Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz
Association

Computational finance – Lecture 8

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- $h_l := TN^{-l}$, $l = 0, \dots, L$, $P_l := f\left(\overline{X}_T^{(h_l)}\right)$, $I_l := \frac{1}{M_l} \sum_{m=1}^{M_l} \left(P_l^{(m)} - P_{l-1}^{(m)}\right)$, with $P_{-1} := 0$.

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- There are constants $\alpha \geq \frac{1}{2}$ (weak rate), $\beta > 0$ (twice strong rate), $C_1, C_2 > 0$ s.t.

$$\left|E\left[f(X_T) - P_l\right]\right| \leq C_1 h_l^{\alpha}, \quad \operatorname{var}(I_l) \leq C_2 \frac{h_l^{\beta}}{M_l}$$

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Theorem

For any error tolerance $\varepsilon > 0$ there are $L, M_0, \ldots, M_L \in \mathbb{N}$ s.t. $I := \sum_{l=0}^{L} I_l$ satisfies $E\left[|E[f(X_T)] - I|^2\right]^{1/2} \leq \varepsilon$ with computational work

$$C \leq \begin{cases} C_3\varepsilon^{-2}, & \beta > 1, \\ C_3\varepsilon^{-2}\log(\varepsilon)^2, & \beta = 1, \\ C_3\varepsilon^{-2-(1-\beta)/\alpha}, & \beta < 1. \end{cases}$$

Corollary

Choose $L = \frac{-\log \varepsilon}{\log N} + \mathcal{O}(1)$ *and* $M_l \simeq (L+1) h_l \varepsilon^{-2}$*. Then the MLMC estimator for the Euler scheme has RMSE* $\mathcal{O}(\varepsilon)$ *with computational cost proportional to* $(\log \varepsilon)^2 \varepsilon^{-2}$*.*

Outline

- The characteristic function of the asset price distribution is often known in closed form.
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$$\psi(u) = -iu \cdot \alpha + \frac{1}{2}\Sigma u \cdot u - \int_{\mathbb{R}^d} \left(\exp(iu \cdot x) - 1 - iu \cdot x\mathbf{1}_{|x|\leq 1}\right) \nu(dx).$$

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- Affine processes: Suppose that X is a Markov process with "generator L affine in the state variable x". Then, for $u \in \mathbb{C}$ s.t. the expectation exists,

$$E[\exp(u \cdot X_t) \mid X_0 = x] = \exp\left(\phi(t,u) + x \cdot \psi(t,u)\right), \quad \text{for } \phi,\ \psi \text{ solving Riccati ODEs.}$$

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- Use suitable generalization of Parseval's identity:
$$E[f(X_T)] = \int_{\mathbb{R}^d} f(x) P_{X_T}(\mathrm{d}x) = \int_{\mathbb{R}^d} \widehat{f}(u)\widehat{P_X}(u)\mathrm{d}u.$$

- For simplicity, we now assume $d = 1$.

Definition

Let $f \in L^1 := L^1(\mathbb{R}, \mathcal{B}(\mathbb{R})\mathrm{d}x; \mathbb{C})$. Then for $u \in \mathbb{R}$, we define

$$\widehat{f}(u) := \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}ux} f(x)\mathrm{d}x, \quad \widecheck{f}(u) := \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}ux} f(x)\mathrm{d}x.$$

Both definitions are extended to $u \in \mathbb{C}$ wherever the integrals exist.

Theorem

For $f \in L^1$ we have $\widehat{f} \in C_0$, the continuous functions vanishing at ∞.

1. If $g(x) = f(x)\mathrm{e}^{\mathrm{i}ax}$ for $a \in \mathbb{R}$, then $\widehat{g}(u) = \widehat{f}(u+a)$.
2. If $g(x) = f(x-a)$ for $a \in \mathbb{R}$, then $\widehat{g}(u) = \mathrm{e}^{\mathrm{i}ua}\widehat{f}(u)$.
3. If $g(x) = f(x/\lambda)$ for $\lambda \in \mathbb{R}_+$, then $\widehat{g}(u) = \lambda \widehat{f}(\lambda u)$.
4. If $g(x) = \overline{f(-x)}$, then $\widehat{g}(u) = \overline{\widehat{f}(u)}$.
5. If $g \in L^1(\mathbb{R})$ and $h = f * g$, i.e., $h(x) = \int_{\mathbb{R}} f(y)g(x-y)\mathrm{d}y$, then $\widehat{h}(u) = \widehat{f}(u)\widehat{g}(u)$.
6. If $g(x) = \mathrm{i}xf(x)$ and $g \in L^1(\mathbb{R})$, then $\widehat{f}$ is continuously differentiable with $\left(\widehat{f}\right)'(u) = \widehat{g}(u)$.
7. Let $f \in C^1(\mathbb{R})$ and assume that $f, f' \in L^1_{\mathrm{bc}}(\mathbb{R}) := L^1(\mathbb{R}) \cap C_b(\mathbb{R})$. Then $\widehat{f'}(u) = -\mathrm{i}u\widehat{f}(u)$ and, in particular, $u \mapsto u\widehat{f}(u)$ is bounded.
8. Let $f \in C^2(\mathbb{R})$ and assume that $f, f', f'' \in L^1_{\mathrm{bc}}(\mathbb{R})$. Then $\widehat{f} \in L^1_{\mathrm{bc}}(\mathbb{R})$.
9. If $f \in L^1_{\mathrm{bc}}(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$, then the Fourier inversion formula $f = \widecheck{\widehat{f}}$ holds, i.e., for every $x \in \mathbb{R}$

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}ux}\widehat{f}(u)\mathrm{d}u.$$

Proofs

- Setting: Let X be the log-price of the asset and f the payoff function (in x).

Example

In the Black-Scholes model, $X = X_T = \log S_0 + \left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T$, and

$$f_{\text{put}}(x) := (K - e^x)^+, \quad f_{\text{call}}(x) := (e^x - K)^+, \quad x \in \mathbb{R}.$$

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- Define the moment generating function (Laplace transform)

$$M_X(u) := E\left[e^{uX}\right], \quad u \in \mathbb{C} \text{ s.t. } M_X \text{ exists.}$$

Further, let $\mathcal{J} := \{ R \in \mathbb{R} : M_X(R) < \infty \}$.

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- For $R \in \mathbb{R}$ define $f_R(x) := e^{-Rx} f(x)$ and let $\mathcal{I} := \left\{ R \in \mathbb{R} : f_R \in L^1_{\text{bc}}(\mathbb{R}) \text{ and } \widehat{f_R} \in L^1 \right\}$.

Theorem

Assume that $\mathcal{R} := \mathcal{I} \cap \mathcal{J} \neq \emptyset$ *and choose* $R \in \mathcal{R}$. *Then*

$$I[f;X] = E[f(X)] = \frac{1}{2\pi} \int_{\mathbb{R}} M_X(R - \mathrm{i}u)\widehat{f}(u + \mathrm{i}R)\mathrm{d}u.$$

- $\mathcal{I} \neq \emptyset$ requires continuity of f, whereas $\mathcal{J} \neq \emptyset$ is a pure integrability condition. Alternatively, we can use

$$\mathcal{I}' := \{R \in \mathbb{R} : f_R \in L^1(\mathbb{R})\} \quad \text{and} \quad \mathcal{J}' := \{R \in \mathbb{R} : M_X(R) < \infty \text{ and } M_X(R - \mathrm{i}\cdot) \in L^1(\mathbb{R})\}.$$

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- Minimal requirements: formula holds in the sense of a pointwise limit for

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- A sufficient condition for $\widehat{f} \in L^1(\mathbb{R})$ is that $f \in H^1(\mathbb{R})$.
- Assume that we extend the definition of the Fourier transform to $u \in \mathbb{C}$, for which the integral exists. Then

$$\widehat{f_R}(u) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}ux - Rx} f(x) \mathrm{d}x = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}(u + \mathrm{i}R)x} f(x) \mathrm{d}x = \widehat{f}(u + \mathrm{i}R).$$

Lemma (Call option)

For $f(x) = f_{call}(x) = (\mathrm{e}^x - K)^+$, *we have* $\mathcal{I} =]1, \infty[$ *and*

$$\widehat{f}(u) = \frac{K^{1+\mathrm{i}u}}{\mathrm{i}u(1+\mathrm{i}u)}, \quad u \in \mathbb{C}, \quad \Im u > 1.$$

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For $f(x) = f_{\mathit{put}}(x) = (K - \mathrm{e}^x)^+$, *we have* $\mathcal{I} =]-\infty, 0[$ *and (same formula!!!)*

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For $f(x) = f_{call}(x) = (e^x - K)^+$, we have $\mathcal{I} =]1, \infty[$ and

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$$\widehat{f}(u) = \frac{K^{1+iu}}{iu(1+iu)}, \quad u \in \mathbb{C}, \quad \Im u < 0.$$

- $\widehat{f}$ has singularities in $u = 0$ and $u = i$. Moving the contour integral in the Fourier pricing formula over both singularities, residual calculus provides the put-call-parity:

$$C(S_0, K, T) = P(S_0, K, T) + S_0 - e^{-rT} K.$$