $H < 1/2$) provide excellent fits with time series of realized variance for essentially all major stock indices and a variety of other indices.
- The rBergomi model, in particular, can fit the full implied volatility surface of SPX with only three free parameters (H, η, ρ).
- So far, we use trivial market price of volatility risk, hence we cannot get a realistic smile for VIX options.
- We can price SPX and VIX options using MC simulation, but accurate asymptotic formulas for calibration are missing.
- There is a clear mis-fit to volatility of volatility (VVIX).
- Model can be obtained as scaling limit from a micro-structure model based on Hawkes processes for order flows.

- *Rough* fractional stochastic volatility models (with $H < 1/2$) provide excellent fits with time series of realized variance for essentially all major stock indices and a variety of other indices.
- The rBergomi model, in particular, can fit the full implied volatility surface of SPX with only three free parameters (H, η, ρ).
- So far, we use trivial market price of volatility risk, hence we cannot get a realistic smile for VIX options.
- We can price SPX and VIX options using MC simulation, but accurate asymptotic formulas for calibration are missing.
- There is a clear mis-fit to volatility of volatility (VVIX).
- Model can be obtained as scaling limit from a micro-structure model based on Hawkes processes for order flows.

- *Rough* fractional stochastic volatility models (with $H < 1/2$) provide excellent fits with time series of realized variance for essentially all major stock indices and a variety of other indices.
- The rBergomi model, in particular, can fit the full implied volatility surface of SPX with only three free parameters (H, η, ρ).
- So far, we use trivial market price of volatility risk, hence we cannot get a realistic smile for VIX options.
- We can price SPX and VIX options using MC simulation, but accurate asymptotic formulas for calibration are missing.
- There is a clear mis-fit to volatility of volatility (VVIX).
- Model can be obtained as scaling limit from a micro-structure model based on Hawkes processes for order flows.

Conclusions and outlook

- *Rough* fractional stochastic volatility models (with $H < 1/2$) provide excellent fits with time series of realized variance for essentially all major stock indices and a variety of other indices.
- The rBergomi model, in particular, can fit the full implied volatility surface of SPX with only three free parameters (H, η, ρ).
- So far, we use trivial market price of volatility risk, hence we cannot get a realistic smile for VIX options.
- We can price SPX and VIX options using MC simulation, but accurate asymptotic formulas for calibration are missing.
- There is a clear mis-fit to volatility of volatility (VVIX).
- Model can be obtained as scaling limit from a micro-structure model based on Hawkes processes for order flows.

- *Rough* fractional stochastic volatility models (with $H < 1/2$) provide excellent fits with time series of realized variance for essentially all major stock indices and a variety of other indices.
- The rBergomi model, in particular, can fit the full implied volatility surface of SPX with only three free parameters (H, η, ρ).
- So far, we use trivial market price of volatility risk, hence we cannot get a realistic smile for VIX options.
- We can price SPX and VIX options using MC simulation, but accurate asymptotic formulas for calibration are missing.
- There is a clear mis-fit to volatility of volatility (VVIX).
- Model can be obtained as scaling limit from a micro-structure model based on Hawkes processes for order flows.

- *Rough* fractional stochastic volatility models (with $H < 1/2$) provide excellent fits with time series of realized variance for essentially all major stock indices and a variety of other indices.
- The rBergomi model, in particular, can fit the full implied volatility surface of SPX with only three free parameters (H, η, ρ).
- So far, we use trivial market price of volatility risk, hence we cannot get a realistic smile for VIX options.
- We can price SPX and VIX options using MC simulation, but accurate asymptotic formulas for calibration are missing.
- There is a clear mis-fit to volatility of volatility (VVIX).
- Model can be obtained as scaling limit from a micro-structure model based on Hawkes processes for order flows.

References

- Bayer, C., Friz, P., Gatheral, J. *Pricing under rough volatility*, preprint, 2015.
- Bergomi, L. *Smile dynamics II*, Risk, October 2005.
- Bergomi, L., Guyon, J. *Stochastic volatility's orderly smiles*, Risk, May 2012.
- Comte, F., Coutin, L., Renault, E. *Affine fractional stochastic volatility models*, Annals of Finance, 2012.
- Fukasawa, M. *Asymptotic analysis for stochastic volatility: Martingale expansion*, Finance and Stochastics, 2011.
- Gatheral, J., Jaisson, T., Rosenbaum, M. *Volatility is rough*, preprint, 2014.