Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz
Association

Rough volatility models

Christian Bayer

EMEA Quant Meeting 2018

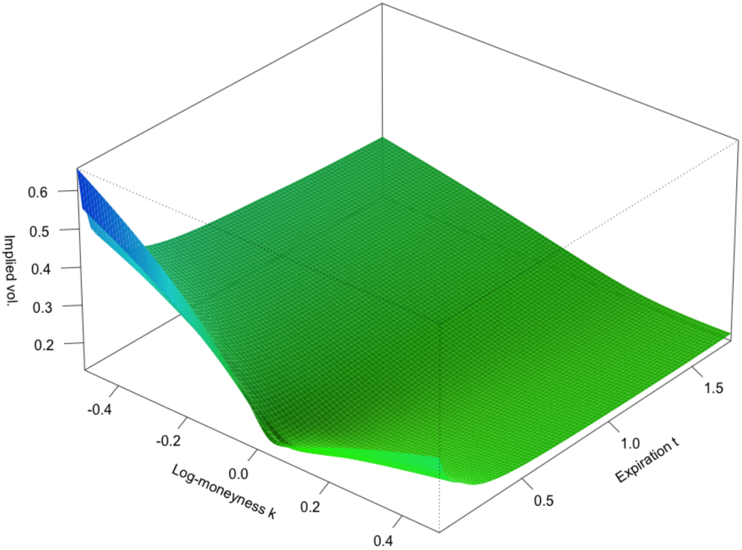
Outline

1 Implied volatility modeling

2 The rough Bergomi model

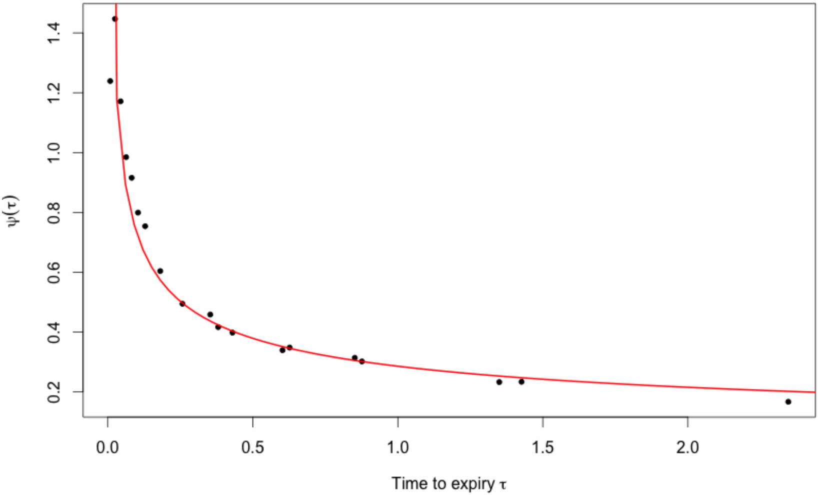
3 Case studies

4 Further challenges and developments



0.6
0.5
0.4
0.3
0.2
Implied vol.
-0.4
-0.2
0.0
0.2
0.4
Log-moneyness k
0.5
1.0
1.5
Expiration t

SPX ATM volatility skew



$$dS_t = \sqrt{v_t} S_t dZ_t,$$
$$dv_t = \dots$$

- We look for time-homogeneous models.
- Term structure of ATM volatility skew ($k = \log(K/S_t)$)

$$\psi(\tau) = \left| \frac{\partial}{\partial k} \sigma_{BS}(k, \tau) \right|_{k=0} \sim 1/\tau^{\alpha}, \quad \alpha \in [0.3, 0.5]$$

- Conventional stochastic volatility models produce ATM skews which are constant for $\tau \ll 1$ and of order $1/\tau$ for $\tau \gg 1$. Hence, conventional stochastic volatility models cannot fit the full volatility surface.
- Do we need jumps?

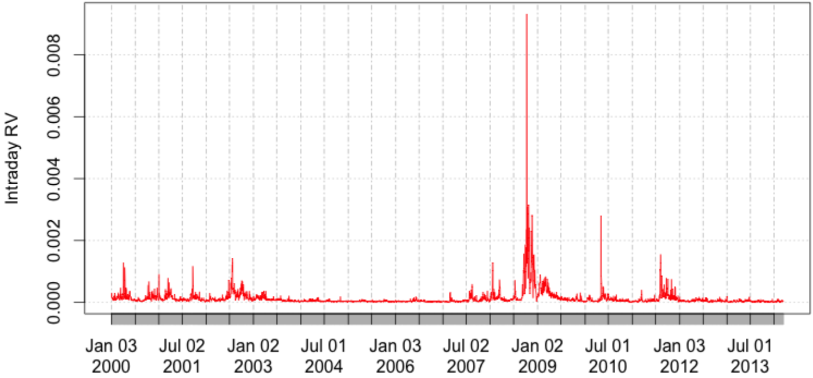
- Given a traded asset S_t satisfying

$$dS_t = \sqrt{v_t} S_t dZ_t$$

- Interest rate $r = 0$; model (and expectations) formulated under Q
- In this talk, S corresponds to the S & P 500 index (SPX).
- *Realized variance* $w_{t,T} = \int_t^T v_s ds$, *forward variance* $\xi_t(u) = E_t[v_u]$
- Log-strip formula:

$$E_t w_{t,T} = -2 \left(\int_0^{S_t} \frac{P(K)}{K^2} dK + \int_{S_t}^{\infty} \frac{C(K)}{K^2} dK \right)$$

Intraday realized variance



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- Recall $\xi_t(u) = E_t v_u$

$$dS_t = \sqrt{\xi_t(t)} S_t dZ_t,$$
$$\xi_t(u) = \xi_0(u) \mathcal{E}\left(\sum_{i=1}^n \eta_i \int_0^t e^{-\kappa_i (u-s)} dW_s^i \right)$$

- $\mathcal{E}(X) := \exp(X - \frac{1}{2} E[|X|^2])$ for Gaussian r.v. X
- Market model
- In practice, $n = 2$ needed for good fit, contains seven parameters
- $\psi(\tau) \sim \sum_{i=1}^n \frac{\eta_i}{\kappa_i \tau} \left(1 - \frac{1 - e^{-\kappa_i \tau}}{\kappa_i \tau} \right)$
- Tempting to replace the exponential kernel by a power law kernel!

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- Gatheral, Jaisson, and Rosenbaum (2014) study time series of realized variance and find amazing fits of a stochastic volatility model based on

$$\log v_u - \log v_t = 2\nu\left(W_u^H - W_t^H\right)$$

- Mandelbrot – Van Ness representation of fBm (with $\gamma = 1/2 - H$)

$$W_t^H = C_H\left(\int_0^t \frac{dW_s^P}{(t-s)^\gamma} + \int_{-\infty}^0 \left[\frac{1}{(t-s)^\gamma} - \frac{1}{(-s)^\gamma}\right] dW_s^P\right)$$

- v_u is not a Markov process.
- With $\widetilde{W}_t^P(u) = \sqrt{2H}\int_t^u \frac{dW_s^P}{(u-s)^\gamma}$, we get

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- $dW_t dZ_t = \rho dt$, $\widetilde{W}_t = \sqrt{2H} \int_0^t \frac{dW_s}{(t-s)^\gamma}$, $\gamma = 1/2 - H$
- $\widetilde{W}$ is a "Volterra" process (or "Riemann-Liouville fBm")
- Covariance:

$$E\left[\widetilde{W}_v \widetilde{W}_u\right] = \frac{2H}{1/2+H} \frac{u^{1/2+H}}{v^{1/2-H}} {}_2F_1\left(1, 1/2-H, 3/2+H, u/v\right), \quad u \le v,$$
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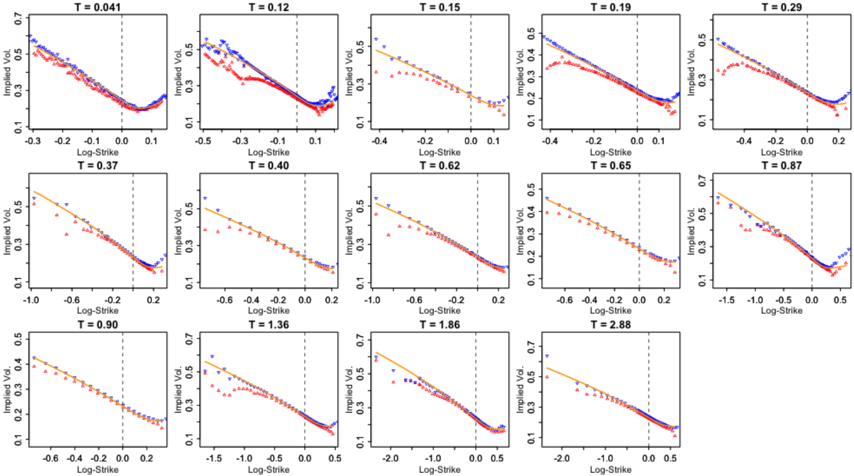
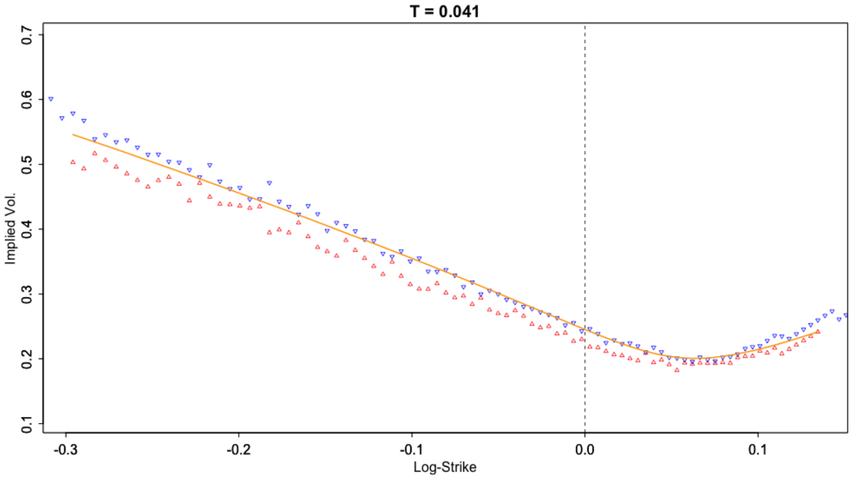
02/04/2010; SPX Vol surface for $H = 0.07$, $\eta = 1.9$, $\rho = -0.9$



02/04/2010; SPX short maturity smile for $H = 0.07$, $\eta = 1.9$, $\rho = -0.9$



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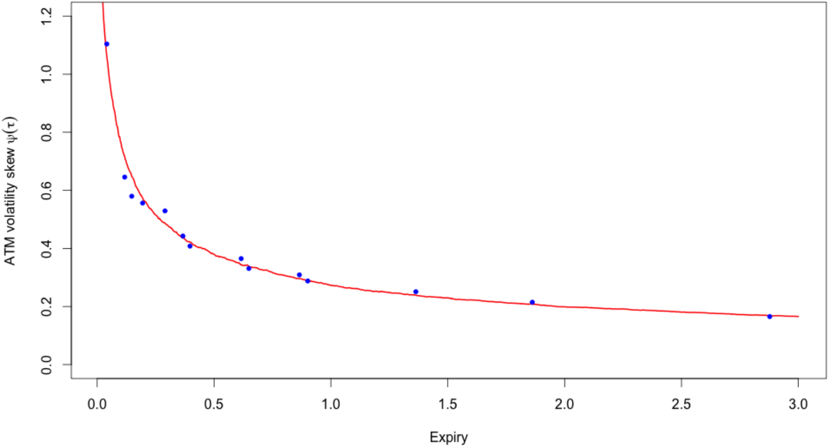

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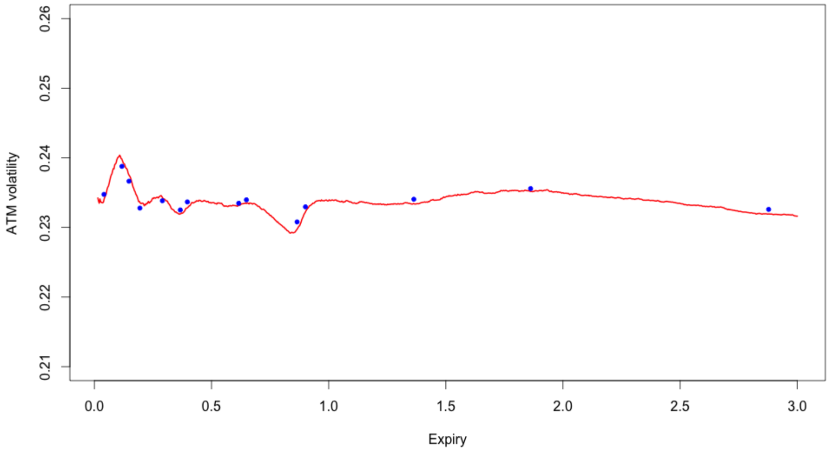

- Variance v is not a martingale, hence non-trivial forecast.
- Using a result in (Nuzman and Poor, 2000), we have

$$E^P\left[\log v_{t+\Delta}|\mathcal{F}_t\right] = \frac{\cos(H\pi)}{\pi}\Delta^{H+1/2}\int_{-\infty}^{t}\frac{\log v_s}{(t-s+\Delta)(t-s)^{H+1/2}}ds$$

$$E^P\left[v_{t+\Delta}|\mathcal{F}_t\right] = \exp\left(E^P\left[\log v_{t+\Delta}|\mathcal{F}_t\right] + 2c\nu^2\Delta^{2H}\right)$$

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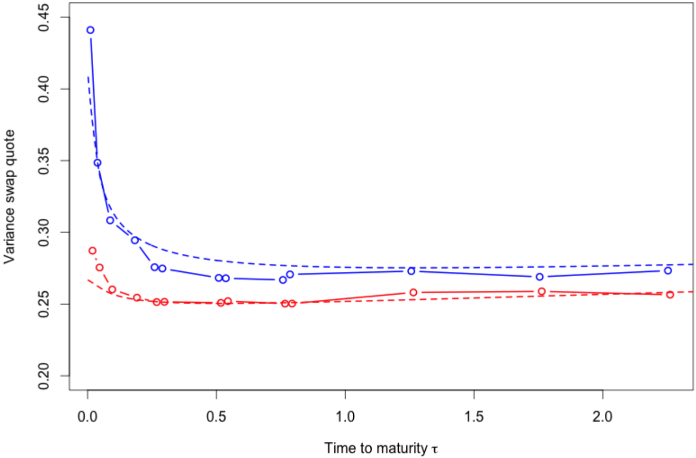

Actual and predicted variance swap curves, 09/12/08 (red) and 09/15/08 (blue), after scaling.

(Bennedsen, Lunde and Pakkanen 2017) compare timeseries data over 10 years of 2000 assets (US equities). They find overwhelming evidence of rough volatility!

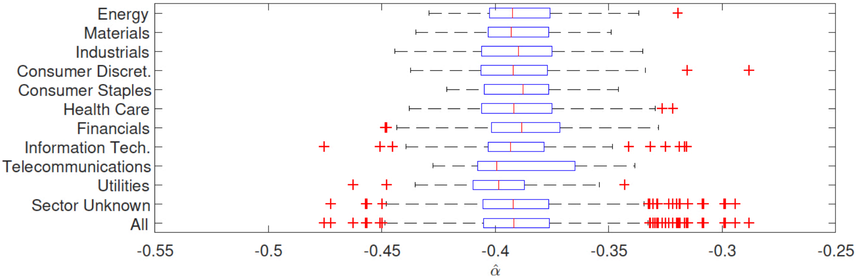

Figure: Estimates for $\alpha := H - 1/2$ according to sector.

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Theory:

- Lack of general fractional stochastic calculus (for instance, no rough path framework for $H \leq 1/4$)
- Difficult to generalize dynamics (needed to capture higher order effects)
- Difficult to analyze even very simple models such as rough Bergomi

Computations:

- No Markov structure, hence no (tractable) pricing PDE or tree approximations
- Large deviations depend on truly infinite dimensional variational problems, making asymptotic analysis more difficult
- Simulation expensive but doable relying on the Gaussian structure

$$dS_t = \sqrt{v_t} S_t dZ_t$$
$$v_t = v_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda(\theta - v_s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda\nu \sqrt{v_s} dW_s$$

Fractional Riccati ODE

$E\left[\exp\left(iu \log(S_t)\right)\right] = \exp\left(g_1(u,t) + v_0 g_2(u,t)\right)$, with

$$g_1(u,t) := \theta\lambda \int_0^t h(u,s)ds, \quad g_2(u,t) := I^{1-\alpha} h(u,t),$$

$$D^\alpha h(u,t) = \frac{1}{2}(-u^2 - iu) + \lambda(iu\rho\nu - 1)h(u,t) + \frac{(\lambda\nu)^2}{2} h^2(u,t), \quad I^{1-\alpha} h(u,0) = 0.$$

$$I^r f(t) := \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s) ds, \quad D^r f(t) := \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_0^t (t-s)^r f(s) ds.$$

Assumption

- *Market orders are indep. Hawkes processes $N^{a/b}$, with intensities*

$$\lambda_t^{a/b} = \mu + \int_0^t \phi(t-s) dN_s^{a/b}$$

- *Market impact exists and has a non-vanishing transient component.*
- *The market is highly endogenous.*

Under some additional assumptions, we obtain a rough Heston type model as scaling limit of price changes obtained from the market orders. ([El Euch, Fukasawa, Rosenbaum, 2016], [Jusselin, Rosenbaum 2018])

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Recall that $(\widehat{W}, Z)$ is a Gaussian process. Hence, we can simulate samples on a grid $0 = t_0 < t_1 < \cdots < t_N = T$ by

- Cholesky factorization of the covariance (exact, but cost $\mathcal{O}(N^2)$ per sample);
- *Hybrid scheme* by [Bennedsen, Lunde, Pakkanen, 2017] (inexact, but cost $\mathcal{O}(N \log N)$).

Leads to Riemann approximation

$$\int_0^T f\left(t, \widehat{W}_t\right) dZ_t \approx \sum_{i=0}^{N-1} f\left(t_i, \widehat{W}_{t_i}\right)\left(Z_{t_{i+1}} - Z_{t_i}\right).$$

Theorem (Neuenkirch and Shalaiko '16)

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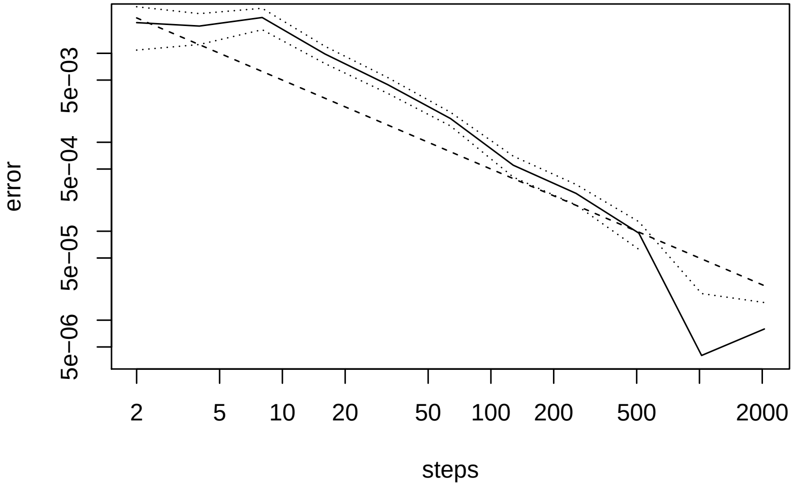
ATM-call with $\xi \equiv 0.04$, $H = 0.06$, $\eta = 2.5$, $\rho = -0.8$, $T = 0.8$



The weak rate of convergence seems unknown even for

$$Y \equiv \int_0^1 f(s, \widehat{W}_s) dW_s.$$

- Standard methods for SDEs rely on PDE arguments.
- Using metrics for weak convergence such as Wasserstein distance seems difficult.
- Techniques based on Malliavin calculus work in principle.
- For Y as above, one can get weak rate $2H$, but numerical experiments suggest much better rates.
- Partial result: for Euler approximation $\overline{Y}$, f "nice"

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Definition

Fractional Brownian motion is a continuous time Gaussian process B^H (with Hurst index $0 < H < 1$) with $B_0^H = 0$, $E[B_t^H] = 0$ and

$$E[B_t^H B_s^H] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

- B^H with $H = \frac{1}{2}$ is classical Brownian motion.
- Increments are neg. corr. for $H < \frac{1}{2}$ and pos. corr. for $H > \frac{1}{2}$.

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