Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz
Association

Computational finance – Lecture 13

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1 Weak solutions

2 Finite element method for elliptic problems

3 Parabolic problems

4 Euler schemes

Consider the elliptic model problem:

$$-\sum_{i,j=1}^{d} \partial_{x_j}\left(a_{ij}(x)\partial_{x_i}u(x)\right) + \sum_{i=1}^{d} b_i(x)\partial_{x_i}u(x) + c(x)u(x) = f(x), \quad x \in G, \quad u(x)|_{\partial G} \equiv 0$$

Conditions

- $a_{ij},\, c \in L^\infty(G),\, b_i \in W^1_\infty(G),\, f \in L^2(G),\, c(\cdot) - \frac{1}{2}\sum_{i=1}^{d} \partial_{x_i} b_i(\cdot) \geq 0.$
- Uniform ellipticity: $\exists C > 0\ \forall \xi \in \mathbb{R}^d,\, x \in \overline{G} : a(x)\xi \cdot \xi \geq C\,|\xi|^2.$

Definition

$u \in H^1_0(G)$ is called weak solution of the PDE, iff for all $v \in H^1_0(G)$ we have

$$\sum_{i,j=1}^{d} \int_G a_{ij}(x)\,\partial_{x_i}u(x)\,\partial_{x_j}v(x)\,\mathrm{d}x + \sum_{i=1}^{d} \int_G b_i(x)\,\partial_{x_i}u(x)\,v(x)\mathrm{d}x + \int_G c(x)u(x)v(x)\mathrm{d}x = \int_G f(x)v(x)\mathrm{d}x.$$

$$A(u,v) = L(v).$$

Theorem (Lax–Milgram lemma)

Let V be a real Hilbert space with norm $\|\cdot\|_V$ and inner product $\langle\cdot,\cdot\rangle_V$. Assume that the bi-linear functional $A : V \times V \to \mathbb{R}$ and the linear functional $L : V \to \mathbb{R}$ satisfy:

(i) *[A is symmetric;]*

(ii) A is *elliptic*, i.e., $\exists \alpha > 0\ \forall v \in V :\ A(v,v) \geq \alpha \|v\|_V^2$;

(iii) A is *continuous*, i.e., $\exists C > 0\ \forall v,w \in V :\ |A(v,w)| \leq C \|v\|_V \|w\|_V$;

(iv) L is *continuous*, i.e, $\exists \Lambda > 0\ \forall v \in V :\ |L(v)| \leq \Lambda \|v\|_V$.

Then there is a unique $u \in V$ such that $\forall v \in V :\ A(u,v) = L(v)$. Moreover, we have the a-priori estimate $\|u\|_V \leq \frac{\Lambda}{\alpha}$.

Corollary

The elliptic model PDE has a unique solution $u \in H_0^1$.

Lemma

Suppose that ∂G is sufficiently smooth. Then there is $c_* = c_*(G)$ s.t.

$$\forall u \in H_0^1(G): \; \|u\|_{L^2(G)}^2 \leq c_* \|\nabla u\|_{L^2(G)}^2 .$$

Proof of the Corollary

Outline

Recall that u solves $\forall v \in V : A(u, v) = L(v)$.

Given $V_h \subset V$, $\dim V_h < \infty$, let $u_h \in V_h$ denote the solution of $\forall v \in V_h : A(u_h, v) = L(v)$.

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- Lax–Milgram implies existence and uniqueness of u_h.
- Galerkin orthogonality: $\forall v \in V_h : A(u - u_h, v) = \langle u - u_h, v \rangle = 0$. u_h is the orthogonal projection w.r.t. the energy norm of u to V_h.

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Theorem (Lemma of Céa)

Let $\mathcal{R}_h : V \to V_h$ be a projection. Then $\|u - u_h\|_V \leq \sqrt{\frac{C}{\alpha}} \|u - \mathcal{R}_h u\|_V$.

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Theorem (Lemma of Céa)

Let $\mathcal{R}_h : V \to V_h$ be a projection. Then $\|u - u_h\|_V \le \sqrt{\frac{C}{\alpha}}\, \|u - \mathcal{R}_h u\|_V$.

- The error depends on regularity of u (a-priori bounds) and approximation properties of V_h. As the constant in $\|u - \mathcal{R}_h u\|_V$ is not known, a-posteriori bounds are useful.

Proof of the lemma of Céa

Fix $0 = x_0 < x_1 < \cdots < x_{N+1} = 1$, $h := \max_{i=0,\ldots,N} |x_{i+1} - x_i|$, and define

$$V_h := \left\{ v \in C([0,1]) \;\middle|\; \forall i \in \{0,\ldots,N\} : \; v|_{[x_i,x_{i+1}]} \text{ is affine}, \; v(0) = v(1) = 0 \right\}.$$

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- Obvious basis of V_h: tent functions $\phi_i(x_j) = \delta_{ij}$, $i = 1,\dots,N$, $j = 0,\dots,N+1$.
- $v \in V_h \Rightarrow v(x) = \sum_{i=1}^N v(x_i)\phi_i(x)$, $0 \le x \le 1$. Define a projection $\mathcal{R}_h : V = H_0^1((0,1)) \to V_h$ by
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Lemma

Assume that $v \in V$ *with* $v'' \in L^2((0,1))$. *Then* $\|v - \mathcal{R}_h v\|_V \le C \|v''\|_{L^2} h$.

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Lemma

Assume that $v \in V$ with $v'' \in L^2((0,1))$. Then $\|v - \mathcal{R}_h v\|_V \le C\,\|v''\|_{L^2}\, h$.

- By the Lemma of Céa, this implies $\|u - u_h\|_V \le Ch$, provided that $u'' \in L^2$.

Proof of the lemma

- Elliptic regularity: For instance, assume that $a \in C^1((0,1)) \cap L^\infty((0,1))$, $c \in L^\infty((0,1))$, $f \in L^2((0,1))$, then $u \in H_0^1((0,1)) \cap H^2((0,1))$.

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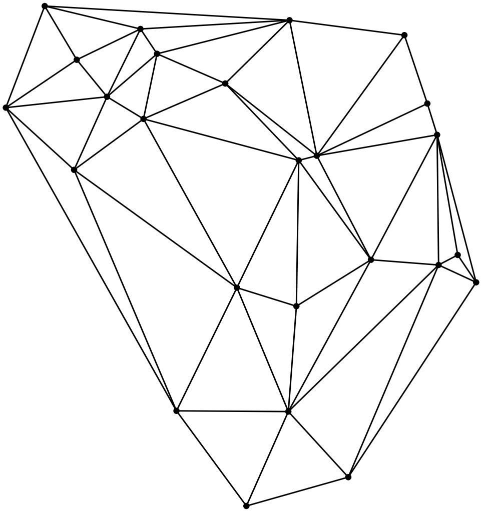
- Higher order finite elements can give higher order convergence rates, provided that there is enough regularity of the solution.

Remarks

- Elliptic regularity: For instance, assume that $a \in C^1((0,1)) \cap L^\infty((0,1))$, $c \in L^\infty((0,1))$, $f \in L^2((0,1))$, then $u \in H_0^1((0,1)) \cap H^2((0,1))$.
- Higher order finite elements can give higher order convergence rates, provided that there is enough regularity of the solution.
- Construction of grids in multi-dimensional domains much more challenging, in particular requiring discretization of the domain boundary.



- Under elliptic regularity, we have seen that $\|u - u_h\|_{H_0^1} \leq Ch$. By weakening the norm, the rate can be improved:

$$\|u - u_h\|_{L^2} \leq Ch^2.$$

- Consider $G \subset \mathbb{R}^d$ open, bounded, $T > 0$. Let $\nabla v(t,x) := (\partial_{x_1} v(t,x), \ldots, \partial_{x_d} v(t,x)) \in \mathbb{R}^d$.

$$\partial_t u(t,x) - \nabla \cdot (a(t,x)\nabla u(t,x)) + b(t,x) \cdot \nabla u(t,x) + c(t,x)u(t,x) = f(t,x), \quad x \in G,\ 0 < t \le T,$$

$$u(t,x) = 0, \quad x \in \partial G,\ 0 \le t \le T,$$

$$u(0,x) = u_0(x), \quad x \in \overline{G}.$$

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Assumptions

- $a \in L^\infty\left((0,T] \times G; \mathbb{R}^{d\times d}\right)$, $b \in W^1_\infty\left((0,T] \times G; \mathbb{R}^d\right)$, $c \in L^\infty\left((0,T] \times G\right)$, $f \in L^2\left((0,T] \times G\right)$, $u_0 \in L^2(G)$
- $c(t,x) - \frac{1}{2}\sum_{i=1}^d \partial_{x_i} b_i(t,x) \ge 0$, $0 < t \le T$, $x \in G$.
- Uniform ellipticity: there is $\widetilde{c} > 0$ s.t. $\forall (t,x) \in (0,T] \times G,\ \forall \xi \in \mathbb{R}^d:\ a(t,x)\xi \cdot \xi \ge \widetilde{c}|\xi|^2$.

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$$\begin{aligned} \partial_t u(t,x) - \nabla \cdot (a(t,x)\nabla u(t,x)) + b(t,x) \cdot \nabla u(t,x) + c(t,x)u(t,x) &= f(t,x), \quad x \in G,\ 0 < t \le T, \\ u(t,x) &= 0, \quad x \in \partial G,\ 0 \le t \le T, \\ u(0,x) &= u_0(x), \quad x \in \overline{G}. \end{aligned}$$

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There is a unique solution $u \in L^2\left([0,T]; H^1_0(G)\right)$ with $\partial_t u \in L^2\left([0,T]; H^{-1}(G)\right)$, i.e.,

$$\forall 0 < t \le T,\ \forall v \in H^1_0(G) :\ \langle \partial_t u(t,\cdot), v\rangle_{H^{-1}(G);H^1_0(G)} + A(u,v;t) = L(v;t), \quad u(0,\cdot) = u_0.$$

Theorem

Let $K := \widetilde{c}/c_*(G)$ *– from the Poincaré–Friedrichs inequality. Then*

$$\|u(t,\cdot)\|_{L^2}^2 \le \mathrm{e}^{-Kt}\,\|u_0\|_{L^2}^2 + \frac{1}{K}\int_0^t \mathrm{e}^{-K(t-s)}\,\|f(s,\cdot)\|_{L^2}^2\,\mathrm{d}s.$$

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- Note that energy dissipation implies uniqueness of solutions: If u_1 and u_2 are solutions, than $u_1 - u_2$ solves the PDE with initial value 0 and source term 0.

Outline

Forward Euler scheme

One-dimensional heat equation: $G = (0,1)$, $a_{ij} \equiv \delta_{ij}$, $b \equiv 0$, $c \equiv 0$, i.e.,

$$\partial_t u(t,x) = \partial_x^2 u(t,x) + f(t,x), \quad u(t,0) = u(t,1) = 0, \quad u(0,x) = u_0(x)$$

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- Uniform grids $x_j := jh$, $0 \le j \le N+1$, $t^m := m\Delta t$, $0 \le m \le M$, V_h as before.
- Notation: for $v = v(t,x)$ denote $v^m := v(t^m, \cdot)$. As before, $A(v,w) := \int_0^1 v'(x)w'(x)\mathrm{d}x$.

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Forward Euler scheme

Find $u_h^m \in V_h$, $m = 0, \ldots, M$, s.t. for every $v \in V_h$, $m = 0, \ldots, M-1$:

$$\left\langle \frac{u_h^{m+1} - u_h^m}{\Delta t}, v \right\rangle_{L^2} + A(u_h^m, v) = \langle f^m, v \rangle_{L^2}, \quad \left\langle u_h^0 - u_0, v \right\rangle_{L^2} = 0,$$

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Backward Euler scheme

Find $u_h^m \in V_h$, $m = 0, \ldots, M$, s.t. for every $v \in V_h$, $m = 0, \ldots, M-1$:

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θ-schemes

For $0 \le \theta \le 1$ and $v = v(t, x)$ define $v^{m+\theta} := \theta v(t^{m+1}, \cdot) + (1 - \theta) v(t^m, \cdot)$.

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- Note that $\theta = 0$ corresponds to forward Euler, $\theta = 1$ to backward Euler.

Theorem

For $1/2 \le \theta \le 1$ *the* θ*–scheme is unconditionally stable, i.e.,*

$$\max_{1\le m\le M} \left\|u_h^m\right\|_{L^2}^2 \le \left\|u_h^0\right\|_{L^2}^2 + \Delta t \sum_{m=0}^{M-1} \left\|f^{m+\theta}\right\|_{L^2}^2 .$$

For $0 \le \theta < 1/2$ *the* θ*-scheme is stable provided that for some* $0 < \epsilon < 1$ *we have* $\Delta t \le \frac{h^2}{6(1-2\theta)}(1-\epsilon)$*. In this case,*

$$\max_{1\le m\le M} \left\|u_h^m\right\|_{L^2}^2 \le \left\|u_h^0\right\|_{L^2}^2 + c_\epsilon \Delta t \sum_{m=0}^{M-1} \left\|f^{m+\theta}\right\|_{L^2}^2 , \quad c_\epsilon := \frac{1}{4\epsilon^2} + \Delta t(1-2\theta)(1+1/\epsilon).$$

Proof of stability in the case $1/2 \leq \theta \leq 1$

Theorem

Let u_h^m denote the solution of the backward Euler scheme. Assume that $\left\|\partial_t^2 u(t,\cdot)\right\|_{L^2} < \infty$ and $\|\partial_t u(t,\cdot)\|_{H^2} < \infty$. Then there is a constant C (depending on u) s.t.

$$\max_{m=1,\dots,M} \left\|u^m - u_h^m\right\|_{L^2} \leq C(\Delta t + h^2).$$

- Under similar assumptions, for the Crank–Nicolson scheme ($\theta = 1/2$) we have

$$\max_{m=1,\dots,M} \left\|u^m - u_h^m\right\|_{L^2} \leq C(\Delta t^2 + h^2).$$

Proof