Research Statement

Christian Bayer

November 14, 2013

Introduction

In my research I am interested in numerics for stochastic differential equations and related questions. The motivation for the problems that I am trying to solve usually comes from finance, but I have also broader interests in economics (such as matching and saving), as well as in statistics (in particular, parameter estimation with the EM algorithm) and physics (molecular dynamics). Regarding the mathematical methods, I mainly use tools from stochastic analysis, but I also combine them with techniques from analysis and numerics. In the next paragraphs, I will give a very short overview about some recent research projects of mine, which will be followed by a more detailed discussion.

Starting with my PhD thesis I have worked on the cubature on Wiener space method, see [36] and, independently, [35], a higher order method for weak approximation of solutions of stochastic differential equations (SDEs) based on ideas from rough path analysis, see the original article [37] and the comprehensive book [30]. As part of my PhD thesis, I have extended the applicability of the cubature on Wiener space method to a certain class of SPDEs together with my thesis advisor Josef Teichmann, see [16]. On the theoretical side I have studied weak convergence of cubature methods on path-space with Peter Friz, see [6]. Moreover, together with Ronnie Loeffen we have also tried to further popularize the method among practitioners in financial engineering by showing that very efficient implementations of cubature methods are possible for a large class of models which are very often used in practice, provided that some additional transformation of the problem is employed, [4]. Following this idea, I have studied the calibration problem for the double mean reverting model in finance together with Jim Gatheral and Morten Karlsmark, obtaining substantial boosts in computational time as compared to classical Euler schemes, see [7]. In another research project strongly relying on rough paths notions and techniques (together with Peter Friz, Sebastian Riedel and John Schoenmakers), we have proved strong convergence of a (computable) Milstein type scheme for SDEs driven by fractional Brownian motion with Hearst index $H > \frac{1}{4}$, see [5]. Moreover, we have proved rates of convergence and have analyzed the multi-level Monte Carlo algorithm ([32]) for this algorithm. There are numerous questions on discretization of SDEs, which I would like to proceed in the future. For instance, I would like to combine the higher order methods constructed so far with adaptivity based on a-posteriori error control following cf. [44] and also [14].

A second recent pillar of my research are asymptotic methods in financial engineering. In a project together with Peter Laurence, we have applied the technique of heat kernel expansions from PDE theory to obtain asymptotic formulas for basket option prices, which are very fast to evaluate even in very high dimensions, while being astonishingly accurate, [10, 11, 12]. Stemming from this research, we got interested in the applicability of Laplace's method for integrals of densities over sub-manifolds of the space, which we have explored together with Peter Friz, see [3]. A generally open (and difficult) problem in the area of asymptotic formulas in financial mathematics is the treatment of the singularities naturally appearing at the boundary of the domain. A reasonable numerical technique to deal with this problem seems to be to apply boundary layer methods.

Together with John Schoenmakers, I have recently contributed to the problem of sampling from a diffusion (or Markov) bridge, i.e., from the path of a diffusion process being conditioned to take on a certain value both at time 0 and at time T. We use the technique of forward-reverse processes introduced by [38, 39], generalize it to this setting, and give a thorough numerical analysis of the corresponding algorithm, see [13]. I think that this new algorithm can be applied for many practical problems, in particular in the context of the EM algorithm in statistics and I would like to explore these possibilities further in the future.

Furthermore, I am very interested in applications in other sciences. In the past, I have contributed to successful research projects in economics and physics, see [20, 19] and [8], collaborations which I would like to continue in the future. More recently, I have started to become interested in applications in statistics.

Numerics for SDEs

Consider a stochastic differential equation

$$dX_t = V(X_t)dt + \sum_{i=1}^{d} V_i(X_t)dB_t^i, \quad X_0 = x, \tag{1}$$

for a stochastic process X evolving in $\mathbb{R}^n$ with vector fields $V, V_1, \ldots, V_d : \mathbb{R}^n \to \mathbb{R}^n$. Here, B typically denotes a d-dimensional Brownian motion (but more general driving processes are also considered, such as processes with jumps or fractional Brownian motion), and the stochastic integral is typically understood in the Ito sense. When using techniques from rough path analysis see [37] and [30], it is often advantageous to use the equivalent Stratonovich formulation

$$dX_t = V_0(X_t)dt + \sum_{i=1}^{d} V_i(X_t) \circ dB_t^i, \quad X_0 = x. \tag{2}$$

Very often, the quantity of interest is of the form

$$u(t, x) := E[f(X_T)|X_t = x], \tag{3}$$

for some function $f : \mathbb{R}^n \to \mathbb{R}$ and some fixed time $T > 0$ – in finance, f would be the payoff function and T the maturity of an option. u can be equally described as the solution of the Cauchy problem

$$\frac{\partial}{\partial t} u(t,x) + Lu(t,x) = 0, \quad u(T,x) = f(x), \tag{4}$$

with L denoting the infinitesimal generator of X. (In finance, u is essentially the price of a European option with payoff function f.)

Cubature on Wiener space

Cubature on Wiener space is a high order numerical method for computing (3) conceptually based on rough path theory introduced by Lyons and Victoir [36] and independently by Kusuoka [35], see also [43] and [42]. Formally, the idea is to replace the Brownian motion B by a process W with paths of bounded variation such that the iterated integrals of B and the iterated integrals of W (up to some level m) have the same expected value. More precisely, W is constructed by pasting together properly re-scaled copies of this bounded-variation process on a suitable grid $\mathcal{D}$ on $[0, T]$. Then one can show that

$$\left| E\left[f(X_T(B))\right] - E\left[f(X_T^{\mathcal{D}}(W))\right]\right| \leq C(\operatorname{mesh}\mathcal{D})^{(m-1)/2},$$

where $X(B)$ denotes the true solution of (2), whereas $X^h(W)$ denotes the solution of (2) with B being replaced by W. (In fact, for this to hold some regularity assumptions either on f or on the vector fields and $\mathcal{D}$ are required.)

In my PhD thesis under the supervision of Josef Teichmann, see [2] and [16], I extended the cubature on Wiener space method to a certain class of SPDEs, i.e., differential equations of the form (1) where the state space is infinite-dimensional and (in our case) V is an unbounded operator, assuming that the driving noise B is, in fact, finite-dimensional. The main challenge here was that in this setting, it is well-known that the solution to (1) is not a semi-martingale. Hence, it is not possible to formulate the Stratonovich equation (2) for B itself. On the other hand, for W equation (2) obviously does make sense, as it can be seen as a PDE with random coefficients – the approximations $X^{\mathcal{D}}(W)$ have to be chosen this way. In the end, the problem can be solved by working in proper abstract Sobolev spaces. In my opinion, the main advantage of the use of cubature methods on Wiener space for SPDEs is that they allow to uncouple the space-time discretization from the random noise. Indeed, as already remarked before, the SPDE for $X^{\mathcal{D}}(W)$ is a standard PDE (only with random coefficients), so standard PDE solvers (assuming they exist for the problem at hand) can be used without worrying about the white noise dB.

Together with Peter Friz, I have analyzed the convergence of the method when the payoff, i.e., the function f in (3) is path-dependent, i.e., depends on the whole trajectory X, not just the terminal value X_T, see [6]. In fact, we proved a Donsker-type theorem for the cubature paths in rough path sense (i.e., enhanced with the iterated integrals), which directly implies the convergence

$$E\left[f\left((X^{\mathcal{D}}(W)_t)_{0\leq t\leq T}\right)\right] \xrightarrow{\operatorname{mesh}\mathcal{D}\to 0} E\left[f\left((X(B)_t)_{0\leq t\leq T}\right)\right].$$

Applications of cubature on Wiener space in finance

With Peter Friz and Ronnie Loeffen, I have found that efficient implementation of the cubature on Wiener space method are available for a large class of models used in financial engineering with the help of an interesting transformation of the drift vector field, see [4]. Indeed, as a natural consequence of financial modelling, the drift V in a model in mathematical finance is often quite simple, whereas the Stratonovich drift $V_0 = V - \frac{1}{2}\sum_{i=1}^d DV_i \cdot V_i$ is considerably more complicated. The efficiency of cubature methods critically depend on fast and accurate solvers for the flows of the vector fields $V_0, V_1, \dots, V_d$, see [43]. In many models in finance, the flows of V and $V_1, \dots, V_d$ are available in closed form, but this is no longer the case for V_0. Due to the structure of the Stratonovich correction, it is often possible to split the Stratonovich drift vector field

$$V_0 = U + \sum_{i=1}^{d} \gamma_i V_i,$$

such that the flow of the vector field U is available in closed form. Replacing the components of the cubature path W_t^i by $W_t^i + \gamma_i t$, we can then apply the cubature on Wiener space method for the vector fields $U, V_1, \dots, V_d$, which allows highly efficient implementations of the algorithm without any loss of accuracy. In the paper, we show that such a decomposition is possible for many widely-applied models in mathematical finance and demonstrate that the cubature method is indeed much more efficient than classical Euler schemes.

As a by-product of the last paper, I have applied the cubature method to the problem of calibrating the double mean reverting model (a popular model among financial engineers introduced by Jim Gatheral in [31]), showing that it can lead to drastically reduced run-times as compared to standard Euler methods for calibration, see [7]. In this work together with Jim Gatheral and Morten Karlsmark, we actually need to use a further splitting of the Stratonovich drift vector field, as the splitting from [4] turned out not to be sufficient for the double mean reverting model.

Numerics for SDEs driven by fractional Brownian motion

Together with Peter Friz, Sebastian Riedel and John Schoenmakers I have studied numerical approximation of the SDE (2) when the driving noise is a fractional Brownian motion. In this case (at least in the "rougher than Brownian motion" regime $H < \frac{1}{2}$), (2) is actually much more natural than (1), since it allows us to interpret the solution in the rough path sense ([37, 30]), at least when the Hearst parameter satisfies $H > \frac{1}{4}$. In that case, the fractional Brownian motion can be extended to an enhanced fractional Brownian motion $\mathbf{B}$, which is a rough path (with its iterated integrals of degree up to three). Thus, the theory of rough paths can be used to give a canonical meaning to the solution of (2) (as solution to a rough differential equation evaluated at the enhanced fractional Brownian motion path). Moreover, it also directly gives approximation schemes (in a.s. sense) together with rates, see [27] and [30]. However, this scheme depends on the increments of the iterated increments of the noise – similar to Milstein's scheme for standard SDEs. Deya, Neuenkirch and Tindel [29] have given a computable version of the scheme, where the increments of the iterated integrals are themselves approximated by polynomials in the increments of B – which can be interpreted as the Davie scheme applied to the Wong-Zakai approximation of (2). They prove a.s. rates when $1/3 < H$, but cannot prove strong convergence. Indeed, the (random) constants in their a.s. convergence results are not integrable. Using recent deep results of [25], we improve the results from [29] to $H > 1/4$. More importantly, we prove strong convergence with strong rates (being equal to $2H - \frac{1}{2}$ up to a logarithmic term) of the scheme. To overcome the rather low convergence rates (which can be observed in numerical experiments), we accompany the scheme with the multi-level Monte Carlo method of Giles [32], which allows to reduce the complexity needed to achieve a mean squared error of $O(\epsilon^2)$ by a factor $O(\epsilon^{-2})$, thereby producing an efficient numerical scheme when H is not too small.

Optimal choice of numerical parameters

When calculating quantities like (3) on a computer, the user has essentially two free numerical parameters, which control both the accuracy and the run-time of the algorithm, namely the grid of the discretization of the SDE and the number of samples in the Monte Carlo or quasi Monte Carlo integration procedure. In most situations, the user would like to minimize the computational cost given a certain, prescribed error tolerance – the situation might be a bit more complicated for Monte Carlo algorithms due to their stochastic nature. An optimal choice of the grid should be based on a-posteriori error estimates by introducing adaptive refinements of an initial grid. Building up on the methods in [44], I have worked on this method for the Euler scheme for reflected SDEs together with Anders Szepessy and Raul Tempone, see [14]. Moreover, in a more recent work with Håkon Hoel, Erik von Schwerin and Raul Tempone, I have proposed an algorithm on efficient, again adaptive choices of the number of samples for a Monte Carlo algorithm, see [9]. In the future, I would like to extend these ideas, in particular adaptive grid construction based on a-posteriori error estimates to cubature on Wiener space. If successful, this would open up the possibility of very high accuracy calculations for problem of type (3) even in high dimensions and even in the presence of singularities of the coefficients.

Approximation formulas for option prices and hedges

Consider a local volatility model in finance. This means that we are given a model of the form (1), where $d = n$ and $(V_1, \ldots, V_d) =: \sigma$ has the property that $\sigma_{ij}(x) = f_i(x_i)\delta_{ij}$. Hence, the individual components of the stock price vector X only depend on each other via the correlation matrix ρ of the n-dimensional Brownian motion B. These models are very popular in finance due to their simplicity – a prominent example being the so-called CEV models where $f_i(x_i) = \xi_i x_i^{\beta_i}$ with $0 < \beta_i \leq 1$. However, even for such models, calculations of basket or spread option prices becomes numerically extremely challenging when the dimension n is high. In practice, $n = 100$ or $n = 500$ are quite common (e.g., S&P 100 and 500 indices), and then basically only Monte Carlo methods can be used any more. Together with Peter Laurence I have used the technique of heat kernel expansions, see for instance [40, 46, 41, 34, 45, 26, 21, 1] to derive an asymptotic expansion for the density of the process X_T (asymptotic in the sense of $T \to 0$). Then we applied Laplace's expansion (again for $T \to 0$) together with a Carr-Jarrow (see [24]) formula for baskets to obtain highly accurate, but yet essentially explicit formulas, which allows for almost instant evaluation [10, 11]. Moreover, we have also applied this method to calculate the greeks for the option ([12]). Together with Peter Friz we have discussed problems arising from the Laplace's expansion when integrating over a sub-manifold of the space, see [3].

There are a number of exciting future research problems in the context of this project. A very hot topic in financial engineering at the moment is the so-called "correlation skew", i.e., the dependence of the correlation between different stocks on the overall dynamics of the whole basket of stocks (or more generally on the vector of all individual stocks). Hence, I would like to try to extend the above asymptotic formula to the case of correlation skew. On a larger scale, a big open problem in the field of asymptotic formulas in finance is how to handle the singularities that naturally appear at the boundary of the domain – in the above basket case: when one of the individual stock prices vanishes. Special, low-dimensional cases have been dealt with in a rigorous manner, but in general one relies on "the principle of not feeling the boundary", i.e., one assumes that the initial price vector is far enough away from the boundary and that the boundary is (consequently) only hit with extremely low probability during the life-time of the option. A rigorous treatment of the singularities is not only desirable from a theoretical point of view, but also from a practical point of view as many asymptotic formulas will loose accuracy when the initial price vector is, in fact, close to the boundary, a quite common situation in interest rate model, for instance. Moreover, asymptotic methods as discussed above are promising as a building block for the numerical treatment of American options (or other stochastic control problems) in high dimensions.

Simulation of diffusion bridges

In a recent research project with John Schoenmakers, we have considered the problem of computing quantities of the form

$$E\left[g\left(X_{t_1}, \dots, X_{t_n}\right) \middle| X_0 = x, X_T = y \right], \tag{5}$$

where X is a diffusion process given by (1). This is a classical problem, for which many different methods have been proposed [22, 28, 33, 23]. Many methods work well in dimension 1, but have difficulties in higher dimensions. We have extended the forward-reverse method of Milstein et al. [38, 39] for the problem at hand and show that this method can, in fact, avoid the curse of dimensionality and is efficient in the sense that it achieves the optimal Monte Carlo rate of convergence with complexity only logarithmically larger than usual, see [13]. This method has many applications, which I plan to study further in the future, for instance for maximum likelihood estimation of parameters of the dynamics of the process X when X is observed at a discrete time grid, and is actually not restricted to diffusion processes, but can be applied to more general classes of Markov processes, such as discrete and continuous time Markov chains.

References

[1] Robert Azencott. Densité des diffusions en temps petit: développements asymptotiques. I. In *Seminar on probability, XVIII*, volume 1059 of *Lecture Notes in Math.*, pages 402–498. Springer, Berlin, 1984.

[2] Christian Bayer. *Selected Topics in Numerics of Stochastic Differential Equations*. PhD thesis, TU Vienna, 2008.

[3] Christian Bayer, Peter Friz, and Peter Laurence. On the probability density function of baskets. Preprint, 2013.

[4] Christian Bayer, Peter Friz, and Ronnie Loeffen. Semi-closed form cubature and applications to financial diffusion models. *Quant. Finance*, 13(5):769–782, 2013.

[5] Christian Bayer, Peter Friz, Sebastian Riedel, and John Schoenmakers. From rough-path estimates to multi-level monte carlo. Preprint, 2013.

[6] Christian Bayer and Peter K. Friz. Cubature on Wiener space: pathwise convergence. *Appl. Math. Optim.*, 67(2):261–278, 2013.

[7] Christian Bayer, Jim Gatheral, and Morten Karlsmark. Fast ninomiya-victoir calibration of the double-mean-reverting model. *Quantitative Finance*, 2013. Available online.

[8] Christian Bayer, Håkon Hoel, Petr Plecháč, Anders Szepessy, and Raul Tempone. How accurate is molecular dynamics? Preprint, 2011.

[9] Christian Bayer, Håkon Hoel, Erik von Schwerin, and Raul Tempone. On non-asymptotic optimal stopping criteria for monte carlo simulation. Preprint, 2013.

[10] Christian Bayer and Peter Laurence. Asymptotics beats monte carlo: The case of correlated cev baskets. *Comm. Pure Appl. Math.*, 2013. Available online.

[11] Christian Bayer and Peter Laurence. Asymptotics for at the money local vol basket options. Preprint, 2013.

[12] Christian Bayer and Peter Laurence. Calculation of greeks for basket options. Working paper, 2013.

[13] Christian Bayer and John Schoenmakers. Simulation of conditional diffusions via forward-reverse stochastic representations. Ann. Appl. Prob., to appear.

[14] Christian Bayer, Anders Szepessy, and Raúl Tempone. Adaptive weak approximation of reflected and stopped diffusions. *Monte Carlo Methods Appl.*, 16(1):1–67, 2010.

[15] Christian Bayer and Josef Teichmann. The proof of Tchakaloff's theorem. *Proc. Amer. Math. Soc.*, 134(10):3035–3040, 2006.

[16] Christian Bayer and Josef Teichmann. Cubature on Wiener space in infinite dimension. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 464(2097):2493–2516, 2008.

[17] Christian Bayer, Josef Teichmann, and Richard Warnung. An implementation of hypo-elliptic simulated annealing. Preprint, 2008.

[18] Christian Bayer and Bezirgen Veliyev. Utility maximization in a binomial model with transaction costs: A duality approach based on the shadow price process. Preprint, 2013.

[19] Christian Bayer and Klaus Wälde. The dynamics of distributions in continuous-time stochastic models. Preprint, 2012.

[20] Christian Bayer and Klaus Wälde. Optimal saving in frictional labour markets. Preprint, 2012.

[21] G. Ben Arous. Développement asymptotique du noyau de la chaleur hypoelliptique hors du cut-locus. *Ann. Sci. École Norm. Sup. (4)*, 21(3):307–331, 1988.

[22] Alexandros Beskos, Gareth Roberts, Andrew Stuart, and Jochen Voss. MCMC methods for diffusion bridges. *Stoch. Dyn.*, 8(3):319–350, 2008.

[23] Mogens Bladt and Michael Sørensen. Simple simulation of diffusion bridges with application to likelihood inference for diffusions. Preprint, 2012.

[24] Peter P. Carr and Robert A. Jarrow. The stop-loss start-gain paradox and option valuation: A new decomposition into intrinsic and time value. *The Review of Financial Studies*, 3(3):469–492, 1990.

[25] Thomas Cass, Christian Litterer, and Terry Lyons. Integrability estimates for Gaussian rough differential equations. *Ann. of Prob.*, 41(4):3026–3050, 2013.

[26] Isaac Chavel. *Eigenvalues in Riemannian geometry*, volume 115 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1984.

[27] A. M. Davie. Differential equations driven by rough paths: an approach via discrete approximation. *Appl. Math. Res. Express. AMRX*, (2):Art. ID abm009, 40, 2007.

[28] B. Delyon and Y. Hu. Simulation of conditioned diffusion and application to parameter estimation. *Stochastic Process. Appl.*, 116(11):1660–1675, 2006.

[29] A. Deya, A. Neuenkirch, and S. Tindel. A Milstein-type scheme without Lévy area terms for SDEs driven by fractional Brownian motion. *Ann. Inst. Henri Poincaré Probab. Stat.*, 48(2):518–550, 2012.

[30] Peter K. Friz and Nicolas B. Victoir. *Multidimensional stochastic processes as rough paths*, volume 120 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

[31] Jim Gatheral. Consistent modeling of SPX and VIX options. Presentation at Bachelier Congress: London, 2008.

[32] Michael B. Giles. Multilevel Monte Carlo path simulation. *Oper. Res.*, 56(3):607–617, 2008.

[33] Martin Hairer, Andrew M. Stuart, and Jochen Voss. Sampling conditioned hypoelliptic diffusions. *Ann. Appl. Probab.*, 21(2):669–698, 2011.

[34] Pierre Henry-Labordère. *Analysis, geometry, and modeling in finance*. Chapman & Hall/CRC Financial Mathematics Series. CRC Press, Boca Raton, FL, 2009.

[35] Shigeo Kusuoka. Approximation of expectation of diffusion processes based on Lie algebra and Malliavin calculus. In *Advances in mathematical economics. Vol. 6*, volume 6 of *Adv. Math. Econ.*, pages 69–83. Springer, Tokyo, 2004.

[36] Terry Lyons and Nicolas Victoir. Cubature on Wiener space. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 460(2041):169–198, 2004.

[37] Terry J. Lyons. Differential equations driven by rough signals. *Rev. Mat. Iberoamericana*, 14(2):215–310, 1998.

[38] G. N. Milstein, J. Schoenmakers, and V. Spokoiny. Transition density estimation for stochastic differential equations via forward-reverse representations. *Bernoulli*, 10(2):281–312, 2004.

[39] G. N. Milstein, J. Schoenmakers, and V. Spokoiny. Forward and reverse representations for Markov chains. *Stochastic Process. Appl.*, 117(8):1052–1075, 2007.

[40] S. Minakshisundaram and Å. Pleijel. Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds. *Canadian J. Math.*, 1:242–256, 1949.

[41] S. A. Molčanov. Diffusion processes, and Riemannian geometry. *Uspehi Mat. Nauk*, 30(1(181)):3–59, 1975.

[42] Mariko Ninomiya and Syoiti Ninomiya. A new higher-order weak approximation scheme for stochastic differential equations and the Runge-Kutta method. *Finance Stoch.*, 13(3):415–443, 2009.

[43] Syoiti Ninomiya and Nicolas Victoir. Weak approximation of stochastic differential equations and application to derivative pricing. *Appl. Math. Finance*, 15(1-2):107–121, 2008.

[44] Anders Szepessy, Raúl Tempone, and Georgios E. Zouraris. Adaptive weak approximation of stochastic differential equations. *Comm. Pure Appl. Math.*, 54(10):1169–1214, 2001.

[45] S. R. S. Varadhan. Diffusion processes in a small time interval. *Comm. Pure Appl. Math.*, 20:659–685, 1967.

[46] Kôsaku Yosida. On the fundamental solution of the parabolic equation in a Riemannian space. *Osaka Math. J.*, 5:65–74, 1953.