Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Computational finance – Lecture 10

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June 18, 2021

Given a payoff $f : \mathbb{R}^d \to \mathbb{R}$ and a d-dimensional r.v. X. Define

- For $u \in \mathbb{R}^d$ – but extended to suitable $u \in \mathbb{C}^d$ – define the Fourier transform of f by

$$\widehat{f}(u) := \int_{\mathbb{R}^d} e^{i\langle u, x\rangle} f(x) dx.$$

- For $R \in \mathbb{R}^d$ define the dampened function $f_R(x) := e^{\langle R, x\rangle} f(x)$, $x \in \mathbb{R}^d$.
- Define the moment generating function $M_X(u) := E\left[e^{\langle u, X\rangle}\right]$, $u \in \mathbb{C}^d$ s.t. the expectation exists.

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- Define the admissible sets

$$\mathcal{I} := \left\{ R \in \mathbb{R}^d \;\middle|\; f_R \in L^1_{\mathrm{bc}}(\mathbb{R}^d) \text{ and } \widehat{f_R} \in L^1(\mathbb{R}^d) \right\}, \quad \mathcal{J} := \left\{ R \in \mathbb{R}^d \;\middle|\; M_X(R) < \infty \right\},$$
$$\mathcal{I}' := \left\{ R \in \mathbb{R}^d \;\middle|\; f_R \in L^1(\mathbb{R}^d) \right\}, \quad \mathcal{J}' := \left\{ R \in \mathbb{R}^d \;\middle|\; M_X(R) < \infty \text{ and } M_X(R - \mathrm{i}\cdot) \in L^1(\mathbb{R}^d) \right\}.$$

Theorem

Suppose that $R \in \mathcal{R} \cup \mathcal{R}'$, $\mathcal{R} := \mathcal{I} \cap \mathcal{J}$, $\mathcal{R}' := \mathcal{I}' \cap \mathcal{J}'$. *Then*

$$E\left[f(X)\right] = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} M_X(R - \mathrm{i}u)\widehat{f}(u + \mathrm{i}R)\mathrm{d}u.$$

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Examples of options in multi-dimensional assets include:

- Best of put options, with

$$f(x) = \left(K - \mathrm{e}^{x_1} \vee \mathrm{e}^{x_2} \vee \cdots \vee \mathrm{e}^{x_d}\right)^+, \quad \widehat{f}(z) = \frac{K^{1+\mathrm{i}\sum_{l=1}^d z_l}}{\left(1 + \mathrm{i}\sum_{l=1}^d z_l\right)\prod_{l=1}^d (\mathrm{i}z_l)}, \quad z \in \mathbb{C}^d, \ \Im z_l < 0$$

- Basket put options, with

$$f(x) = \left(K - \sum_{l=1}^d \mathrm{e}^{x_l}\right)^+, \quad \widehat{f}(z) = \frac{\prod_{l=1}^d \Gamma(\mathrm{i}z_l)}{\Gamma\left(\mathrm{i}\sum_{l=1}^d z_l + 2\right)}, \quad z \in \mathbb{C}^d, \ \Im z_l < 0$$

1. Typical models of the form $\log S_T = s + X_T$ for, say, affine X_T and $s := \log S_0$. Then,

$$E\left[f(X)\right] = \frac{1}{2\pi}\int_{\mathbb{R}} e^{(R-iu)s} M_{X_T}(R-iu)\widehat{f}(u+iR)du =: \frac{e^{Rs}}{2\pi}\int_{\mathbb{R}} e^{-ius}\psi(u)du.$$

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3. Discretize, e.g., by the trapezoidal rule with $u_\ell := a + \eta\ell$, $\ell = 0, \ldots, N-1$, $\eta := \frac{b-a}{N-1}$:

$$\int_a^b \mathrm{e}^{-\mathrm{i}us}\psi(u)\mathrm{d}u \approx \eta\left(\frac{\mathrm{e}^{-\mathrm{i}as}\psi(a)}{2} + \sum_{\ell=1}^{N-2} \mathrm{e}^{-\mathrm{i}u_\ell s}\psi(u_\ell) + \frac{\mathrm{e}^{-\mathrm{i}bs}\psi(b)}{2}\right) =: \eta\sum_{\ell=0}^{N-1} \mathrm{e}^{-\mathrm{i}u_\ell s}\widetilde{\psi}(u_\ell)$$

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4. For prices on a grid of s – or, equivalently, strike – values. $s_j := -\beta + \lambda j$, $j = 0, \ldots, N-1$:

$$\eta\mathrm{e}^{-\mathrm{i}a\lambda j}\sum_{\ell=0}^{N-1}\mathrm{e}^{-\mathrm{i}\eta\ell\lambda j}\mathrm{e}^{\mathrm{i}\beta u_\ell}\widetilde{\psi}(u_\ell), \qquad j = 0, \ldots, N-1.$$

5. Under the Nyquist relation $\lambda\eta = 2\pi/N$, we obtain

$$\Phi_j := \sum_{\ell=0}^{N-1} \mathrm{e}^{-\mathrm{i}\eta\ell\lambda j} \mathrm{e}^{\mathrm{i}\beta u_\ell} \widetilde{\psi}(u_\ell) = \sum_{\ell=0}^{N-1} \mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}\ell j} \phi_\ell, \quad \phi_\ell := \mathrm{e}^{\mathrm{i}\beta u_\ell} \widetilde{\psi}(u_\ell), \quad \ell, j = 0, \ldots, N-1.$$

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Definition (Discrete Fourier transform)

Fix $N \in \mathbb{N}$. Then $\mathrm{DFT} : \mathbb{C}^N \to \mathbb{C}^N$ is given by $\mathrm{DFT}(x)_j = \sum_{\ell=0}^{N-1} \mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}\ell j} x_\ell$, $j = 0, \ldots, N-1$.

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Lemma

Let $\omega_N := \mathrm{e}^{-2\pi\mathrm{i}/N}$. *Then* $\mathrm{DFT}(x) = T_N x$ *for* $T_N \in \mathbb{C}^{N\times N}$ *given by*

$$T_N := \begin{pmatrix} \omega_N^0 & \omega_N^0 & \omega_N^0 & \cdots & \omega_N^0 \\ \omega_N^0 & \omega_N^1 & \omega_N^2 & \cdots & \omega_N^{N-1} \\ \omega_N^0 & \omega_N^2 & \omega_N^4 & \cdots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_N^0 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \cdots & \omega_N^{(N-1)(N-1)} \end{pmatrix}.$$

Lemma

Let $x \in \mathbb{C}^{2N}$, $y := \mathrm{DFT}(x)$. *Denote* $x' := (x_1, x_3, \dots, x_{2N-1})$, $x'' := (x_2, x_4, \dots, x_{2N})$, $y' := (y_1, \dots, y_N)$, $y'' := (y_{N+1}, \dots, y_{2N})$, *and* $D_N := \mathrm{diag}\left(\omega_{2N}^0, \dots, \omega_{2N}^{N-1}\right) \in \mathbb{C}^{N \times N}$. *Then*

$$y' = T_N x' + D_N T_N x'', \quad y'' = T_N x' - D_N T_N x''.$$

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The lemma is the basis of a famous divide-and-conquer algorithm, the FFT:

Fast Fourier Transform

Given $x \in \mathbb{C}^N$ with $N = 2^J$ for some $J \in \mathbb{N}$. Recursively compute $\mathrm{DFT}(x)$ by:

1. If $N = 2$, go to 2. Otherwise, return $\left(\mathrm{DFT}(x') + D_N\,\mathrm{DFT}(x''), \mathrm{DFT}(x') - D_N\,\mathrm{DFT}(x'')\right)$.
2. Return $T_2 x$ – computed by ordinary matrix multiplication.

- This is the most basic version, variants exist for powers of 2, 4, 8 or even general N.

Lemma

For $N = 2^J$, let $\mathcal{W}(N)$ denote the work of computing the DFT of a vector $x \in \mathbb{C}^N$ by FFT. Denoting C the work for one floating point operation (addition, subtraction, multiplication), we have

$$\mathcal{W}(N) \leq C\left(\frac{3}{2}\log_2(N) + \frac{1}{2}\right)N.$$

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Definition (Fourier transform)

For $f \in L^1(G, \mu; \mathbb{C})$ we define $\widehat{f} \in C_b\left(\widehat{G}; \mathbb{C}\right)$ by $\widehat{f}(\chi) := \int_G f(x)\overline{\chi(x)}\mu(\mathrm{d}x)$.

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Example: $G = \mathbb{R}$

- $\mu = \lambda$ (Lebesgue measure)
- Characters: $\chi(x) = e^{iux}$, $u \in \mathbb{R}$, hence $\widehat{G} \simeq (\mathbb{R}, +)$.
- Classical (inverse) Fourier transform!

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Example: $G = [-\pi, \pi]$ $(\simeq \mathbb{T})$

- $\mu = \lambda$ (up to normalization)
- Characters: $\chi(x) = \mathrm{e}^{inx}$, $n \in \mathbb{Z}$, hence $\widehat{G} \simeq (\mathbb{Z}, +)$.
- $\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\mathrm{e}^{-inx}\mathrm{d}x$

Cosine series

Let $g : [0, \pi] \to \mathbb{R}$. Then, under suitable conditions (e.g., g Hölder)

$$g(x) = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k \cos(kx) =: \sum_{k=0}^{\infty}{}' A_k \cos(kx), \quad A_k := \frac{2}{\pi}\int_0^{\pi} g(x)\cos(kx)\,\mathrm{d}x.$$

More generally, let $g : [a, b] \to \mathbb{R}$. Then,

$$g(x) = \sum_{k=0}^{\infty}{}' A_k \cos\left(k\pi\frac{x-a}{b-a}\right), \quad A_k := \frac{2}{b-a}\int_a^b g(x)\cos\left(k\pi\frac{x-a}{b-a}\right)\mathrm{d}x.$$

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4. Truncation of the series: $E[f(X)] \approx {\sum_{k=0}^{N-1}}' F_k \int_a^b f(x)\cos\left(k\pi\frac{x-a}{b-a}\right)\mathrm{d}x =: {\sum_{k=0}^{N-1}}' F_k C_k$

$$f_{\mathsf{call}}(x) = \left(e^x - K\right)^+, \quad f_{\mathsf{put}}(x) = \left(K - e^x\right)^+$$

$$f_{\text{call}}(x) = (\mathrm{e}^x - K)^+ , \quad f_{\text{put}}(x) = (K - \mathrm{e}^x)^+$$

$$\chi_k(c,d) := \frac{1}{1+\left(\frac{k\pi}{b-a}\right)^2}\left[\cos\left(k\pi\frac{d-a}{b-a}\right)\mathrm{e}^d - \cos\left(k\pi\frac{c-a}{b-a}\right)\mathrm{e}^c + \frac{k\pi}{b-a}\left\{\sin\left(k\pi\frac{d-a}{b-a}\right)\mathrm{e}^d - \sin\left(k\pi\frac{c-a}{b-a}\right)\mathrm{e}^c\right\}\right]$$

$$\psi_k(c,d) := \begin{cases}\left(\sin\left(k\pi\frac{d-a}{b-a}\right) - \sin\left(k\pi\frac{c-a}{b-a}\right)\right)\frac{b-a}{k\pi}, & k \neq 0,\\ d-c, & k = 0.\end{cases}$$

$$C_k^{\text{call}} = \frac{2}{b-a}K(\chi_k(0,b) - \psi_k(0,b)), \quad C_k^{\text{put}} = \frac{2}{b-a}K(\psi_k(a,0) - \chi_k(a,0)).$$