Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Computational finance – Lecture 4

Christian Bayer

Mohrenstrasse 39 · 10117 Berlin · Germany · Tel. +49 30 20372 0 · www.wias-berlin.de
May 7, 2021

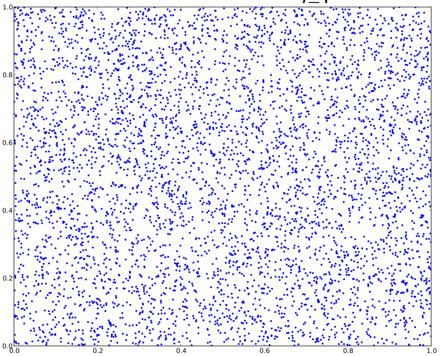
1 Quasi Monte Carlo

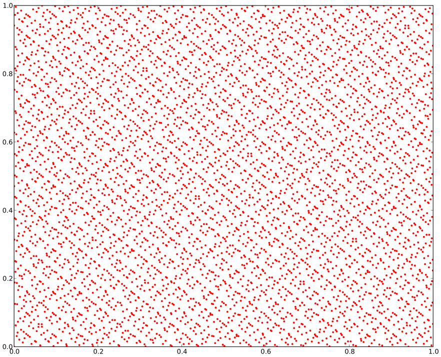
2 Sample path generation

More evenly distributed numbers

$$I[f] := \int_{[0,1]^d} f(x)dx$$

- Quadrature: $J_M[f] := \frac{1}{M}\sum_{i=1}^{M} f(x_i)$, $x_1, \dots, x_M \in [0,1]^d$.





- Intuition: discrepancy is the quadrature error for indocator functions of rectangles R.
- Consider point sets $(x_i)_{i=1}^M \subset (x_i)_{i\in\mathbb{N}}$ or $(x_i^M)_{i=1}^M$, $x_i \in [0,1]^d$.

- Intuition: discrepancy is the quadrature error for indocator functions of rectangles R.
- Consider point sets $(x_i)_{i=1}^M \subset (x_i)_{i\in\mathbb{N}}$ or $(x_i^M)_{i=1}^M$, $x_i \in [0,1]^d$.

Definition (Discrepancy)

The discrepancy D_M and star-discrepancy D_M^* for a point set $(x_i)_{i=1}^M$ are defined as

$$D_M = \sup_{\text{rectangles } R\subset[0,1]^d} \left| \frac{1}{M} \#\{\, 1 \le i \le M : x_i \in R \,\} - \lambda(R) \right|,$$

$$D_M^* = \sup\left\{ \left| \frac{1}{M} \#\{\, 1 \le i \le M : x_i \in R \,\} - \lambda(R) \right| \;:\; R = \bigtimes_{j=1}^{d} [0, b_j[,\ b_1, \dots, b_d \in [0,1] \right\}.$$

- Intuition: discrepancy is the quadrature error for indocator functions of rectangles R.
- Consider point sets $(x_i)_{i=1}^M \subset (x_i)_{i\in\mathbb{N}}$ or $(x_i^M)_{i=1}^M$, $x_i \in [0,1]^d$.

Definition (Discrepancy)

The discrepancy D_M and star-discrepancy D_M^* for a point set $(x_i)_{i=1}^M$ are defined as

$$D_M = \sup_{\text{rectangles } R\subset[0,1]^d} \left| \frac{1}{M} \#\{ 1 \le i \le M : x_i \in R \} - \lambda(R) \right|,$$

$$D_M^* = \sup\left\{ \left| \frac{1}{M} \#\{ 1 \le i \le M : x_i \in R \} - \lambda(R) \right| \;:\; R = \bigtimes_{j=1}^{d} [0, b_j[,\ b_1, \dots, b_d \in [0,1] \right\}.$$

Definition (Low discrepancy)

A sequence of point sets $(x_i^M)_{i=1}^M$, $M \in \mathbb{N}$, has low discrepancy iff $D_M^* \le c \frac{\log(M)^d}{M}$.

- Regularity of integrand f classically measured by variation.

Definition (Variation in the sense of Hardy-Krause)

For a one-dimensional function $f : [0,1] \to \mathbb{R}$

$$V[f] := \int_0^1 \left| \frac{df}{dx}(x) \right| dx$$

and for $f : [0,1]^d \to \mathbb{R}$

$$V[f] := \int_{[0,1]^d} \left| \frac{\partial^d f}{\partial x^1 \cdots \partial x^d}(x) \right| dx + \sum_{j=1}^{d} V[f_1^{(j)}],$$

where $f_1^{(j)}$ denotes the restriction of f to the boundary $x^j = 1$.

Theorem (Koksma-Hlawka inequality)

$$|I[f] - J_M[f]| \leq V[f] D_M^*.$$

Theorem (Koksma-Hlawka inequality)

$$|I[f] - J_M[f]| \leq V[f] D^*_M.$$

- Error factorization into term depending on regularity of integrand and uniformity of point set.
- In contrast to MC error, the Koksma-Hlawka inequality is deterministic, but usually far from sharp! (Note that it essentially require f to be C^d!)

Theorem (Koksma-Hlawka inequality)

$$|I[f] - J_M[f]| \leq V[f] D_M^*.$$

- Error factorization into term depending on regularity of integrand and uniformity of point set.
- In contrast to MC error, the Koksma-Hlawka inequality is deterministic, but usually far from sharp! (Note that it essentially require f to be C^d!)

Example (Van der Corput sequence)

Let p prime and for $k \in \mathbb{N}_0$ let $(a_j(k))_{j \in \mathbb{N}_0}$ denote the p-ary expansion of k, i.e., $k = \sum_{j=0}^{\infty} a_j(k) p^j$. A one-dimensional low-discrepancy sequence is given by

$$x_i := \psi_p(i) := \sum_{j=0}^{\infty} \frac{a_j(i)}{p^{j+1}}, \quad i \in \mathbb{N}_0.$$

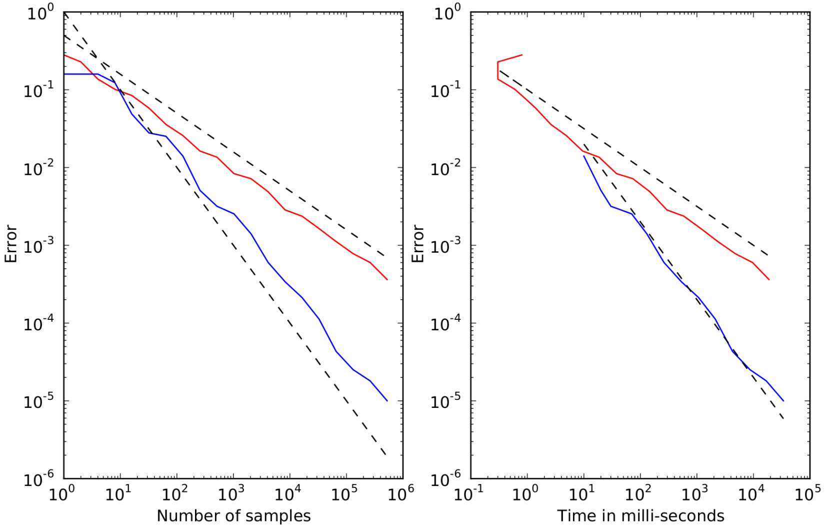


Figure: A call option in the Black-Scholes model using Monte Carlo and Quasi Monte Carlo simulation. Red: MC simulation, blue: QMC simulation, black: Reference lines proportional to $1/M$ and $1/\sqrt{M}$.

- For uniform (pseudo) r.v.s: If $U_i \sim \mathcal{U}([0,1])$, (U_i) independent, then $(U_{(i-1)d+1}, \ldots, U_{id})_{i \in \mathbb{N}}$ is a sequence of independent r.v.s from $\mathcal{U}([0,1]^d)$.
- In contrast, if $(x_i)_{i \in \mathbb{N}}$ is one-dimensional low-discrepancy sequence, then $(x_{(i-1)d+1}, \ldots, x_{id})_{i \in \mathbb{N}}$ is most likely not a d-dimensional low-discrepancy sequence.
- QMC convergence is asymptotically much faster than MC convergence, but note that $\log(M)^d/M \gg M^{-1/2}$ for all reasonably sized M even in fairly moderate dimensions d. E.g., for $d = 8$ roughly from $M \geq 1.8 \times 10^{29}$.

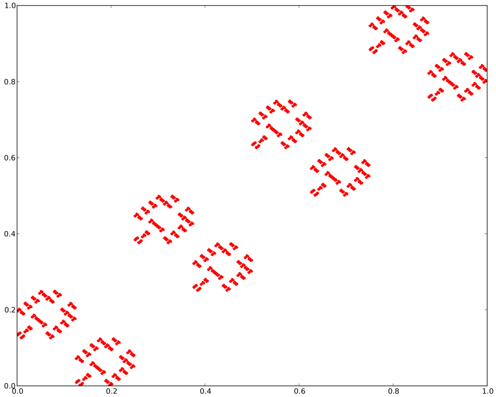


Figure: Pairs of one-dimensional Sobol numbers

Setting: $(x_i)_{i=1}^M$ low-discrepancy sequence in dim. d, $(U_l)_{l\in\mathbb{N}}$ i.i.d. sequence, $U_l \sim \mathcal{U}\left([0,1]^d\right)$.

Definition (Randomized quasi Monte Carlo simulation)

$$J^R_{M;m}[f] := \frac{1}{m}\sum_{l=1}^{m}\frac{1}{M}\sum_{i=1}^{M} f(x_i + U_l \pmod 1)$$

Setting: $(x_i)_{i=1}^M$ low-discrepancy sequence in dim. d, $(U_l)_{l\in\mathbb{N}}$ i.i.d. sequence, $U_l \sim \mathcal{U}\left([0,1]^d\right)$.

Definition (Randomized quasi Monte Carlo simulation)

$$J^R_{M;m}[f] := \frac{1}{m}\sum_{l=1}^{m}\frac{1}{M}\sum_{i=1}^{M} f(x_i + U_l \pmod 1)$$

- Provides variance reduction for the Monte Carlo estimator $\frac{1}{m}\sum_{l=1}^{m} f(U_l)$
- Provides sharp, computable error control for the quasi Monte Carlo approximation $\frac{1}{M}\sum_{i=1}^{M} f(x_i)$:

$$E\left[\left(I[f] - J^R_{M;m}[f]\right)^2\right] = \frac{\operatorname{var}\left(\frac{1}{M}\sum_{i=1}^{M} f(x_i + U \pmod 1)\right)}{m}$$

- Typically $m \ll M$, as accuracy is more important than error control.

Goal

Simulate a trajectory $W_{t_1}, \dots, W_{t_N}$ of a Brownian motion W, $0 \le t_1 < t_2 < \dots < t_N$.

Cholesky method

- Recall: if $X \sim \mathcal{N}(0, \Sigma)$ and $\Sigma = AA^\top$, then $X \overset{D}{=} AZ$ for $Z \sim \mathcal{N}(0, I)$.
- For $(W_{t_1}, \dots, W_{t_N})$: $\Sigma = (t_n \wedge t_m)_{n,m=1}^N$ and its Cholesky factorization $AA^\top$:

$$A = \begin{pmatrix} \sqrt{t_1} & 0 & \dots & 0 \\ \sqrt{t_1} & \sqrt{t_2 - t_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{t_1} & \sqrt{t_2 - t_1} & \dots & \sqrt{t_N - t_{N-1}} \end{pmatrix}.$$

Random walk construction

- $\Delta W_n := W_{t_n} - W_{t_{n-1}} \sim \sqrt{t_n - t_{n-1}} Z_n$, $Z \sim \mathcal{N}(0, I_N)$
- Same result as Cholesky method!

Brownian bridge construction

- $W_{t_N} \sim \mathcal{N}(0, t_N)$, $W_{t_0} := W_0 = 0$.
- $(W_s | W_u = x, W_t = y) \sim \mathcal{N}\left(\frac{(t-s)x + (s-u)y}{t-u}, \frac{(s-u)(t-s)}{t-u} \right)$.

Brownian bridge construction

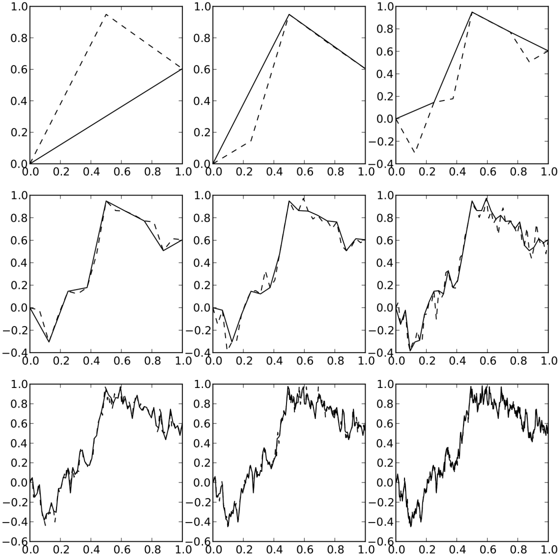


Figure: Brownian motion constructed by the Brownian bridge approach. Dashed lines correspond to the newly inserted Brownian bridge

- Construct functional approximations $(W_t^{(N)})_{t\in[0,T]}$ of the Bm $(W_t)_{t\in[0,T]}$.
- For simplicity, restriction to $T = 1$ and Haar wavelets, i.e., Lévy's construction of Bm.

- Construct functional approximations $(W_t^{(N)})_{t\in[0,T]}$ of the Bm $(W_t)_{t\in[0,T]}$.
- For simplicity, restriction to $T = 1$ and Haar wavelets, i.e., Lévy's construction of Bm.
- Construct basis functions $\psi_{n,k}$ by shifting and rescaling of a mother wavelet ψ

$$\psi_{n,k}(t) := 2^{n/2}\psi\left(2^n t - k\right), \quad \psi(t) := \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t < 1, \\ 0, & \text{else}, \end{cases} \quad n \ge 0, \quad 0 \le k \le 2^n - 1, \quad t \in [0, 1].$$

- Note that $\{\psi_{n,k}\}$ form an orthonormal basis of $L^2\left([0, 1]\right)$.

- Construct functional approximations $(W_t^{(N)})_{t\in[0,T]}$ of the Bm $(W_t)_{t\in[0,T]}$.
- For simplicity, restriction to $T = 1$ and Haar wavelets, i.e., Lévy's construction of Bm.
- Construct basis functions $\psi_{n,k}$ by shifting and rescaling of a mother wavelet ψ

$$\psi_{n,k}(t) := 2^{n/2}\psi\left(2^n t - k\right), \quad \psi(t) := \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t < 1, \\ 0, & \text{else}, \end{cases} \quad n \ge 0, \quad 0 \le k \le 2^n - 1, \quad t \in [0,1].$$

- Note that $\{\psi_{n,k}\}$ form an orthonormal basis of $L^2\left([0,1]\right)$.
- Given a i.i.d. sequence $X_0, X_{n,k}$ of standard normal r.v.s, a Bm is defined by

$$W_t := X_0 t + \sum_{n=0}^{\infty}\sum_{k=0}^{2^n-1} X_{n,k}\Psi_{n,k}(t), \quad t \in [0,1], \quad \Psi_{n,k}(t) := \int_0^t \psi_{n,k}(s)ds = 2^{-n/2}\Psi(2^n t - k)$$

- Define $W^{(N)}$ by truncation at $n = N$. Note that $W_t^{(N)} = W_t$ for $t \in \left\{ k2^{-N-1} \,\middle|\, 0 \le k \le 2^{N+1} \right\}$

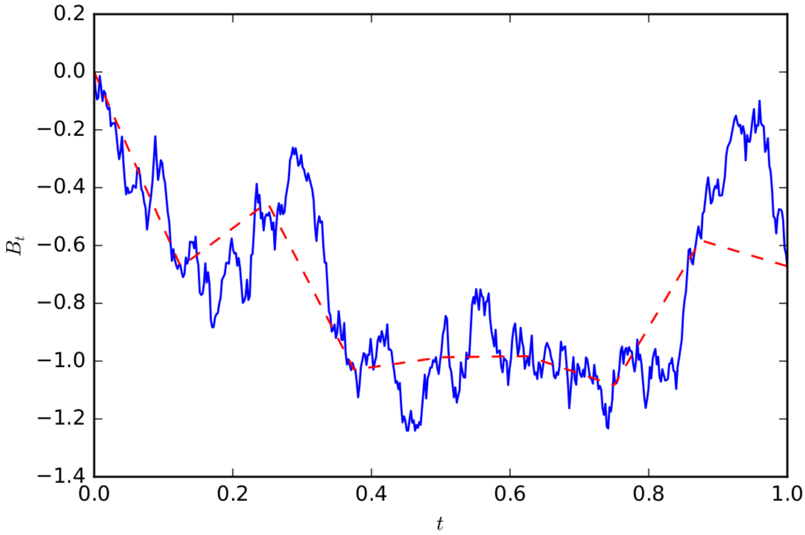


Figure: Approximate Brownian motion $B_t^{(N)}$, $0 \le t \le 1$, for $N = 10$ (blue) and $N = 2$ (red) superimposed

- Important as building blocks: Recall that finite activity (pure jump) Lévy processes are compound Poisson processes, i.e., of the form $Z_t = Z_0 + \sum_{i=1}^{N_t} X_i$.

- Important as building blocks: Recall that finite activity (pure jump) Lévy processes are compound Poisson processes, i.e., of the form $Z_t = Z_0 + \sum_{i=1}^{N_t} X_i$.
- Values on a grid: Note that increments are independent. Hence, $N_{t_1} \sim \mathrm{Poi}(\lambda t_1)$, $N_{t_2} - N_{t_1} \sim \mathrm{Poi}(\lambda(t_2 - t_1))$, ...
- Poisson bridge: $(N_s \mid N_t = n) \sim \mathrm{Bin}(n, p = s/t)$, $0 < s < t$.

- Important as building blocks: Recall that finite activity (pure jump) Lévy processes are compound Poisson processes, i.e., of the form $Z_t = Z_0 + \sum_{i=1}^{N_t} X_i$.
- Values on a grid: Note that increments are independent. Hence, $N_{t_1} \sim \mathrm{Poi}(\lambda t_1)$, $N_{t_2} - N_{t_1} \sim \mathrm{Poi}(\lambda(t_2 - t_1))$, ...
- Poisson bridge: $(N_s \mid N_t = n) \sim \mathrm{Bin}(n, p = s/t)$, $0 < s < t$.

Exact simulation

Exact simulation of the trajectory $(N_t)_{t\in[0,T]}$ by simulating the jump times $(T_1, \ldots, T_{N_T})$.

1. Interarrival times $\tau_n := T_n - T_{n-1}$ are i.i.d. $\sim \mathrm{Exp}(\lambda)$.
   - $T_0 := 0$, and iterate: simulate $\tau_n \sim \mathrm{Exp}(\lambda)$, set $T_n := T_{n-1} + \tau_n$, until $T_n > T$.

- Important as building blocks: Recall that finite activity (pure jump) Lévy processes are compound Poisson processes, i.e., of the form $Z_t = Z_0 + \sum_{i=1}^{N_t} X_i$.
- Values on a grid: Note that increments are independent. Hence, $N_{t_1} \sim \mathrm{Poi}(\lambda t_1)$, $N_{t_2} - N_{t_1} \sim \mathrm{Poi}(\lambda(t_2 - t_1))$, ...
- Poisson bridge: $(N_s \mid N_t = n) \sim \mathrm{Bin}(n, p = s/t)$, $0 < s < t$.

Exact simulation

Exact simulation of the trajectory $(N_t)_{t\in[0,T]}$ by simulating the jump times $(T_1, \dots, T_{N_T})$.

1. Interarrival times $\tau_n := T_n - T_{n-1}$ are i.i.d. $\sim \mathrm{Exp}(\lambda)$.
   - $T_0 := 0$, and iterate: simulate $\tau_n \sim \mathrm{Exp}(\lambda)$, set $T_n := T_{n-1} + \tau_n$, until $T_n > T$.
2. Given $N_T = n$, jump times $(T_1, \dots, T_n)$ are uniformly distributed on $[0, T]$.
   - Simulate $N_T \sim \mathrm{Poi}(\lambda T)$ and independent $U_1, \dots, U_{N_T} \sim \mathcal{U}([0, T])$.
   - Set $(T_1, \dots, T_{N_T}) := (U_{(1)}, \dots, U_{(N_T)})$, the order statistics of $U_1, \dots, U_{N_T}$.

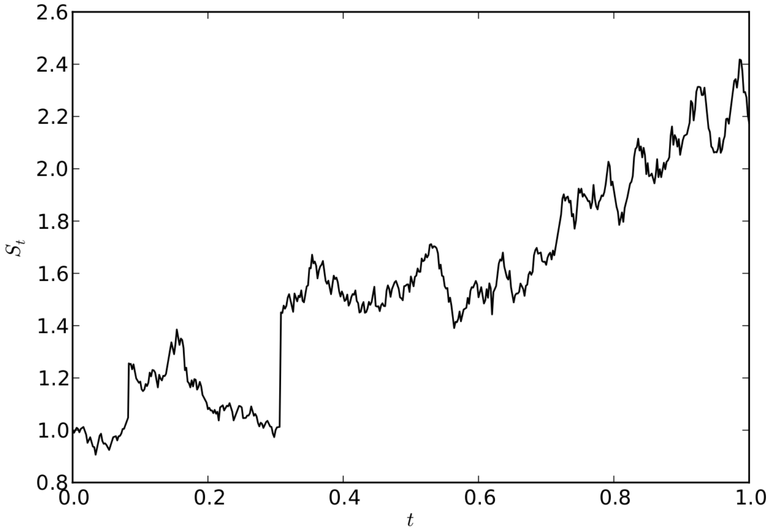


Figure: Trajectory of Merton's model $S_t = S_0 \exp\left(\sigma B_t + \left(\mu - \frac{\sigma^2}{2}\right) t\right) \prod_{j=1}^{N_t} X_j$ generated by exact simulation of jump times and simulation of Bm by random walk construction inbetween.