Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz
Association

# Computational finance – Lecture 12

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- Let $G \subset \mathbb{R}^d$ open. Integration by parts: for $u \in C^k(G)$, $v \in C_0^\infty(G) := \{ f \in C^\infty(G) \mid \operatorname{supp} f \subset G \text{ and bounded} \}$, $\alpha \in \mathbb{N}^d$, $|\alpha| := \sum_{i=1}^d \alpha_i \leq k$, we have

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**Example**

Let $G = \mathbb{R}$, $u(x) := (1 - |x|)^+$. Then $u$ is weakly differentiable with weak derivative

$$u'(x) = \begin{cases} 0, & |x| > 1, \\ 1, & -1 < x < 0, \\ -1, & 0 < x < 1. \end{cases}$$

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**Example**

Let $u : \mathbb{R}^d \to \mathbb{R}$, $x \mapsto |x|^{-\alpha}$. If $\alpha + 1 < d$, then $u$ is weakly differentiable with derivative
$$\partial_{x_i} u(x) = -\alpha \frac{x_i}{|x|^{\alpha+2}} = \left(-\alpha \frac{x_i}{|x|}\right) \frac{1}{|x|^{\alpha+1}}.$$

## Definition (Sobolev space)

For $1 \le p < \infty$, $k \in \mathbb{N}$, define:

$$W_p^k(G) := \left\{ u \in L^p(G) \,\middle|\, \forall\, |\alpha| \le k : D^\alpha u \in L^p(G) \right\}, \; \|u\|_{W_p^k}^p := \sum_{|\alpha| \le k} \left\|D^\alpha u\right\|_{L^p}^p, \; H^k(G) := W_2^k(G).$$

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- Definition of $W_\infty^k(G)$ similar.
- $W_p^k(G)$ is a Banach space, $H^k(G)$ is a Hilbert space with $\langle u, v \rangle_{H^k} := \sum_{|\alpha| \leq k} \langle D^\alpha u, D^\alpha v \rangle_{L^2}$.

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- We define $H_0^k(G)$ as the closure in $\|\cdot\|_{H^k}$ of $C_0^\infty(G) \subset H^k(G)$. In a specific sense (of *traces*), this means that $H_0^k(G) = \left\{ u \in H^k(G) \,\middle|\, u|_{\partial G} = 0 \right\}$.

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- Alternative definitions in terms of Fourier transform allows extension of definition to $k \in \mathbb{R}$.
- Sobolev embedding theorems study embeddings of $W_p^k$ in $L^{p'}$, $W_q^m$, $C^{n,\alpha}$.

Consider the elliptic model problem:

$$-\sum_{i,j=1}^{d} \partial_{x_j}\Big(a_{ij}(x)\partial_{x_i}u(x)\Big) + \sum_{i=1}^{d} b_i(x)\partial_{x_i}u(x) + c(x)u(x) = f(x), \quad x \in G, \quad u(x)|_{\partial G} \equiv 0$$

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## Conditions

- $a_{ij},\, c \in L^\infty(G),\, b_i \in W^1_\infty(G),\, f \in L^2(G),\, c(\cdot) - \frac{1}{2}\sum_{i=1}^{d} \partial_{x_i} b_i(\cdot) \geq 0.$
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## Definition

$u \in H^1_0(G)$ is called weak solution of the PDE, iff for all $v \in H^1_0(G)$ we have

$$\sum_{i,j=1}^{d} \int_G a_{ij}(x)\,\partial_{x_i}u(x)\,\partial_{x_j}v(x)\,\mathrm{d}x + \sum_{i=1}^{d} \int_G b_i(x)\,\partial_{x_i}u(x)\,v(x)\mathrm{d}x + \int_G c(x)u(x)v(x)\mathrm{d}x = \int_G f(x)v(x)\mathrm{d}x.$$

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$$A(u,v) = L(v).$$

## Theorem (Lax–Milgram lemma)

*Let $V$ be a real Hilbert space with norm $\|\cdot\|_V$ and inner product $\langle\cdot,\cdot\rangle_V$. Assume that the bi-linear functional $A : V \times V \to \mathbb{R}$ and the linear functional $L : V \to \mathbb{R}$ satisfy:*

**(i)** *[$A$ is symmetric;]*

**(ii)** *$A$ is elliptic, i.e., $\exists\alpha > 0\ \forall v \in V :\ A(v,v) \geq \alpha\,\|v\|_V^2$ ;*

**(iii)** *$A$ is continuous, i.e., $\exists C > 0\ \forall v,w \in V :\ |A(v,w)| \leq C\,\|v\|_V\,\|w\|_V$ ;*

**(iv)** *$L$ is continuous, i.e, $\exists\Lambda > 0\ \forall v \in V :\ |L(v)| \leq \Lambda\,\|v\|_V$ .*

*Then there is a unique $u \in V$ such that $\forall v \in V :\ A(u,v) = L(v)$. Moreover, we have the a-priori estimate $\|u\|_V \leq \frac{\Lambda}{\alpha}$.*

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## Corollary

*The elliptic model PDE has a unique solution $u \in H_0^1$.*

**Lemma**

*Assume that the bi-linear form $A$ is symmetric and positive semi-definite, i.e.,*

$$\forall v, w \in V : \ A(v,w) = A(w,v), \quad \forall v \in V : \ A(v,v) \geq 0.$$

*Then $u \in V$ is a weak solution iff $u$ minimizes the functional $F(v) := \frac{1}{2}A(v,v) - L(v)$, $v \in V$.*

# Proof of the Theorem