Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Computational finance – Lecture 11

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$$dS_t = rS_t dt + \sigma S_t dW_t$$

European option pricing PDE

$$u(t,x) := E\left[e^{-r(T-t)} f(S_T) \mid S_t = x\right],$$

$$\frac{\partial}{\partial t}u(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}u(t,x) + rx\frac{\partial}{\partial x}u(t,x) - ru(t,x) = 0, \quad u(T,x) = f(x).$$

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American option pricing PDE (HJB equation in form of variational inequality)

$$v(t,x) := \sup_{\tau \in \mathcal{T}_{t,T}} E\left[e^{-r(\tau-t)} f(S_\tau) \mid S_t = x\right], \quad \mathcal{T}_{t,T} := \{ \text{Stopping times } t \le \tau \le T \},$$

$$\frac{\partial}{\partial t}v(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}v(t,x) + rx\frac{\partial}{\partial x}v(t,x) - rv(t,x) \le 0, \quad v(t,x) \ge f(x), \quad v(T,x) = f(x),$$

with equality in the PDE on $\{ x \mid v(t,x) > f(x) \}$.

A simplified European pricing PDE

$$\frac{\partial}{\partial t}u(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}u(t,x) + rx\frac{\partial}{\partial x}u(t,x) - ru(t,x) = 0, \quad u(T,x) = (K-x)^+$$

- Change of variables: $y := \log(x/K)$, $\tau := \frac{1}{2}\sigma^2(T-t)$, $q := 2r/\sigma^2$ and $\widetilde{u}(\tau,y) := \frac{1}{K}\exp\left(\frac{1}{2}(q-1)y + \left(\frac{1}{4}(q-1)^2 + q\right)\tau\right)u(t,x)$, satisfying the heat equation

$$\frac{\partial}{\partial \tau}\widetilde{u}(\tau,y) = \frac{\partial^2}{\partial y^2}\widetilde{u}(\tau,y), \quad \widetilde{u}(0,y) = \left(e^{\frac{1}{2}(q-1)y} - e^{\frac{1}{2}(q+1)y}\right)_+ .$$

$$\frac{\partial}{\partial t}u(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}u(t,x) + rx\frac{\partial}{\partial x}u(t,x) - ru(t,x) = 0, \quad u(T,x) = (K-x)^+$$

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- Boundary conditions: Natural boundary condition for put option at $y \to \infty$ and from put-call-parity:

$$\widetilde{u}(\tau,y) = \exp\left(\frac{1}{2}(q-1)y + \frac{1}{4}(q-1)^2\tau\right) \text{ for } y \to -\infty, \quad \widetilde{u}(\tau,y) = 0 \text{ for } y \to \infty$$

$$\frac{\partial}{\partial t}v(t,x)+\frac{1}{2}\sigma^2x^2\frac{\partial^2}{\partial x^2}v(t,x)+rx\frac{\partial}{\partial x}v(t,x)-rv(t,x)\le 0,\quad v(t,x)\ge (K-x)^+,\quad v(T,x)=(K-x)^+$$

The same transformation as above gives:

$$\left(\frac{\partial}{\partial \tau}\widetilde{v}(\tau,y)-\frac{\partial^2}{\partial y^2}\widetilde{v}(\tau,y)\right)(\widetilde{v}(\tau,y)-g(\tau,y))=0,\quad \frac{\partial}{\partial \tau}\widetilde{v}(\tau,y)-\frac{\partial^2}{\partial y^2}\widetilde{v}(\tau,y)\ge 0,$$

$$\widetilde{v}(\tau,y)\ge g(\tau,y),\quad \widetilde{v}(0,y)=g(0,y),$$

$$\widetilde{v}(\tau,y)=g(\tau,y) \text{ for } y\to-\infty,\quad \widetilde{v}(\tau,y)=0 \text{ for } y\to\infty,$$

where

$$g(y,\tau):=\exp\left(\frac{1}{4}(q+1)^2\tau\right)\left(e^{\frac{1}{2}(q-1)y}-e^{\frac{1}{2}(q+1)y}\right)^+$$

$$\frac{\partial}{\partial t}v(t,x)+\frac{1}{2}\sigma^2x^2\frac{\partial^2}{\partial x^2}v(t,x)+rx\frac{\partial}{\partial x}v(t,x)-rv(t,x) \le 0, \quad v(t,x) \ge (K-x)^+, \quad v(T,x) = (K-x)^+$$

The same transformation as above gives:

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$$\widetilde{v}(\tau,y) \ge g(\tau,y), \quad \widetilde{v}(0,y) = g(0,y),$$

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Outline

- Forward difference quotient: $f'(x) = \dfrac{f(x+h) - f(x)}{h} + \mathcal{O}(h)$

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Replace derivatives in time and space by finite difference quotients based on grids.

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Replace derivatives in time and space by finite difference quotients based on grids.

- Notation: $t_i := i\Delta t$, $i = 0, \dots, N$, $\Delta t := T/N$. $x_j := a + j\Delta x$, $j = 0, \dots, M$, $\Delta x := (b-a)/M$.
- Solving heat equation with appropriate boundary conditions on $[a, b]$, setting $u_{i,j} := u(t_i, x_j)$ – and similarly its FD approximation $\overline{u}_{i,j}$.

Based on the approximations:

$$\frac{\partial}{\partial t}u(t_i, x_j) = \frac{u_{i+1,j} - u_{i,j}}{\Delta t} + \mathcal{O}(\Delta t), \quad \frac{\partial^2}{\partial x^2}u(t_i, x_j) = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta x^2} + \mathcal{O}(\Delta x^2).$$

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Explicit FD scheme for the heat equation

With $\lambda := \frac{\Delta t}{\Delta x^2}$ set $\overline{u}_{i+1,j} := \overline{u}_{i,j} + \lambda(\overline{u}_{i,j+1} - 2\overline{u}_{i,j} + \overline{u}_{i,j-1})$, $i = 0, \ldots, N-1$, $j = 1, \ldots, M-1$.

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- Boundary and initial conditions for the European put option:

$$\overline{u}_{0,j} = \left(e^{\frac{1}{2}(q-1)x_j} - e^{\frac{1}{2}(q+1)x_j}\right)^+, \quad \overline{u}_{i+1,0} = \exp\left(\frac{1}{2}(q-1)a + \frac{1}{4}(q-1)^2 t_{i+1}\right), \quad \overline{u}_{i+1,M} = 0$$

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- Up to the boundary conditions,

$$\overline{u}_{i+1,:} = A(\lambda)\overline{u}_{i,:}, \quad A(\lambda) := \begin{pmatrix} 1-2\lambda & \lambda & 0 & \cdots & 0 \\ \lambda & 1-2\lambda & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \lambda \\ 0 & \cdots & 0 & \lambda & 1-2\lambda \end{pmatrix}.$$

Example (Instability of the explicit FD scheme)

Consider the heat equation with $u(0,x) = \sin(\pi x)$, $x \in [0,1]$, $u(t,0) \equiv u(t,1) \equiv 1$. Then $u(t,x) = \sin(\pi x)\mathrm{e}^{-\pi^2 t}$. Compute $u(0.5, 0.2)$:

- Explicit solution: $u(0.5, 0.2) = 0.004227$.

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- FD with $\Delta x = 0.1$, $\Delta t = 0.0005$: $u(0.5, 0.2) \approx \overline{u}_{1000,2} = 0.00435$.
- FD with $\Delta x = 0.1$, $\Delta t = 0.01$: $u(0.5, 0.2) \approx \overline{u}_{50,2} = -1.5 \times 10^8$.

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- The explicit FD scheme is prone to instability, i.e., explosive error propagation.
- $x \mapsto Ax$ is stable iff the spectral radius is bounded by 1. For $A(\lambda)$ this can be proved to be the case when $\lambda \leq 1/2$.

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Theorem

The explicit FD scheme converges is stable and converges when $\Delta t \leq \frac{1}{2}\Delta x^2$ (plus technical conditions). In this case, the error behaves like $\mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2)$.

Based on the approximations:

$$\frac{\partial}{\partial t}u(t_i, x_j) = \frac{u_{i,j} - u_{i-1,j}}{\Delta t} + \mathcal{O}(\Delta t), \quad \frac{\partial^2}{\partial x^2}u(t_i, x_j) = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta x^2}$$

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Implicit FD scheme for the heat equation

Define $\overline{u}_{i,:}$ as solution of the system $\overline{u}_{i-1,j} = \overline{u}_{i,j} + \frac{\Delta t}{\Delta x^2}\left(-\overline{u}_{i,j+1} + 2\overline{u}_{i,j} - \overline{u}_{i,j-1}\right)$, $i = 1, \ldots, N$, $j = 1, \ldots, M-1$.

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- Up to boundary conditions:

$$A\overline{u}_{i,:} = \overline{u}_{i-1,:}, \quad A := \begin{pmatrix} 1+2\lambda & -\lambda & 0 & \cdots & 0 \\ -\lambda & 1+2\lambda & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -\lambda \\ 0 & \cdots & 0 & -\lambda & 1+2\lambda \end{pmatrix}.$$

Based on the approximations:

$$\frac{\partial}{\partial t}u(t_i,x_j)=\frac{u_{i,j}-u_{i-1,j}}{\Delta t}+\mathcal{O}(\Delta t),\qquad \frac{\partial^2}{\partial x^2}u(t_i,x_j)=\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta x^2}$$

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Theorem

The implicit FD scheme is unconditionally stable and converges with $\mathcal{O}(\Delta t)+\mathcal{O}(\Delta x^2)$.

- Assume that the Cauchy problem is well-posed.
- Notation: u . . . solution of the PDE, u^i . . . u at time t_i discretized on the x-grid, $\overline{u}^i$ . . . FD approximation, given by $B_1\overline{u}^{i+1} = B_0\overline{u}^i + f^i$.

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- Consistency: $\left\|B_1 u^{i+1} - (B_0 u^i + f^i)\right\| \to 0$ as $\Delta t = T/N, \Delta x = (b-a)/M \to 0$, $i = 0, \ldots, N$.

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- Consistency: $\left\| B_1 u^{i+1} - (B_0 u^i + f^i) \right\| \to 0$ as $\Delta t = T/N, \Delta x = (b-a)/M \to 0$, $i = 0, \ldots, N$.
- Stability: there is a constant C s.t. $\left\| (B_1^{-1} B_0)^N \right\| \leq C$ uniformly in N. (Note: B_1, B_0 depend on N via $\Delta t, \Delta x$.)

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- Convergence: Consider $i(N)$ s.t. $t_i \to t$ as $N \to \infty$. Then $\left\|\overline{u}^{i(N)} - u^{i(N)}\right\| \to 0$ as $N \to \infty$.

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Theorem (Lax–Richtmyer; Lax equivalence principle; Fundamental theorem of numerical analysis)

For a consistent scheme, stability is equivalent to convergence, provided the problem is linear and well-posed.

$$\frac{\partial}{\partial t} u(t_{i+1}, x_j) = \frac{u_{i+1,j} - u_{i,j}}{\Delta t} + \mathcal{O}(\Delta t)$$

can be seen as forward or backward difference quotient, leading to

$$\frac{\overline{u}_{i+1,j} - \overline{u}_{i,j}}{\Delta t} = \frac{\overline{u}_{i,j+1} - 2\overline{u}_{i,j} + \overline{u}_{i,j-1}}{\Delta x^2} \quad \text{or} \quad \frac{\overline{u}_{i+1,j} - \overline{u}_{i,j}}{\Delta t} = \frac{\overline{u}_{i+1,j+1} - 2\overline{u}_{i+1,j} + \overline{u}_{i+1,j-1}}{\Delta x^2}$$

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Instead, take the mean of the right hand sides:

Crank–Nicolson scheme

Define $\overline{u}_{i+1,:}$ as solution of the system

$\overline{u}_{i+1,j} - \overline{u}_{i,j} = \frac{\Delta t}{2\Delta x^2} \left(\overline{u}_{i,j+1} - 2\overline{u}_{i,j} + \overline{u}_{i,j-1} + \overline{u}_{i+1,j+1} - 2\overline{u}_{i+1,j} + \overline{u}_{i+1,j-1} \right)$.

$$\frac{\partial}{\partial t} u(t_{i+1}, x_j) = \frac{u_{i+1,j} - u_{i,j}}{\Delta t} + \mathcal{O}(\Delta t)$$

can be seen as forward or backward difference quotient, leading to

$$\frac{\overline{u}_{i+1,j} - \overline{u}_{i,j}}{\Delta t} = \frac{\overline{u}_{i,j+1} - 2\overline{u}_{i,j} + \overline{u}_{i,j-1}}{\Delta x^2} \text{ or } \frac{\overline{u}_{i+1,j} - \overline{u}_{i,j}}{\Delta t} = \frac{\overline{u}_{i+1,j+1} - 2\overline{u}_{i+1,j} + \overline{u}_{i+1,j-1}}{\Delta x^2}$$

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Crank–Nicolson scheme

Define $\overline{u}_{i+1,:}$ as solution of the system
$\overline{u}_{i+1,j} - \overline{u}_{i,j} = \frac{\Delta t}{2\Delta x^2} \left(\overline{u}_{i,j+1} - 2\overline{u}_{i,j} + \overline{u}_{i,j-1} + \overline{u}_{i+1,j+1} - 2\overline{u}_{i+1,j} + \overline{u}_{i+1,j-1} \right)$.

- System of equations of the form $A\overline{u}_{i+1,:} = B\overline{u}_{i,:}$.
- Unconditionally stable and converges with error $\mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2)$.

$$\left(\frac{\partial}{\partial\tau}\widetilde{v}(\tau,y)-\frac{\partial^2}{\partial y^2}\widetilde{v}(\tau,y)\right)(\widetilde{v}(\tau,y)-g(\tau,y))=0,\quad \frac{\partial}{\partial\tau}\widetilde{v}(\tau,y)-\frac{\partial^2}{\partial y^2}\widetilde{v}(\tau,y)\geq 0,$$

$$\widetilde{v}(\tau,y)\geq g(\tau,y),\quad \widetilde{v}(0,y)=g(0,y),$$

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- Have to solve linear inequality systems of the form $Aw-b\geq 0$, $w\geq g$, $(aw-b)^\top(w-g)=0$ for the approximate solution w. Projection SOR (Successive over-relaxation algorithm).

$$\left(\frac{\partial}{\partial \tau}\widetilde{v}(\tau,y) - \frac{\partial^2}{\partial y^2}\widetilde{v}(\tau,y)\right)(\widetilde{v}(\tau,y) - g(\tau,y)) = 0, \quad \frac{\partial}{\partial \tau}\widetilde{v}(\tau,y) - \frac{\partial^2}{\partial y^2}\widetilde{v}(\tau,y) \geq 0,$$

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- Poor man's algorithm: Use standard FD iterations, but take maximum with payoff function at each iteration step.

Outline

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1. Variational (weak) formulation: u is the only element of $V := H_0^1$ such that for every test function $v \in V$:

$$A(u, v) = L(v), \quad A(u, v) := -\int_0^1 u'(x)v'(x)\,\mathrm{d}x, \quad L(v) := \int_0^1 f(x)v(x)\,\mathrm{d}x$$

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2. Projection onto finite dimensional space: Choose $V_h \subset V$, $\dim V_h < \infty$, $h > 0$, and consider the projected problem $\forall v \in V_h : \; A(u_h, v) = L(v)$, with solution $u_h \in V_h$. E.g.,
$$V_h := \left\{ v \in C([0,1]) \;\middle|\; v|_{[x_i,x_{i+1}]} \text{ affine}, i = 0,\dots,N,\; v(0) = v(1) = 0 \right\},\; x_i := ih,\; h := \frac{1}{N+1}.$$

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4. Solve the system: For $u_h = \sum_{i=1}^N \xi_i \phi_i$, $\overline{A}\xi = \overline{L}$, where $\overline{A}_{i,j} := A(\phi_i, \phi_j)$, $\overline{L}_i := L(\phi_i)$, $i, j = 1,\dots,N$.