Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Computational finance – Lecture 7

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&= {\color{red} x + \beta(x)W_h} + \frac{1}{2}\int_0^h \int_0^t \beta''(X_s)\beta(X_s)^2 \mathrm{d}s\mathrm{d}W_t + \int_0^h \int_0^t \beta'(X_s)\beta(X_s)\mathrm{d}W_s\mathrm{d}W_t
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&= x + \beta(x)W_h + \beta'(x)\beta(x)\int_0^h W_t\mathrm{d}W_t + \frac{1}{2}\beta''(x)\beta(x)^2\int_0^h t\mathrm{d}W_t + \text{H.O.T.}
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\end{aligned}$$

- The above scheme is the Milstein scheme (strong order 1, weak order 1).
- Higher order scheme require simulation of iterated integrals of $(t, W_t^1, \dots, W_t^d)$. This is not feasible in the strong sense for $d > 1$. (Exception: commuting vector fields.)

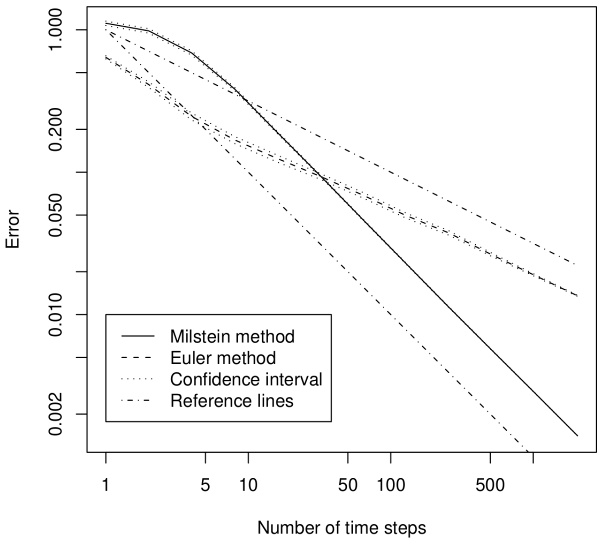
Example

Linear vector fields $V_i(x) = A_i x$, $i = 1, 2$, $n = 3$,

$$dX_t = V_1(X_t) \circ dW_t^1 + V_2(X_t) \circ dW_t^2,$$

$$A_1 = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix},$$

$$f(x) := (|x| - 1)^+, \quad X_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$



$X = X_T$ is the solution of an SDE, $f : \mathbb{R}^n \to \mathbb{R}$ a payoff. Compute $E[f(X_T)] = I[f; X_T]$.

Euler – Monte Carlo method

$$E[f(X_T)] \approx I_M\left[f; \overline{X}_T\right], \quad \overline{X}_T \text{ based on grid of size } N.$$

- Computing $E[f(\overline{X}_T)]$ is a $N \times d$-dimensional integration problem (difficult for QMC?).
- Error decomposition:

$$\left|E[f(X_T)] - I_M\left[f; \overline{X}_T\right]\right| \le \underbrace{\left|E[f(X_T)] - E\left[f\left(\overline{X}_T\right)\right]\right|}_{=: e_{\text{disc}}} + \underbrace{\left|E\left[f\left(\overline{X}_T\right)\right] - I_M\left[f; \overline{X}_T\right]\right|}_{=: e_{\text{stat}}}$$

- Generically, $e_{\text{disc}} \lesssim C_{\text{disc}}/N$, $e_{\text{stat}} \lesssim C_{\text{stat}}/\sqrt{M}$. Hence, given error tolerance $\text{TOL} > 0$, choose $N \simeq \text{TOL}^{-1}$, $M \simeq \text{TOL}^{-2}$, leading to computational cost $\simeq \text{TOL}^{-3}$.

1 Multilevel Monte Carlo

Idea

Let $\overline{X}_T$, $\overline{X}'_T$ be the Euler approximations of X_T based on different step sizes $h < h'$, but the same Brownian paths. Then $\overline{X}_T$ and $\overline{X}'_T$ are highly correlated. Hence, $f(\overline{X}'_T)$ can serve as control variate for $f(\overline{X}_T)$.

By strong convergence, $\mathrm{var}\left(f(\overline{X}_T^{(h_L)}) - f(\overline{X}_T^{(h_{L-1})})\right) \lesssim h_L$, allowing us to choose $M_L \to 0$.

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- Let $h_0 > h_1 > \ldots > h_L > 0$ and $\overline{X}^{(h)}$ the Euler scheme with step size h.

By strong convergence, $\mathrm{var}\left(f(\overline{X}_T^{(h_L)}) - f(\overline{X}_T^{(h_{L-1})})\right) \lesssim h_L$, allowing us to choose $M_L \to 0$.

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- Let $h_0 > h_1 > \ldots > h_L > 0$ and $\overline{X}^{(h)}$ the Euler scheme with step size h.

1. Compute $E[f(\overline{X}_T^{(h_L)}) - f(\overline{X}_T^{(h_{L-1})})]$ by Monte Carlo with M_L samples, at cost $\simeq \frac{M_L}{h_L}$.
2. Compute $E[f(\overline{X}_T^{(h_{L-1})}) - f(\overline{X}_T^{(h_{L-2})})]$ by Monte Carlo with M_{L-1} samples, at cost $\simeq \frac{M_{L-1}}{h_{L-1}}$.

   $\ldots$

L. Compute $E[f(\overline{X}_T^{(h_0)})]$ by Monte Carlo with M_0 samples, at cost $\simeq \frac{M_0}{h_0}$.

By strong convergence, $\mathrm{var}\left(f(\overline{X}_T^{(h_L)}) - f(\overline{X}_T^{(h_{L-1})})\right) \lesssim h_L$, allowing us to choose $M_L \to 0$.

- $h_l := TN^{-l}$, $l = 0, \dots, L$, $P_l := f\left(\overline{X}_T^{(h_l)}\right)$, $I_l := \frac{1}{M_l} \sum_{m=1}^{M_l} \left(P_l^{(m)} - P_{l-1}^{(m)}\right)$, with $P_{-1} := 0$.

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- There are constants $\alpha \geq \frac{1}{2}$ (weak rate), $\beta > 0$ (twice strong rate), $C_1, C_2 > 0$ s.t.

$$\left|E\left[f(X_T) - P_l\right]\right| \leq C_1 h_l^{\alpha}, \quad \operatorname{var}(I_l) \leq C_2 \frac{h_l^{\beta}}{M_l}$$

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Theorem

For any error tolerance $\varepsilon > 0$ *there are* $L, M_0, \dots, M_L \in \mathbb{N}$ *s.t.* $I := \sum_{l=0}^{L} I_l$ *satisfies* $E\left[|E[f(X_T)] - I|^2 \right]^{1/2} \leq \varepsilon$ *with computational work*

$$C \leq \begin{cases} C_3 \varepsilon^{-2}, & \beta > 1, \\ C_3 \varepsilon^{-2} \log(\varepsilon)^2, & \beta = 1, \\ C_3 \varepsilon^{-2-(1-\beta)/\alpha}, & \beta < 1. \end{cases}$$

Corollary

Choose $L = \frac{-\log \varepsilon}{\log N} + O(1)$ *and* $M_l \simeq (L+1)h_l\varepsilon^{-2}$. *Then the MLMC estimator for the Euler scheme has RMSE* $O(\varepsilon)$ *with computational cost proportional to* $(\log \varepsilon)^2\varepsilon^{-2}$.