Adaptive weak approximation of reflected diffusions

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Outline

The Skorohod problem

Given a domain $D \subset \mathbb{R}^d$, consider an SDE reflected at the boundary ∂D.

$$\begin{cases} dX_t^x = V(X_t^x)dt + \sum_{i=1}^{d} V_i(X_t^x)dB_t^i + n(X_t^x)dZ_t^x, \\ X_0^x = x \in D, \quad Z_0^x = 0. \end{cases} \tag{1}$$

- $B_t = (B_t^1, \dots, B_t^d)$ d-dimensional Brownian motion
- $V, V_1, \dots, V_d : \overline{D} \to \mathbb{R}^d$ vector fields
- $X_t^x \in \overline{D}$, $n(X_t^x)$ inward pointing normal vector at $X_t^x \in \partial D$
- Z_t^x increasing process with $dZ_t^x = \mathbf{1}_{\partial D}(X_t^x)dZ_t^x$ ("*local time*")

Kolmogorov equation

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = -Lu(t,x), \quad (t,x) \in [0,T] \times D, \\ u(T,x) = f(x), \quad x \in D, \\ \frac{\partial}{\partial n} u(t,x) = h(x), \quad x \in \partial D. \end{cases} \tag{2}$$

L is the infinitesimal generator of the SDE.

Proposition

Under suitable regularity condition, the solution of (2) has the stochastic representation

$$u(t,x) = E\left[f(X_T) - \int_t^T h(X_s) dZ_s \,\middle|\, X_t = x \right]. \tag{3}$$

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The Euler scheme for reflected diffusions

Given a partition $0 = t_0 < t_1 < \ldots < t_N$, let $\Delta t_i = t_{i+1} - t_i$, $\Delta B_i = B_{t_{i+1}} - B_{t_i}$.

Algorithm

1. *Set* $\overline{X}_0 = x$, $\overline{Z}_0 = 0$, $i = 0$.
2. $\hat{X}_{i+1} = \overline{X}_i + V(\overline{X}_i)\Delta t_i + \sum_{j=1}^{d} V_j(\overline{X}_i)\Delta B_i^j$
3. *If* $\hat{X}_{i+1} \in \overline{D}$ *set*
$$\overline{X}_{i+1} = \hat{X}_{i+1}, \quad \overline{Z}_{i+1} = \overline{Z}_i,$$
else set
$$\overline{X}_{i+1} = \Pi(\hat{X}_{i+1}), \quad \overline{Z}_{i+1}$$
4. *If* $i < N - 1$ *increase* i *by* 1 *and return to (1), else stop.*

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The Euler scheme for reflected diffusions – 2

- Order of convergence: $u(0,x) - \overline{u}(0,x) = \mathcal{O}(\sqrt{\Delta t_{\mathrm{max}}})$
- Algorithm by Gobet [2001] based on approximation of domain by half-spaces (order 1 for particular direction of reflection)
- Algorithm by Bossy, Gobet, Talay [2004] with order 1 for $h \equiv 0$

Turning adaptive

- Goal: given a tolerance level TOL, minimize the computational work such that the computational error remains smaller than TOL.
- Concentrate *only* on the error of the time discretization of the SDE, not the integration error
- Adaptive algorithms for SDEs (without reflection) given by Szepessy, Tempone and Zouraris [2001]; we concentrate on the error introduced by the reflection.

The minimization problem

- The error representation has leading order term

$$E\left[\sum_{i=0}^{N-1} \Delta\overline{Z}_i^2 \left|\frac{\partial^2}{\partial n^2} u(t_{i+1}, \overline{X}_{i+1})\right|\right]. \qquad (4)$$

- Corresponding work given by $\widetilde{N}$, the number of hits at the boundary. Use

$$E\left[\widetilde{N}\right] \approx E\left[\sum_{i=0}^{N-1} \frac{\Delta\overline{Z}_i}{\sqrt{\Delta t_i}}\right]. \qquad (5)$$

- Problem: minimize (5) given that (4) is bounded by TOL among the set of partitions of $[0, T]$.
- Solve by Lagrangian relaxation.

The refinement algorithm

Given a partition $t_0 = 0 < t_1 < \cdots < t_N = T$ and corresponding values for ΔB_i, $\Delta \overline{Z}_i$, $\overline{X}_i$, $i = 0, \dots, N-1$.

Algorithm

For $i = 0, \dots, N$ **do:**

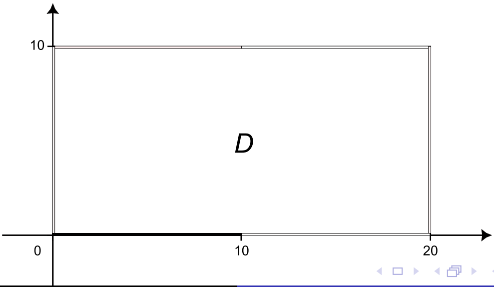
- *Compute*

$$\ell_i = \frac{1}{2\sqrt{2\pi}} E\left(\widetilde{N}\right) \left| \frac{\partial^2}{\partial n^2} v\left(t_i, \overline{X}_i\right) \right| \exp\left(-\frac{d\left(\overline{X}_i\right)^2}{2\Delta t_i} \right) \Delta t_i .$$

- *If* $\ell_i > \mathrm{TOL}$ *refine the mesh by inserting* $t = \frac{t_{i+1}+t_i}{2}$ *in the partition. Sample the corresponding value of the Brownian motion* B_t *using the Brownian bridge.*

A mixed boundary condition

$$
\begin{cases}
\frac{\partial}{\partial t} u(t,x) = -\frac{1}{2}\Delta u(t,x), & (t,x) \in [0,2] \times D, \\
u(2,x) = 10 \exp\left(-\sqrt{(10-x_1)^2 + x_2^2}\right), & x \in D, \\
\frac{\partial}{\partial n} u(t,x) = x_1, & (t,x) \in [0,2] \times D_N, \\
u(t,x) = 10 \exp\left(-\sqrt{(10-x_1)^2 + x_2^2}\right), & x \in \times(\partial D \setminus D_N).
\end{cases}
$$



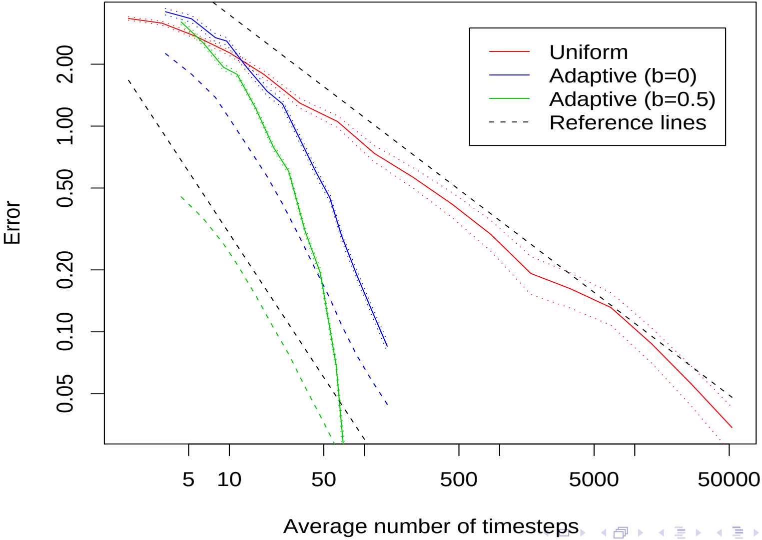
Implementation

- Combine the adaptive Euler scheme for reflected diffusions with the adaptive Euler scheme for stopped diffusions by Dzougoutov et al. [2005].
- Approximate

$$\left| \frac{\partial^2}{\partial n^2} v(t_i, \overline{X}_i) \right| \approx \frac{1}{\left\| \overline{X}_i - x_{\text{sing}} \right\|^{\beta} + \text{TOL}^{\alpha}},$$

  where $x_{\text{sing}} = (10, 0)$.
- Used $\alpha = 2$ and $\beta = 0, 1/2$.

Result


Uniform
Adaptive (b=0)
Adaptive (b=0.5)
Reference lines
Error
2.00
1.00
0.50
0.20
0.10
0.05
5
10
50
500
5000
50000
Average number of timesteps

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- C. Costantini, B. Pacchiarotti and F. Sartoretto. *Numerical approximation for functionals of reflecting diffusion processes.* SIAM J. Appl. Math. 58(1), 1998.
- A. Dzougoutov, K.-S. Moon, E. von Schwerin, A. Szepessy and R. Tempone. *Adaptive Monte-Carlo algorithms for stopped diffusion.* in Lect. Notes Comp. Sci. Eng. 44, 2005.
- E. Gobet. *Euler schemes and half-space approximation for the simulation of diffusion in a domain.*, ESAIM Prob. Statist. 5, 2001.