Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Computational finance – Lecture 9

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Outline

1 **Affine processes**

2 Fourier method for option pricing

- Consider a time-homogeneous Markov process X on the state space $D := \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$, $d := m + n$.
- Associated semigroup and infinitesimal generator defined by

$$P_t f(x) := E\left[f(X_t) \mid X_0 = x \right], \qquad \mathcal{A} f(x) := \lim_{t \to 0} \frac{P_t f(x) - f(x)}{t}.$$

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Definition (Affine process)

X is called affine iff there are functions $\phi : \mathbb{R}_{\geq 0} \times \mathrm{i}\mathbb{R}^d \to \mathbb{C}$, $\psi : \mathbb{R}_{\geq 0} \times \mathrm{i}\mathbb{R}^d \to \mathbb{C}^d$ s.t.

$$\forall x \in D,\ t \geq 0,\ u \in \mathrm{i}\mathbb{R}^d :\ E\left[\exp\left(\langle u, X_t \rangle\right) \mid X_0 = x\right] = \exp\left(\phi(t,u) + \langle \psi(t,u), x \rangle\right).$$

Lemma

$$\phi(t+s,u) = \phi(t,u) + \phi(s,\psi(t,u)), \quad \psi(t+s,u) = \psi(s,\psi(t,u))$$

Definition

The affine process X is called regular iff there are F, R continuous s.t.

$$F(u) = \frac{\partial}{\partial t}\phi(t,u)\Big|_{t=0}, \quad R(u) = \frac{\partial}{\partial t}\psi(t,u)\Big|_{t=0}.$$

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Lemma

For a regular affine process, the following system of ODEs holds:

$$\frac{\partial}{\partial t}\phi(t,u) = F(\psi(t,u)), \quad \phi(0,u) = 0,$$
$$\frac{\partial}{\partial t}\psi(t,u) = R(\psi(t,u)), \quad \psi(0,u) = u.$$

Theorem

The Markov process X with generator $\mathcal{A}$ is affine iff (up to some admissibility conditions)

$$\mathcal{A}f(x) = \sum_{k,l=1}^{d} A_{kl}(x)\frac{\partial^2 f(x)}{\partial x_k \partial x_l} + \langle B(x),\, \nabla f(x)\rangle - C(x)f(x)$$
$$+ \int_{D\setminus\{0\}} \left(f(x+\xi) - f(x) + \langle \nabla f(x),\, \chi(\xi)\rangle\right) M(x, \mathrm{d}\xi),$$

for A, B, C affine functions and $M(x,\mathrm{d}\xi) = m(\mathrm{d}\xi) + \sum_{i=1}^{m} x_i \mu_i(\mathrm{d}\xi)$, $\chi \in C_b$, $\chi(x) = x$ around 0.
F and R can be explicitly expressed in terms of A, B, C, M. Indeed,

$$F(u) = \langle A(0)u,\, u\rangle + \langle B(0),\, u\rangle - C(0) + \int_{D\setminus\{0\}} \left(\mathrm{e}^{\langle u,\xi\rangle} - 1 - \langle u,\, \chi(\xi)\rangle\right) m(\mathrm{d}\xi),$$

$$R_i(u) = \langle \partial_{x_i} A(0)u,\, u\rangle + \langle \partial_{x_i} B(0),\, u\rangle - \partial_{x_i} C(0) + \int_{D\setminus\{0\}} \left(\mathrm{e}^{\langle u,\xi\rangle} - 1 - \langle u,\, \chi(\xi)\rangle\right) \mu_i(\mathrm{d}\xi) \mathbf{1}_{i\le m}.$$

Proof

1. Lévy processes are affine. By the Lévy-Khintchine formula,
$E\left[\exp\left(\mathrm{i}\langle u, X_t\rangle\right) \mid X_0 = x\right] = \exp\left(-t\varphi(u) + \mathrm{i}\langle u, x\rangle\right)$, $u \in \mathbb{R}$, with

$$\varphi(u) = -i\langle u, \alpha\rangle + \frac{1}{2}\langle \Sigma u, u\rangle - \int_{\mathbb{R}^d} \left(\exp(i\langle u, x\rangle) - 1 - i\langle u, x\rangle \mathbf{1}_{|x|\leq 1}\right) \nu(\mathrm{d}x).$$

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2. The Heston model is affine. Letting $Y_t := \log S_t$, we have
$$\mathrm{d}Y_t = -\frac{1}{2}v_t\mathrm{d}t + \sqrt{v_t}\mathrm{d}Z_t, \quad \mathrm{d}v_t = \kappa(\xi - v_t)\mathrm{d}t + \eta\sqrt{v_t}\mathrm{d}W_t.$$
Indeed, the characteristic function can be expressed in closed form.

Lemma

For $a, b, c, u \in \mathbb{C}$, $a \neq 0$, $b^2 + 4ac \in \mathbb{C} \setminus \mathbb{R}_{<0}$, let $\lambda := \sqrt{b^2 + 4ac}$. Consider

$$\dot{y}(t) = ay(t)^2 + by(t) - c, \quad y(0) = u.$$

Then, for t s.t. the solution exists up to t, we have

$$y(t) = -\frac{2c(e^{\lambda t} - 1) - \left(\lambda(e^{\lambda t} + 1) + b(e^{\lambda t} - 1)\right)u}{\lambda(e^{\lambda t} + 1) - b(e^{\lambda t} - 1) - 2a(e^{\lambda t} - 1)u},$$

$$\int_0^t y(s)ds = \frac{1}{a}\log\left(\frac{2\lambda e^{(\lambda - b)t/2}}{\lambda(e^{\lambda t} + 1) - b(e^{\lambda t} - 1) - 2a(e^{\lambda t} - 1)u}\right).$$

Additionally, if $a > 0$, $b \in \mathbb{R}$, $\Re c \geq 0$, $\Re u \leq 0$, then y is globally defined and $\Re y \leq 0$.

Characteristic function for the Heston example

- Setting: Let X be the log-price of the asset and f the payoff function (in x).

Example

In the Black-Scholes model, $X = X_T = \log S_0 + \left(r - \frac{\sigma^2}{2}\right) T + \sigma W_T$, and

$$f_{\text{put}}(x) := (K - e^x)^+, \quad f_{\text{call}}(x) := (e^x - K)^+, \quad x \in \mathbb{R}.$$

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- Define the moment generating function (Laplace transform)

$$M_X(u) := E\left[e^{uX}\right], \quad u \in \mathbb{C} \text{ s.t. } M_X \text{ exists.}$$

Further, let $\mathcal{J} := \{ R \in \mathbb{R} : M_X(R) < \infty \}$.

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- For $R \in \mathbb{R}$ define $f_R(x) := e^{-Rx} f(x)$ and let $\mathcal{I} := \left\{ R \in \mathbb{R} : f_R \in L^1_{\text{bc}}(\mathbb{R}) \text{ and } \widehat{f_R} \in L^1 \right\}$.

Theorem

Assume that $\mathcal{R} := \mathcal{I} \cap \mathcal{J} \neq \emptyset$ *and choose* $R \in \mathcal{R}$. *Then*

$$I[f;X] = E[f(X)] = \frac{1}{2\pi} \int_{\mathbb{R}} M_X(R - \mathrm{i}u)\widehat{f}(u + \mathrm{i}R)\mathrm{d}u.$$

- $\mathcal{I} \neq \emptyset$ requires continuity of f, whereas $\mathcal{J} \neq \emptyset$ is a pure integrability condition. Alternatively, we can use

$$\mathcal{I}' := \{R \in \mathbb{R} : f_R \in L^1(\mathbb{R})\} \quad \text{and} \quad \mathcal{J}' := \{R \in \mathbb{R} : M_X(R) < \infty \text{ and } M_X(R - \mathrm{i}\cdot) \in L^1(\mathbb{R})\}.$$

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- Minimal requirements: formula holds in the sense of a pointwise limit for

$$\mathcal{I}_{\min} := \{R \in \mathbb{R} : f_R \in L^1(\mathbb{R})\} \quad \text{and} \quad \mathcal{J}_{\min} := \{R \in \mathbb{R} : M_X(R) < \infty\}.$$

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- A sufficient condition for $\widehat{f} \in L^1(\mathbb{R})$ is that $f \in H^1(\mathbb{R})$.
- Assume that we extend the definition of the Fourier transform to $u \in \mathbb{C}$, for which the integral exists. Then

$$\widehat{f_R}(u) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}ux - Rx} f(x)\mathrm{d}x = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}(u+\mathrm{i}R)x} f(x)\mathrm{d}x = \widehat{f}(u + \mathrm{i}R).$$

Lemma (Call option)

For $f(x) = f_{call}(x) = (e^x - K)^+$, *we have* $\mathcal{I} =]1, \infty[$ *and*

$$\widehat{f}(u) = \frac{K^{1+iu}}{iu(1+iu)}, \quad u \in \mathbb{C}, \quad \Im u > 1.$$

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Lemma (Put option)

For $f(x) = f_{\mathit{put}}(x) = (K - \mathrm{e}^x)^+$, *we have* $\mathcal{I} =]-\infty, 0[$ *and (same formula!!!)*

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- $\widehat{f}$ has singularities in $u = 0$ and $u = i$. Moving the contour integral in the Fourier pricing formula over both singularities, residual calculus provides the put-call-parity:

$$C(S_0, K, T) = P(S_0, K, T) + S_0 - e^{-rT} K.$$