Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Computational finance – Lecture 5

Christian Bayer

- Notation: Fix a time grid $\mathcal{D} := \{ 0 = t_0 < t_1 < \cdots < t_N = T \}$, $\Delta t_i := t_i - t_{i-1}$, $\Delta Y_i := Y_{t_i} - Y_{t_{i-1}}$,

$$|\mathcal{D}| := \max_{1 \le i \le N} |t_i - t_{i-1}|, \quad \lfloor t \rfloor := \sup \{ t_i \mid i = 0, \ldots, N, \ t_i \le t \}.$$

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- ODE $\dot{x}(t) = V(x(t)), \ x(0) = x_0 \in \mathbb{R}^n$
- By Taylor expansion: $x(t_{i+1}) = x(t_i) + \dot{x}(t_i)\Delta t_i + \mathcal{O}(\Delta t_i^2) = x(t_i) + V(x(t_i))\Delta t_i + \mathcal{O}(\Delta t_i^2)$

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(Explicit) Euler scheme

- $\overline{x}_0 := x_0$, $\overline{x}_{t_{i+1}} := \overline{x}_{t_i} + \Delta t_i V(\overline{x}_{t_i})$
- Error: ignoring error propagation, we have $|x(T) - \overline{x}_T| \le \sum_{i=1}^N \mathcal{O}(\Delta t_i^2) = \mathcal{O}(|\mathcal{D}|)$.

- Many other variants: implicit Euler scheme, Runge-Kutta schemes, multi-step schemes, splitting schemes, ...

$$\mathrm{d}X_t = V(X_t)\mathrm{d}t + \sum_{i=1}^{d} V_i(X_t)\mathrm{d}W_t^i, \quad X_0 = x_0 \in \mathbb{R}^n$$

Euler–Maruyama scheme

$$\overline{X}_0 := x_0, \quad \overline{X}_{t_{j+1}} := \overline{X}_{t_j} + V\left(\overline{X}_{t_j}\right)\Delta t_j + \sum_{i=1}^{d} V_i\left(\overline{X}_{t_i}\right)\Delta W_j^i, \quad j = 0, \ldots, N-1.$$

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- Implicit schemes: Generally only drift implicit schemes possible, since

$$\int_0^T Y_s \mathrm{d}W_s = \lim_{|\mathcal{D}|\to 0} \sum_{j=0}^{N-1} Y_{t_j}\Delta W_j \neq \lim_{|\mathcal{D}|\to 0} \sum_{j=0}^{N-1} Y_{t_{j+1}}\Delta W_j$$

- Naive error control: $\mathcal{O}(\Delta W_j) = \mathcal{O}(\sqrt{\Delta t_j})$, hence

$$\text{error} \approx \sum_{j=0}^{N-1} \left(\mathcal{O}(\Delta t_j^2) + \mathcal{O}(\Delta W_j^2)\right) \approx \sum_{j=0}^{N-1} \mathcal{O}(\Delta t_j) \approx \mathcal{O}(1)$$

- Test functions $\mathcal{G}$, e.g., $\mathcal{G} = C^k_{\mathrm{pol}}$ or $\mathcal{G} = L^\infty$

Strong convergence

$\overline{X}_T$ converges strongly to X_T iff

$$\lim_{|\mathcal{D}|\to 0} E\left[\left|\overline{X}_T - X_T\right|\right] = 0,$$

with rate $\gamma > 0$ iff $\exists C$ indep. of $\mathcal{D}$ s.t.

$$E\left[\left|\overline{X}_T - X_T\right|\right] \le C\,|\mathcal{D}|^\gamma$$

(for $|\mathcal{D}|$ small enough).

Weak convergence

$\overline{X}_T$ converges weakly to X_T iff

$$\forall f \in \mathcal{G}: \lim_{|\mathcal{D}|\to 0} E\left[f(\overline{X}_T)\right] = E[f(X_T)],$$

with rate $\gamma > 0$ iff $\exists C = C(f)$ indep. of $\mathcal{D}$ s.t.

$$\forall f \in \mathcal{G}: \left|E\left[f(\overline{X}_T)\right] - E[f(X_T)]\right| \le C\,|\mathcal{D}|^\gamma$$

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- Weak convergence does not require $\overline{X}$ to be defined on the same probability space as X.

- Adapted continuous extension of the approximate solution: for $0 \le t \le T$ let

$$\overline{X}_t := \overline{X}_{\lfloor t \rfloor} + V\left(\overline{X}_{\lfloor t \rfloor}\right)(t - \lfloor t \rfloor) + \sum_{i=1}^{d} V_i\left(\overline{X}_{\lfloor t \rfloor}\right)\left(W_t^i - W_{\lfloor t \rfloor}^i\right)$$

$$\overline{X}_t = x_0 + \int_0^t V\left(\overline{X}_{\lfloor s \rfloor}\right) \mathrm{d}s + \sum_{i=1}^{d} \int_0^t V_i\left(\overline{X}_{\lfloor s \rfloor}\right) \mathrm{d}W_s^i$$

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Theorem

Suppose that $V, V_1, \dots V_d$ are Lipschitz and have linear growth with constant K. Then

$$E\left[\sup_{0 \le t \le T} \left|X_t - \overline{X}_t\right|\right] \le C\sqrt{|\mathcal{D}|}$$

for some constant C only depending on the coefficients, the initial value and the time horizon T. In particular, the Euler-Maruyama method has strong order $1/2$.

- Note that uniform Lipschitz continuity implies linear growth:

$$|V(x)| \leq |V(x) - V(0)| + |V(0)| \leq \widetilde{K}(1 + |x|)$$

Lemma (Grönwall's inequality)

Let u be real-valued, continuous, $\alpha, \beta \geq 0$. If

$$\forall 0 \leq t \leq T: \ u(t) \leq \alpha + \beta \int_0^t u(s)ds \quad \Rightarrow \quad \forall 0 \leq t \leq T: \ u(t) \leq \alpha e^{\beta t}.$$

Lemma (Doob's inequality, L^2 case)

Let $(M_t)_{t \in [0,T]}$ be a square integrable martingale. Then $E\left[\sup_{0 \leq t \leq T} |M_t|^2\right] \leq 4E\left[|M_T|^2\right]$.

- Notation: $V_0 := V$, $W_t^0 := t$
- Known estimates from existence and uniqueness proof:

$$E\left[\sup_{0\le t\le T} |X_t|^2\right] \le C(1+|x_0|^2), \quad E\left[\sup_{0\le t\le T} \left|\overline{X}_t\right|^2\right] \le C(1+|x_0|^2)$$

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$e_t := E\left[\sup_{0\le s\le t}\left|X_s - \overline{X}_s\right|^2\right] \le C\sum_{i=0}^d \left(c_t^i + d_t^i\right)$, where

$$c_t^i := E\left[\sup_{0\le s\le t}\left|\int_0^s \left(V_i(X_u) - V_i(X_{\lfloor u\rfloor})\right)\mathrm{d}W_u^i\right|^2\right], \quad d_t^i := E\left[\sup_{0\le s\le t}\left|\int_0^s \left(V_i(X_{\lfloor u\rfloor}) - V_i(\overline{X}_{\lfloor u\rfloor})\right)\mathrm{d}W_u^i\right|^2\right]$$

$$d_t^0 \leq K^2 T \int_0^t e_u \mathrm{d}u$$

$$d_t^i \leq 4K^2 \int_0^t e_u \mathrm{d}u, \quad 1 \leq i \leq d$$

$$c_t^i \leq CE\left[\int_0^t \left|X_u - X_{\lfloor u \rfloor}\right|^2 \mathrm{d}u\right] \leq C(1 + |x_0|^2)\,|\mathcal{D}|, \quad 0 \leq i \leq d$$

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- By the Cauchy-Schwarz inequality, we have

$$E\left[\sup_{0\leq s\leq T}\left|X_s - \overline{X}_s\right|\right] \leq E\left[\sup_{0\leq s\leq T}\left|X_s - \overline{X}_s\right|^2\right]^{1/2} \leq \widetilde{C}\sqrt{|\mathcal{D}|}.$$

Existence and uniqueness

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- Similarly, the above convergence proof implies existence & uniqueness for SDEs.

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Strategy:

1. Uniqueness clearly follows, since $\overline{X}^{\mathcal{D}}$ converges to any solution X.
2. For two grids $\mathcal{D}, \mathcal{D}'$, we have $E\left[\sup_{0\le t\le T}\left|\overline{X}_t^{\mathcal{D}} - \overline{X}_t^{\mathcal{D}'}\right|^2\right] \le C \max\left(|\mathcal{D}|, |\mathcal{D}'|\right).$

   (Proof: Compare both with $\overline{X}^{\mathcal{D}''}$, $\mathcal{D}'' := \mathcal{D} \cup \mathcal{D}'$ replacing the exact solution.)

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3. It follows that there is a unique limit process $\widetilde{X} := \lim_{|D|\to 0} \overline{X}^{\mathcal{D}}$.
4. $\widetilde{X}$ solves the SDE by similar estimates, showing that integrals in $V(\overline{X}^{\mathcal{D}})$, $V_i(\overline{X}^{\mathcal{D}})$ converge to the corresponding integrals in $\widetilde{X}$.