Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz
Association

Computational finance – Lecture 14

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Outline

- Consider $G \subset \mathbb{R}^d$ open, bounded, $T > 0$. Let $\nabla v(t,x) := (\partial_{x_1} v(t,x), \ldots, \partial_{x_d} v(t,x)) \in \mathbb{R}^d$.

$$
\begin{aligned}
\partial_t u(t,x) - \nabla \cdot (a(t,x)\nabla u(t,x)) + b(t,x) \cdot \nabla u(t,x) + c(t,x)u(t,x) = f(t,x), &\quad x \in G,\ 0 < t \leq T,\\
u(t,x) = 0, &\quad x \in \partial G,\ 0 \leq t \leq T,\\
u(0,x) = u_0(x), &\quad x \in \overline{G}.
\end{aligned}
$$

Assumptions

- $a \in L^\infty\left((0,T] \times G; \mathbb{R}^{d\times d}\right)$, $b \in W^1_\infty\left((0,T] \times G; \mathbb{R}^d\right)$, $c \in L^\infty((0,T] \times G)$, $f \in L^2((0,T] \times G)$, $u_0 \in L^2(G)$
- $c(t,x) - \frac{1}{2}\sum_{i=1}^d \partial_{x_i} b_i(t,x) \geq 0$, $0 < t \leq T$, $x \in G$.
- Uniform ellipticity: there is $\widetilde{c} > 0$ s.t. $\forall (t,x) \in (0,T] \times G,\ \forall \xi \in \mathbb{R}^d :\ a(t,x)\xi \cdot \xi \geq \widetilde{c}\,|\xi|^2$.

There is a unique solution $u \in L^2\left([0,T]; H^1_0(G)\right)$ with $\partial_t u \in L^2\left([0,T]; H^{-1}(G)\right)$, i.e.,

$$
\forall 0 < t \leq T,\ \forall v \in H^1_0(G) :\ \langle \partial_t u(t,\cdot), v\rangle_{H^{-1}(G);H^1_0(G)} + A(u,v;t) = L(v;t), \quad u(0,\cdot) = u_0.
$$

Theorem

Let $K := \widetilde{c}/c_*(G)$ *– from the Poincaré–Friedrichs inequality. Then*

$$\|u(t,\cdot)\|_{L^2}^2 \leq \mathrm{e}^{-Kt}\,\|u_0\|_{L^2}^2 + \frac{1}{K}\int_0^t \mathrm{e}^{-K(t-s)}\,\|f(s,\cdot)\|_{L^2}^2\,\mathrm{d}s.$$

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- Note that energy dissipation implies uniqueness of solutions: If u_1 and u_2 are solutions, than $u_1 - u_2$ solves the PDE with initial value 0 and source term 0.

Outline

1 Parabolic problems

2 Euler schemes

One-dimensional heat equation: $G = (0,1)$, $a_{ij} \equiv \delta_{ij}$, $b \equiv 0$, $c \equiv 0$, i.e.,

$$\partial_t u(t,x) = \partial_x^2 u(t,x) + f(t,x), \quad u(t,0) = u(t,1) = 0, \quad u(0,x) = u_0(x)$$

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- Uniform grids $x_j := jh$, $0 \le j \le N+1$, $t^m := m\Delta t$, $0 \le m \le M$, V_h as before.
- Notation: for $v = v(t,x)$ denote $v^m := v(t^m, \cdot)$. As before, $A(v,w) := \int_0^1 v'(x)w'(x)\mathrm{d}x$.

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Forward Euler scheme

Find $u_h^m \in V_h$, $m = 0, \ldots, M$, s.t. for every $v \in V_h$, $m = 0, \ldots, M-1$:

$$\left\langle \frac{u_h^{m+1} - u_h^m}{\Delta t}, v \right\rangle_{L^2} + A(u_h^m, v) = \langle f^m, v \rangle_{L^2}, \quad \left\langle u_h^0 - u_0, v \right\rangle_{L^2} = 0,$$

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$$\left\langle u_h^{m+1}, v \right\rangle_{L^2} = \left\langle u_h^m, v \right\rangle_{L^2} - \Delta t A(u_h^m, v) + \Delta t \langle f^m, v \rangle_{L^2}, \quad \left\langle u_h^0, v \right\rangle = \langle u_0, v \rangle.$$

Backward Euler scheme

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θ-schemes

For $0 \leq \theta \leq 1$ and $v = v(t, x)$ define $v^{m+\theta} := \theta v(t^{m+1}, \cdot) + (1-\theta) v(t^m, \cdot)$.

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- Note that $\theta = 0$ corresponds to forward Euler, $\theta = 1$ to backward Euler.

Theorem

For $1/2 \le \theta \le 1$ *the* θ*–scheme is unconditionally stable, i.e.,*

$$\max_{1\le m\le M} \left\|u_h^m\right\|_{L^2}^2 \le \left\|u_h^0\right\|_{L^2}^2 + \Delta t \sum_{m=0}^{M-1} \left\|f^{m+\theta}\right\|_{L^2}^2 .$$

For $0 \le \theta < 1/2$ *the* θ*-scheme is stable provided that for some* $0 < \epsilon < 1$ *we have* $\Delta t \le \frac{h^2}{6(1-2\theta)}(1-\epsilon)$*. In this case,*

$$\max_{1\le m\le M} \left\|u_h^m\right\|_{L^2}^2 \le \left\|u_h^0\right\|_{L^2}^2 + c_\epsilon \Delta t \sum_{m=0}^{M-1} \left\|f^{m+\theta}\right\|_{L^2}^2 , \quad c_\epsilon := \frac{1}{4\epsilon^2} + \Delta t(1-2\theta)(1+1/\epsilon).$$

Proof of stability in the case $1/2 \leq \theta \leq 1$

Theorem

Let u_h^m denote the solution of the backward Euler scheme. Assume that $\left\|\partial_t^2 u(t,\cdot)\right\|_{L^2} < \infty$ and $\|\partial_t u(t,\cdot)\|_{H^2} < \infty$. Then there is a constant C (depending on u) s.t.

$$\max_{m=1,\ldots,M} \left\|u^m - u_h^m\right\|_{L^2} \leq C(\Delta t + h^2).$$

- Under similar assumptions, for the Crank–Nicolson scheme ($\theta = 1/2$) we have

$$\max_{m=1,\ldots,M} \left\|u^m - u_h^m\right\|_{L^2} \leq C(\Delta t^2 + h^2).$$

Proof