Pricing under rough volatility

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Joint work with Peter Friz and Jim Gatheral.
Outline

1 Models for variance swaps and VIX

2 The rough Bergomi model

3 Volatility is rough: the econometric evidence

4 Case studies

5 Towards calibration of the rough Bergomi model
Variance swaps and VIX

- Given a traded asset $S_t$ satisfying
  \[ dS_t = \sqrt{v_t} S_t dZ_t \]

- Interest rate $r = 0$; model formulated under $Q$

- In this talk, $S$ corresponds to the S & P 500 index (SPX).

- Realized variance $w_{t,T} = \int_t^T v_s ds$

- Variance swaps are swaps on realized variances.

- Allow direct trades in volatility, not indirect via options

- For convenience, CBOE introduced an index (VIX) for the square root of (annualized) one month variance swaps.

- $VIX_t \approx \sqrt{\frac{1}{\Delta} E_t w_{t,t+\Delta}}, \Delta = 1/12$
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- $VIX_t \approx \sqrt{\frac{1}{\Delta} E_t w_{t,t+\Delta}}$, $\Delta = 1/12$
Ito’s formula gives for the payoff $\log(S_T)$

$$\log S_T - \log S_t = \int_t^T \frac{dS_u}{S_u} - \frac{1}{2} \int_t^T \nu_u du$$

Breeden-Litzenberger formula: $p(S_T, T, S_t, t) = \frac{\partial^2 C/P(S_t, K, t, T)}{\partial K^2}\bigg|_{K=S_t}$

$p$ ... density, $C, P$ call and put prices

Integration by parts, put-call-parity give for smooth payoff $g$

$$E[g(S_T)|S_t] = g(S_t) + \int_0^{S_t} P(K)g''(K)dK + \int_{S_t}^\infty C(K)g''(K)dK$$

For $g(S) = -2 \log S$, we have $g''(K) = \frac{2}{K^2}$ and

$$E_t w_{t,T} = -2 \left( \int_0^{S_t} \frac{P(K)}{K^2} dK + \int_{S_t}^\infty \frac{C(K)}{K^2} dK \right)$$
The log-strip

- Ito’s formula gives for the payoff \(\log(S_T)\)

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E_t w_{t,T} = E_t \int_t^T v_u du = 2 \log S_t - 2E_t \log S_T
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- Ito's formula gives for the payoff $\log(S_T)$

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Stochastic volatility models

\[ dS_t = \sqrt{v_t} S_t dZ_t, \]
\[ dv_t = \ldots \]

- \( Z, W \) Brownian motions with correlation \( \rho \)
- Goal: model consistent with the full SPX implied volatility surface
  - \( \text{VIX}_t \approx \sqrt{v_t} \) (with \( \Delta \approx 0 \))
  - \( \text{VIX} \) itself is not traded, but the following are:
    - \( \text{VIX} \) futures (rate given by \( E_t \text{VIX}_T \); traded on CBOE)
    - \( \text{VIX} \) options (i.e., options on \( \text{VIX} \) futures; traded on CBOE)
    - Variance swaps (swap rate \( E_t \xi_{t,T} \); traded over the counter)
  - Fundamental object: forward variance \( \xi_t(u) = E_t v_u, \, t \leq u \)
  - Variance swap \( E_t \xi_{t,T} = E_t \int_t^T v_s ds = \int_t^T \xi_t(s) ds \)
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dS_t = \sqrt{\nu_t} S_t dZ_t,

d\nu_t = \ldots

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Some SPX implied volatility surfaces

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Conclusions

▶ Since the rough shape of volatility surfaces seems pretty stable, we look for time-homogeneous models.

▶ Term structure of ATM volatility skew ($k = \log(K/S_t)$)

$$\psi(\tau) = \left| \frac{\partial}{\partial k} \sigma_{BS}(k, \tau) \right|_{k=0} \sim 1/\tau^{\alpha}, \quad \alpha \in [0.3, 0.5]$$

▶ Conventional stochastic volatility models produce ATM skews which are constant for $\tau \ll 1$ and of order $1/\tau$ for $\tau \gg 1$. Hence, conventional stochastic volatility models cannot fit the full volatility surface.

▶ Do we need jumps?

▶ Stochastic variance has log-normal distribution (under $P$).
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Recall $\xi_t(u) = E_t v_u$

$$dS_t = \sqrt{\xi_t(t)} S_t dZ_t,$$

$$\xi_t(u) = \xi_0(u) E \left( \sum_{i=1}^n \eta_i \int_0^t e^{-\kappa_i(u-s)} dW^i_s \right)$$

$E(X) = \exp(X - \frac{1}{2} E[|X|^2])$ for Gaussian r.v. $X$

Market model

In practice, $n = 2$ needed for good fit, contains seven parameters

$$\psi(\tau) \sim \sum_{i=1}^n \frac{\eta_i}{\kappa_i \tau} \left( 1 - \frac{1 - e^{-\kappa_i \tau}}{\kappa_i \tau} \right)$$

Tempting to replace the exponential kernel by a power law kernel!
Bergomi model

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Tempting to replace the exponential kernel by a power law kernel!
Gatheral, Jaisson, and Rosenbaum (2014) study time series of realized variance and find amazing fits of a stochastic volatility model based on

\[ \log v_u - \log v_t = 2\nu \left( W_u^H - W_t^H \right) \]

Mandelbrot – Van Ness representation of fBm (with \( \gamma = 1/2 - H \))

\( \nu_u \) is not a Markov process (neither under \( P \) or \( Q \)).

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\[ W^H_t = C_H \left( \int_0^t \frac{dW^P_s}{(t-s)^\gamma} + \int_{-\infty}^0 \left[ \frac{1}{(t-s)^\gamma} - \frac{1}{(-s)^\gamma} \right] dW^P_s \right) \]

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With \( \tilde{W}_t^P(u) = \sqrt{2H} \int_t^u \frac{dW_s^P}{(u - s)^\gamma} \), we get

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With $\tilde{W}^P_t(u) = \sqrt{2H} \int_t^u \frac{dW^P_s}{(u - s)^\gamma}$, we get

$$v_u = \xi_t(u) \mathcal{E}(\eta \tilde{W}^Q_t(u))$$
The Rough Bergomi model (under $Q$)

\begin{align*}
    dS_t &= \sqrt{v_t} S_t dZ_t \\
    v_t &= \xi_0(t) \mathcal{E}(\eta \tilde{W}_t)
\end{align*}

- $dW_t dZ_t = \rho dt, \tilde{W}_t = \sqrt{2H} \int_0^t \frac{dW_s}{(t-s)^\gamma}, \gamma = 1/2 - H$

- $\tilde{W}$ is a “Volterra” process (or “Riemann-Liouville fBm”)

- Covariance:

\begin{align*}
    E\left[\tilde{W}_v \tilde{W}_u\right] &= \frac{2H}{1/2 + H} \frac{u^{1/2+H}}{\nu^{1/2-H}} \binom{1, 1/2 - H, 3/2 + H, u/v}, \ u \leq \nu, \\
    E\left[\tilde{W}_v Z_u\right] &= \rho \frac{\sqrt{2H}}{1/2 + H} \left(\nu^{1/2+H} - [\nu - \min(u, \nu)]^{1/2+H}\right)
\end{align*}

- $\psi(\tau) \sim 1/\tau^\gamma$

- Typical parameter values: $H \approx 0.05, \eta \approx 2.5$
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  \tilde{W} \text{ is a “Volterra” process (or “Riemann-Liouville fBm”)} \\
  \text{Covariance:} \\
  E\left[\tilde{W}_v \tilde{W}_u\right] &= \frac{2H}{1/2 + H} \frac{u^{1/2+H}}{v^{1/2-H}} _2F_1\left(1, 1/2 - H, 3/2 + H, u/v\right), \quad u \leq v, \\
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KRV estimates of SPX realized variance from 2000 to 2014
Moments of differences of realized volatility

- The Oxford Man Institute provides estimated realized variances $v_t$ for numerous indices on a daily bases.
- Let $\sigma_t = \sqrt{v_t}$.
- For some lag $\Delta > 0$ fix a corresponding time-grid $t_i$ (with $t_{i+1} - t_i = \Delta$) and define the moment of the log-differences by

$$m(q, \Delta) := \langle |\log \sigma_{t+\Delta} - \log \sigma_t|^q \rangle$$

- $\langle \cdot \rangle$ denotes taking sample average.
- $m(q, \Delta)$ measures smoothness of realized volatility at various lags.
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Scaling of $m$ in $\Delta$
Monofractal scaling of $m(q, \Delta)$

- We see fractal behaviour: for each moment order $q$ there is a coefficient $\zeta_q$ such that

$$m(q, \Delta) \sim \Delta^{\zeta_q}$$

- Different $q$ show the same fractal behaviour in the sense that for some $H \approx 0.1$, $\zeta_q \approx qH$.
- Log-volatility is also approximately normal.
- These observations hold for all 21 indices in the Oxford Man database.

Log-volatility seems to be described by a fractional Brownian motion with Hurst index $H \approx 0.1$. This suggests models of the form

$$dS_t = S_t \exp\left(\eta W_t^H\right) dZ_t + \cdots$$

for $0 < H < 1/2$. 
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for $0 < H < 1/2$. 
Several fractional stochastic volatility models have been proposed, inevitably with $H > 1/2$.

Fractional Brownian motion with $H > 1/2$ has long memory, i.e., the auto-correlation function $\rho(\Delta)$ (at lag $\Delta$) has power law decay as $\Delta \to \infty$.

It was an accepted stylized fact that volatility has long memory.

In our rough model:

$$\rho(\Delta) \sim \exp\left(-\frac{\eta^2}{2} \Delta^{2H}\right)$$

Hence, no long term memory!

Estimates and comparisons by Gatheral, Jaisson, Rosenbaum suggest that there really is no long term memory in volatility.

Might be an effect of new, better (high-frequency) data.
Fractional models in the literature

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Estimates and comparisons by Gatheral, Jaisson, Rosenbaum suggest that there really is no long term memory in volatility.

Might be an effect of new, better (high-frequency) data.
Empirical auto-correlation against exponential decay

![Graph showing empirical auto-correlation against exponential decay. The x-axis represents \( \log(\Delta) \) and the y-axis represents \( \log(\text{Autocovariance}) \). The trendline suggests an exponential decay pattern.]
Comparing with Comte-Renault model

Fractional stochastic volatility model:

\[ dS_t = \sigma_t S_t dZ_t, \]
\[ d\log \sigma_t = -\alpha (\log \sigma_t - \theta) \, dt + \gamma d\hat{W}_t^H \]

with \( \hat{W}_t^H = \int_0^t \frac{(t-s)^{H-1/2}}{\Gamma(H+1/2)} dW_s, \, \langle Z, W \rangle_t = \rho t, \, 1/2 \leq H < 1. \)

- Related to Hull-White stochastic volatility model
- FSV model equivalent to RFSV model of Gatheral, Jaisson, Rosenbaum (up to choice of \( H \))
- rBergomi: replace fOU-process by fBm
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Moment comparison for realized variance

Blue: FSV model with $H = 0.53$, orange: rBergomi, $H = 0.15$
Outline

1. Models for variance swaps and VIX
2. The rough Bergomi model
3. Volatility is rough: the econometric evidence
4. Case studies
5. Towards calibration of the rough Bergomi model
02/04/2010; SPX Vol surface for $H = 0.07$, $\eta = 1.9$, $\rho = -0.9$
02/04/2010; SPX short maturity smile for $H = 0.07$, $\eta = 1.9$, $\rho = -0.9$
02/04/2010; SPX volatility skew for $H = 0.07$, $\eta = 1.9$, $\rho = -0.9$
02/04/2010; SPX ATM volatility for $H = 0.07, \eta = 1.9, \rho = -0.9$
Variance swap forecast

- Variance ν is not a martingale, hence non-trivial forecast.
- Formulate in RFSV model.

\[
E^P \left[ \log \nu_{t+\Delta} | \mathcal{F}_t \right] = \frac{\cos(H\pi)}{\pi} \Delta^{H+1/2} \int_{-\infty}^t \frac{\log \nu_s}{(t-s+\Delta)(t-s)^{H+1/2}} ds,
\]

\[
E^P [\nu_{t+\Delta} | \mathcal{F}_t] = \exp \left( E^P \left[ \log \nu_{t+\Delta} | \mathcal{F}_t \right] + 2c\nu^2 \Delta^{2H} \right).
\]

- Use realized variance as proxy for ν
- Problem: realized variance only available from opening to close, not from close to close
- Forecasts must be re-scaled by (time-varying) factor; hence should predict variance swap curve up to a factor
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E^P \left[ \log v_{t+\Delta} | \mathcal{F}_t \right] = \frac{\cos(H\pi)}{\pi} \Delta^{H+1/2} \int_{-\infty}^{t} \frac{\log v_s}{(t - s + \Delta)(t - s)^{H+1/2}} ds,
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Forecasts for the Lehman weekend

Actual and predicted variance swap curves, 09/12/08 (red) and 09/15/08 (blue), after scaling.
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Bergomi-Guyon implied volatility expansion


- Expansion is based on auto-covariance \( C = E [\langle \log S \cdot \xi(u) \rangle_t] \)

- We derived the formula for the rBergomi model. In the special case \( \xi_0(\cdot) \equiv \bar{\sigma} \), we obtain

\[
\psi(\tau) = \rho \eta F_H \frac{1}{\tau^\gamma} + \rho^2 \eta^2 \bar{\sigma} \tau^{2H} G_H + o(\eta^3 \tau^{3H})
\]

- Very high accuracy for \( \lambda = \eta \tau^H \ll 1 \).

- Unsurprisingly, poor accuracy for \( \lambda = \eta \tau^H \) not sufficiently small, as typically the case for real-life situations.

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Bergomi-Guyon formula for $H = 0.1, \eta = 0.4, \rho = -0.85, \bar{\sigma} = 0.235$
VIX options and VVIX

- Maybe we can calibrate against VIX options, in particular VIX variance swaps / VVIX?
- Let $\sqrt{\zeta(T)}$ be the terminal value of VIX futures, i.e.,

$$\zeta(T) = \frac{1}{\Delta} \int_T^{T+\Delta} E_T \nu_u du$$

- Similar to the construction of VIX, we use the log-strip to construct VVIX (based on VIX options)

$$\text{VVIX}^2_{t,T}(T - t) = -2E_t \left[ \log \sqrt{\zeta(T)} - \log \sqrt{\zeta(t)} \right]$$

- Heuristic approximation gives

$$\text{VVIX}^2_{t,T} \tau \approx \frac{1}{4} \eta^2 \tau^{2H} f_H \left( \frac{\Delta}{\tau} \right),$$

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Comparison to market VVIX term structure

04-Feb-2010

Variance of VIX

Time to expiry $\tau$

Pricing under rough volatility · November 3, 2015 · Page 33 (35)
Conclusions and outlook

- *Rough* fractional stochastic volatility models (with $H < 1/2$) provide excellent fits with time series of realized variance for essentially all major stock indices and a variety of other indices.

- The rBergomi model, in particular, can fit the full implied volatility surface of SPX with only three free parameters ($H, \eta, \rho$).

- So far, we use trivial market price of volatility risk, hence we cannot get a realistic smile for VIX options.

- We can price SPX and VIX options using MC simulation, but accurate asymptotic formulas for calibration are missing.

- There is a clear mis-fit to volatility of volatility (VVIX).

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